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Primer of spherical harmonic analysis on $SL_2(\mathbb{C})$

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We derive basic L^2 properties of $SU(2)$-bi-invariant functions on $SL_2(\mathbb{C})$.

One important thing *neglected* here is development of global Sobolev theory for $SU(2)$-bi-invariant functions on $SL_2(\mathbb{C})$. Nevertheless, the L^2 theory suggests the correct approach to such a Sobolev theory, and our heuristic for solving

$$\big(\Delta - s(z-1)\big)^2 u_z \;=\; \delta \qquad\qquad (\text{on } SL_2(\mathbb{C})/SU(2))$$

motivates the Sobolev theory.

Throughout, corresponding phenomena for $SL_2(\mathbb{R})$ are compared.

[1.1] Hyperbolic three-space With $G = SL_2(\mathbb{C})$ and $K = SU(2)$, the quotient G/K is a model of hyperbolic three-space. [1] Similarly, $SL_2(\mathbb{R})/SO(2)$ is a model of hyperbolic two-space.

The standard *split component* in both $SL_2(\mathbb{R})$ and $SL_2(\mathbb{C})$ is

$$A \;=\; \{a_r = \begin{pmatrix} e^{r/2} & 0 \\ 0 & e^{-r/2} \end{pmatrix} \;:\; r \in \mathbb{R}\}$$

Let

$$A^+ \;=\; \{a_r = \begin{pmatrix} e^{r/2} & 0 \\ 0 & e^{-r/2} \end{pmatrix} \;:\; r \geq 0\}$$

The Cartan decomposition is [2]

$$G \;=\; KA^+K$$

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[1] We have no reason to give any definition of hyperbolic three-space other than as this quotient of $SL_2(\mathbb{C})$ by $SU(2)$, as we invoke no other properties than those that follow from this model, so there is no immediate reason to elaborate on any comparison with other definitions.

[2] If we knew $g \in SL_2(\mathbb{C})$ had an expression $g = ka_rk'$, then $gg^* = ka_{2r}k^*$, where g^* is conjugate transpose. This suggests how to determine components k, a_r, k': by the *spectral theorem* for positive-definite hermitian operators gg^*, we can find $k \in SU(2)$ and a diagonal matrix a_{2r} of positive real eigenvalues, so that $gg^* = ka_{2r}k^*$. We claim that $k' = (ka_r)^{-1}g \in SU(2)$. Indeed,

$$\Big((ka_r)^{-1}g\Big)\Big((ka_r)^{-1}g\Big)^* \;=\; a_r^{-1}\,k^{-1}\,(gg^*)\,k^*\,a_r^{-1} \;=\; a_r^{-1}\,k^{-1}\,ka_{2r}k^*\,k\,a_r^{-1} \;=\; 1$$

The same argument works to give a Cartan decomposition for $SL_n(\mathbb{C})$.

The non-negative r-coordinate gives a left G-invariant *metric* d on G/K by

$$d(gK, hK) \;=\; r \qquad \text{where} \qquad h^{-1}g \;\in\; Ka_rK$$

[1.2] Invariant volume for $SL_2(\mathbb{R})$

We recall how to express a G-invariant measure in Cartan/radial coordinates. Since K acts on left and right in Cartan coordinates, and Haar measure on G is left and right invariant, the only question is determination of the dependence on radius. That is, *a priori* we know that for some function f of radius the G-invariant measure is

$$d(ka_rk') \;=\; f(r)\, dk\, dr\, dk' \qquad\qquad (\text{with } k, k' \in K,\ r > 0)$$

The smaller case of $SL_2(\mathbb{R})$ is worth reviewing first. Let $\mathfrak{a}$ be the Lie algebra of the split component A, and $\mathfrak{k}$ the Lie algebra of $K = SO(2)$. For fixed $r > 0$, the mapping $\mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{k} \to G$ by

$$\theta \oplus \alpha \oplus \theta' \;\to\; \exp(\theta) \cdot a_r \exp(\alpha) \cdot \exp(\theta')$$

is a linear map to the tangent space to G at a_r, identified with $\mathfrak{g}$ by left-translating back to $1 \in G$ by left multiplication by a_r^{-1}:

$$\theta \oplus \alpha \oplus \theta' \;\to\; a_r^{-1} \cdot \exp(\theta) \cdot a_r \exp(\alpha) \cdot \exp(\theta')$$

The Adjoint action of a_r on θ moves it enough so that the resulting vectors, together with $\mathfrak{a}$ and $\mathfrak{k}$, span $\mathfrak{g}$. Thus, to determine the function f, fix an arbitrary basis for $\mathfrak{g}$ and compute the Jacobian of this map, as $r > 0$ varies, in terms of it. Obviously it is convenient to take

$$h \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \;\in\; \mathfrak{a} \qquad\qquad \theta \;=\; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathfrak{k}$$

and then choose

$$\sigma \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The image

$$a_r^{-1}\theta a_r \;=\; \begin{pmatrix} 0 & e^{-r} \\ e^{r} & 0 \end{pmatrix}$$

of θ under $\operatorname{Ad} a_r^{-1}$ is expressible as a linear combination $x\sigma + y\theta$ of σ and θ, by solving

$$\begin{cases} x + y &= \;\; e^{-r} \\ x - y &= \;\; -e^{r} \end{cases}$$

yielding $x = \sinh r$ and $y = -\cosh r$. That is, the map of $\mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{k}$ to the tangent space at a_r sends

$$a\theta \oplus bh \oplus c\theta \;\longrightarrow\; a \sinh r \cdot \sigma \oplus bh \oplus (c - a\cosh r)\theta \qquad\qquad (\text{with } a, b, c \in \mathbb{R})$$

That is, mapping from $\mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{k}$ to $\mathbb{R}\sigma \oplus \mathfrak{a} \oplus \mathfrak{k}$ in a, b, c coordinates,

$$(a, b, c) \;\longrightarrow\; (a \sinh r,\ b,\ c - a\cosh r)$$

Thus, up to an irrelevant constant,

$$\text{Haar measure} \;=\; d(ka_rk') \;=\; |\sinh r|\, dk\, dr\, dk' \qquad\qquad (\text{for } SL_2(\mathbb{R}))$$

[1.3] Invariant volume for $SL_2(\mathbb{C})$

Returning to $SL_2(\mathbb{C})$, the same sort of argument is applied, but now the Lie algebra $\mathfrak{k}$ of $K = SU(2)$ is larger, namely,

$$\mathfrak{k} \;=\; \mathbb{R} \cdot \theta + \mathbb{R} \cdot i\sigma + \mathbb{R} \cdot ih$$

The Adjoint action of a_r on the third of these summands is trivial, on the first is the case of $SL_2(\mathbb{R})$ already considered. On the second summand, considerations essentially the same as for the first apply to the subspace $\mathbb{R}\cdot i\sigma + \mathbb{R}\cdot i\theta$, with the roles of θ and σ reversed, as follows. Write [3] $\alpha = i\theta$ and $\beta = i\sigma$. Then, as in the argument for $SL_2(\mathbb{R})$, solve for the coefficients x, y such that

$$x\cdot\alpha + y\cdot\beta \;=\; a_r^{-1}\beta a_r$$

This is

$$\begin{cases} x\cdot i + y\cdot i &=& e^{-r}\cdot i \\ x\cdot(-i) + y\cdot i &=& e^{r}\cdot i \end{cases}$$

Then $x = -\sinh r$, we have a *further* factor of $\sinh r$, altogether giving *two* factors of $\sinh r$, and

$$\text{Haar measure} \;=\; d(ka_rk') \;=\; |\sinh r|^2\, dk\, dr\, dk' \;=\; \sinh^2 r\, dk\, dr\, dk' \qquad (\text{for } SL_2(\mathbb{C}))$$

[1.4] Invariant Laplacian

The **Laplacian** Δ on G/K is the restriction of the Casimir operator on G to right K-invariant functions. Let $\langle,\rangle$ be the $\mathbb{R}$-valued Ad G-invariant pairing on the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ given by

$$\langle x, y\rangle \;=\; \mathrm{Re}\big(\mathrm{tr}(xy)\big)$$

As usual, this non-degenerate pairing gives a natural identification of $\mathfrak{g}$ with its $\mathbb{R}$-linear dual $\mathfrak{g}^*$. The chain of G-equivariant natural maps

$$\mathrm{End}_{\mathbb{R}}(\mathfrak{g}) \;\to\; \mathfrak{g}\otimes_{\mathbb{R}}\mathfrak{g}^* \;\approx\; \mathfrak{g}\otimes_{\mathbb{R}}\mathfrak{g} \;\subset\; \bigotimes{}^{\bullet}\mathfrak{g} \;\to\; U\mathfrak{g}$$

maps the identity endomorphism $1_{\mathfrak{g}}$, which certainly commutes with the action of G on $\mathfrak{g}$, to an element Ω of the universal enveloping algebra $U\mathfrak{g}$ therefore commuting with the action of G. Thus, certainly Ω is in the *center* [4] $\mathfrak{z}$ of the enveloping algebra, and is a multiple [5] of the *Casimir element*. It is not immmediate that this is not accidentally 0, but the computation below of the effect of Ω on certain representations will incidentally prove non-vanishing. The description via the maps above implies that, for any basis x_i of $\mathfrak{g}$, letting x_i^* be the dual basis with respect to $\langle,\rangle$,

$$\Omega \;=\; \sum_i x_i\, x_i^* \;\in\; \mathfrak{z} \;\subset\; U\mathfrak{g}$$

The Laplacian Δ is the restriction of Ω to right K-invariant functions. For a decomposition $\mathfrak{g} = \mathfrak{p}\oplus\mathfrak{k}$, on right K-invariant functions $\mathfrak{k}$ acts by 0, so for any basis x_i of $\mathfrak{p}$, and

$$\Delta \;=\; \sum_i x_i\, x_i^* \qquad (\text{on } G/K)$$

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[3] Without *some* notational device, one might inadvertently imagine that there is $\mathbb{C}$-linearity sufficient to allow us to pull out the i as a scalar!

[4] To refer to the center $\mathfrak{z}$ is slightly misleading, since the argument shows that Ω is in the subalgebra $(U\mathfrak{g})^G$ of G-invariant elements of $U\mathfrak{g}$, in general possibly slightly smaller than the center $\mathfrak{z}$. Nevertheless, this abuse of language is common.

[5] That this construction produces a multiple of Casimir follows whenever $\mathfrak{g}$ is irreducible as a G-representation under Adjoint, by the *adjunction*

$$\mathrm{Hom}_G(\mathfrak{g}\otimes\mathfrak{g}^*,\,\mathrm{triv}) \;\approx\; \mathrm{Hom}_G(\mathfrak{g},\mathrm{Hom}(\mathfrak{g}^*,\mathrm{triv})) \;\approx\; \mathrm{Hom}_G(\mathfrak{g},\mathfrak{g})$$

and invoking Schur's lemma to know that the latter Hom is one-dimensional. For simple G, the highest-weight criterion, together with the fact that root spaces $\mathfrak{g}_\alpha$ in $\mathfrak{g}$ interact by $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$, almost immediately proves irreducibility. Irreducibility in small cases such as $SL_2(\mathbb{R})$ can be verified even more directly.

[1.5] Casimir on principal series The (unramified) *principal series* representations of $SL_2(\mathbb{R})$ and $SL_2(\mathbb{C})$ are the simplest representations to construct. Let

$$P = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \qquad \text{(for both } SL_2(\mathbb{R}) \text{ and } SL_2(\mathbb{C})\text{)}$$

The s^{th} (unramified) character χ_s on P is[6]

$$\chi_s \begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix} = \begin{cases} |a|^{2s} & \text{(for } SL_2(\mathbb{R})\text{)} \\ |a|^{4s} & \text{(for } SL_2(\mathbb{R})\text{)} \end{cases}$$

The s^{th} smooth principal series representation I_s for $G = SL_2(\mathbb{R})$ or $SL_2(\mathbb{C})$ is

$$I_s = \{\text{smooth } f \text{ on } G : f(pg) = \chi_s(p) \cdot f(g), \text{ for } p \in P \text{ and } g \in G\}$$

A useful basis for the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ is

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The dual basis with respect to the pairing $\langle \alpha, \beta \rangle = \text{tr}(\alpha\beta)$ is $h^* = \frac{1}{2}h$, $x^* = y$, and $y^* = x$. The Casimir operator on I_s can be computed by letting it act on the *left*, where x acts by 0, as

$$\Omega = \tfrac{1}{2}h^2 + xy + yx = \tfrac{1}{2}h^2 + [x, y] + 2yx \longrightarrow \tfrac{1}{2}h^2 + h + 0 \qquad \text{(acting on the left)}$$

Noting that the left action is $(g \cdot f)(x) = f(g^{-1}x)$, including an inverse, the action of Casimir on I_s for $SL_2(\mathbb{R})$ is $\frac{1}{2}h^2 + h$ on the left. The action of h on the left is

$$f(x) \longrightarrow \frac{\partial}{\partial t}\Big|_{t=0} f(e^{-t \cdot h} \cdot x) = \frac{\partial}{\partial t}\Big|_{t=0} |e^{-t}|^{2s} \cdot f(x) = -2s \cdot f(x)$$

Thus, Casimir on I_s for $SL_2(\mathbb{R})$ is

$$\tfrac{1}{2}(-2s)^2 + (-2s) = 2s^2 - 2s = 2 \cdot s(s-1) \qquad \text{(for } SL_2(\mathbb{R})\text{)}$$

For $SL_2(\mathbb{C})$, we take three further basis elements $\gamma = ih$, $\xi = ix$, and $\eta = iy$. The further dual basis elements are $\gamma^* = -\frac{1}{2}\gamma$, $\xi^* = -\eta$, $\eta^* = -\xi$. Thus,

$$\Omega = \tfrac{1}{2}h^2 - \tfrac{1}{2}\gamma^2 + xy + yx - \xi\eta - \eta\xi \qquad \text{(for } SL_2(\mathbb{C})\text{)}$$

All of γ, x, and ξ act on the left by 0 on I_s, and we use commutators:

$$\Omega = \tfrac{1}{2}h^2 - \tfrac{1}{2}\gamma^2 + [x, y] + 2yx - [\xi, \eta] - 2\eta\xi = \tfrac{1}{2}h^2 - 0 + h - 0 - (-h) = \tfrac{1}{2}h^2 + 2h$$

On the *left* on I_s, h acts by $-4s$, so this is

$$\tfrac{1}{2}(-4s)^2 + 2(-4s) = 8s^2 - 8s = 8 \cdot s(s-1) \qquad \text{(for } SL_2(\mathbb{C})\text{)}$$

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[6] The exponents in the definition of χ_s are s^{th} powers of the modular function on the parabolic P. Then the unramified principal series I_s and I_{1-s} generally admit non-trivial G intertwining operators between them. That is, we have fixed on a functional equation $s \to 1 - s$ as opposed to other possibilities $s \to a - bs$ for other constants a, b.

The point of these computations is verification that on both $SL_2(\mathbb{R})$ and $SL_2(\mathbb{C})$, the normalization of the character as s^{th} power of the *modular function* on the parabolic P gives eigenvalues that are constant multiples of $s(s-1)$.

[1.5.1] Remark: It may prove useful to divide the $SL_2(\mathbb{R})$ operator by 4, and to divide the $SL_2(\mathbb{C})$ by 8, so that the renormalized Casimirs' eigenvalues on principal series I_s are *exactly* $s(s-1)$, rather than constant multiples, but this is a secondary issue. More important is the symmetry under $s \leftrightarrow 1-s$.

[1.6] Radial Laplacian for $SL_2(\mathbb{R})$

On left-and-right K-invariant functions

$$F(k \cdot a_r \cdot k') \;=\; f(r) \qquad\qquad (\text{with } k, k' \in K \text{ and } r > 0)$$

on G, the Laplacian becomes a differential operator in the radius r, determined as follows. Let $G = SL_2(\mathbb{R})$ and $K = SO(2)$ and

$$h \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \theta \;=\; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \sigma \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

as above. These are mutually orthogonal with respect to the pairing $\langle x, y\rangle = \mathrm{tr}(xy)$, and have lengths $2, -2, 2$, respectively. Thus, Casimir is

$$\Omega \;=\; \tfrac{1}{2}h^2 - \tfrac{1}{2}\theta^2 + \tfrac{1}{2}\sigma^2$$

Since Casimir necessarily preserves K-bi-invariance, to see how it acts on a K-bi-invariant function $F(k \cdot a_r \cdot k') = f(r)$ it suffices to evaluate the outcome at a_r.

Under the *right* action of $\mathfrak{g}$ on K-bi-invariant functions, θ obviously acts by 0. Similarly, on the *left*, $\mathfrak{k}$ acts by 0. Then, moving θ across a_r by Adjoint, it acts on the *right* by $a_r^{-1}\theta a_r$. The element σ can be expressed as a linear combination of $a_r^{-1}\theta a_r$ and θ, with coefficients depending on r: solve $x a_r^{-1}\theta a_r + y\theta = \sigma$ by solving

$$\begin{cases} e^{-r}x + y & = \;\; 1 \\ -e^{r}x - y & = \;\; 1 \end{cases}$$

Thus, $x = -1/\sinh r$ and

$$y \;=\; 1 - e^{-r}x \;=\; 1 + \frac{2e^{-r}}{e^r - e^{-r}} \;=\; \coth r$$

Thus, letting $\theta_r = a_r^{-1}\theta a_r$,

$$\sigma \;=\; \frac{-1}{\sinh r} \cdot \theta_r \;+\; \coth r \cdot \theta$$

Then, in $U\mathfrak{g}$,

$$\sigma^2 \;=\; (x\theta_r + y\theta)^2 \;=\; x^2\theta_r^2 + xy\theta\theta_r + xy\theta_r\theta + y^2\theta^2$$

Since the first-order operators θ and θ_r do not preserve right K-invariance, we want to evaluate the second-order operators θ^2, θ_r^2, $\theta\theta_r$, and $\theta_r\theta$ at a_r on left-and-right K-invariant operators. The operators θ^2, θ_r^2, and $\theta_r\theta$ act by 0 on left-and-right K-invariant functions. The only term with an obstruction is $\theta \cdot \theta_r$: in the expression

$$F(a_r\, e^{s\theta}\, e^{t\theta_r})$$

the θ term cannot be moved to the right, and the θ_r term cannot be moved to the left. Thus, on K-bi-invariant functions

$$\sigma^2 \;\longrightarrow\; 0 + xy\theta\theta_r + 0 + 0 \qquad\qquad (\text{on } K\text{-bi-invariant functions})$$

As usual, although the two elements do not commute, their commutator is intelligible:

$$\theta\theta_r \;=\; [\theta, \theta_r] + \theta_r\theta \;=\; -2\sinh r \cdot h + \theta_r\theta$$

Then $\theta_r\theta$ acts on the right by 0 on K-bi-invariant functions evaluated at points a_r. Thus, on K-bi-invariant functions, $\frac{1}{2}\sigma^2$ acts on the right by

$$\tfrac{1}{2}xy(-2\sinh r)\cdot h \;=\; \tfrac{1}{2}\frac{-1}{\sinh r}\cdot\coth r\cdot(-2\sinh r)\cdot h \;=\; \coth r\cdot h$$

Gathering all the summands of Ω, on K-bi-invariant functions $\Omega = \frac{1}{2}h^2 + \frac{1}{2}\sigma^2 - \frac{1}{2}\theta^2$ acts by

$$\tfrac{1}{2}h^2 + \coth r\cdot h \qquad\qquad (\text{on } K\backslash G/K, \text{ for } SL_2(\mathbb{R}))$$

The action of h is

$$(h\cdot F)(a_r) \;=\; \frac{\partial}{\partial t}\Big|_{t=0}F(a_r\cdot\exp(th)) \;=\; \frac{\partial}{\partial t}\Big|_{t=0}F(a_{r+2t}) \;=\; \frac{\partial}{\partial t}\Big|_{t=0}f(r+2t) \;=\; 2f'(r)$$

Similarly, $\frac{1}{2}h^2$ sends f to $\frac{1}{2}(2\cdot 2\cdot f'') = 2f''$. Thus, removing the irrelevant common factor of 2 from all terms, on K-bi-invariant functions $F(ka_rk') = f(r)$ Casimir Ω is (up to a constant)

$$f \;\longrightarrow\; f'' + \coth r\cdot f' \qquad\qquad (\text{for } SL_2(\mathbb{R}))$$

[1.7] Radial Laplacian for $SL_2(\mathbb{C})$ Determination of the Laplacian in radial coordinates for $SL_2(\mathbb{R})$ is half the computation for $SL_2(\mathbb{C})$. Indeed, in addition to the basis elements h,θ,σ for $\mathfrak{sl}_2(\mathbb{R})$, of lengths $2,-2,2$, we have $\eta = ih, \alpha = i\theta$, and $\beta = i\sigma$, of lengths $-2,2,-2$. Thus, Casimir is

$$\Omega \;=\; \tfrac{1}{2}\cdot\Big(h^2+\sigma^2-\theta^2-\eta^2+\alpha^2-\beta^2\Big)$$

To see the effect of Ω on K-bi-invariant functions F, it suffices to evaluate $\Omega F(a_r)$. Parallel to the argument for $SL_2(\mathbb{R})$, η,θ,β are in $\mathfrak{k}$, so under the right action on K-bi-invariant functions those summands act by 0.

Just as a linear combination θ and of θ conjugated across a_r gives σ, a linear combination of β and β conjugated across a_r gives α: with $\beta_r = a_r^{-1}\beta a_r$, solve

$$x\cdot\beta + y\cdot\beta_r \;=\; \alpha$$

This is

$$\begin{cases} x\cdot i + y\cdot ie^{-r} &= \quad i \\ x\cdot i + y\cdot ie^{r} &= \; -i \end{cases}$$

Then $y = -1/\sinh r$ and $x = \coth r$. Then on K-bi-invariant functions, under the right action,

$$\alpha^2 \;=\; \big(x\beta + y\beta_r\big)^2 \;=\; x^2\beta^2 + xy\beta\beta_r + xy\beta_r\beta + y^2\beta_r^2 \;\longrightarrow\; 0 + xy\beta\beta_r + 0 + 0$$

Further,

$$\beta\beta_r \;=\; [\beta,\beta_r] + \beta_r\beta \;\longrightarrow\; -2\sinh r\cdot h + 0$$

Thus, on K-bi-invariant functions,

$$xy\beta\beta_r \;\longrightarrow\; (\coth r)\cdot\frac{-1}{\sinh r}\cdot(-2\sinh r)\cdot h \;=\; 2\coth r\cdot h$$

Multiplying through by the $\frac{1}{2}$ from the expression for Ω, on K-bi-invariant functions

$$\Omega \;\longrightarrow\; \tfrac{1}{2}h^2 + \coth r\cdot h + \coth r\cdot h \;=\; \tfrac{1}{2}h^2 + 2\coth r\cdot h$$

As in the computation for $SL_2(\mathbb{C})$, the action of h on $f(r) = F(a_r)$ is $(h \cdot f)(r) = 2f'(r)$. Thus, removing the irrelevant common factor of 2, Casimir on K-bi-invariant functions is

$$f \longrightarrow f''(r) + 2\coth r \cdot f'(r) \qquad \text{(on functions } F(ka_rk') = f(r))$$

[1.7.1] Remark: Similarly, and by the same sort of computation, on $G = O(n,1)$ the Casimir operator on K-bi-invariant functions is a constant multiple of

$$f''(r) + (n-1)\coth r \cdot f'(r)$$

[1.8] Spherical functions on $SL_2(\mathbb{C})$

A **spherical function** on G is a smooth K-bi-invariant eigenfunction for Δ, that is, a function F such that

$$\Delta F \;=\; \lambda \cdot F$$

At this moment, renormalize by adjusting Ω by dividing by 8, so that its eigenvalue on the s^{th} principal series I_s is exactly $s(s-1)$, rather than the $8s(s-1)$. Thus, writing $f(r) = F(Ka_rK)$, the computation of the Laplacian/Casimir in radial coordinates converts the spherical condition to

$$\frac{1}{8}\big(2f'' + 4\coth r \cdot f\big) \;=\; \lambda \cdot f \qquad \text{(with } \lambda = s(s-1))$$

or

$$\tfrac{1}{4}f'' + \tfrac{1}{2}\coth r \cdot f \;=\; \lambda \cdot f \qquad \text{(where } \lambda = s(s-1))$$

The coincidence [7] we now examine is the elementariness of the spherical functions for $SL_2(\mathbb{C})$. There is a heuristic that spherical functions with eigenvalues $\lambda = s(s-1)$ with $s = \frac{1}{2} + it$, t real, should just *fail* to be in L^2. Thus, in light of

$$d(ka_rk') \;=\; |\sinh r|^2\, dk\, dr\, dk' \qquad \text{(Haar measure on } SL_2(\mathbb{C}))$$

f might profitably be rewritten as $f = \varphi/\sinh r$ for $r > 0$. Up to a constant, the differential operator on f becomes

$$\Big(\frac{\varphi}{\sinh r}\Big)'' + 2\coth r \cdot \Big(\frac{\varphi}{\sinh r}\Big)'$$

$$= \frac{\varphi''}{\sinh r} - \frac{2\varphi'\cosh r}{\sinh^2 r} + \varphi\Big(-\frac{\sinh r}{\sinh^2 r} + \frac{2\cosh^2 r}{\sinh^3 r}\Big) + 2\frac{\cosh r}{\sinh r}\Big(\frac{\varphi'}{\sinh r} - \frac{\varphi\cosh r}{\sinh^2 r}\Big)$$

$$= \frac{\varphi''}{\sinh r} - \frac{\varphi}{\sinh r}$$

The original constant would give $2\varphi''/\sinh r - 2\varphi/\sinh r$, and dividing by 8 gives

$$\tfrac{1}{8}\Omega\Big(\frac{\varphi}{\sinh r}\Big) \;=\; \frac{\varphi''}{4\sinh r} - \frac{\varphi}{4\sinh r}$$

Thus, the eigenvalue problem becomes

$$\frac{\varphi''}{4\sinh r} - \frac{\varphi}{4\sinh r} \;=\; s(s-1)\cdot\frac{\varphi}{\sinh r}$$

[7] The elementariness of the spherical functions for $SL_2(\mathbb{C})$ does appear to be a peculiar artifact of all these computations. However, Gelfand-Naimark and Harish-Chandra and others showed that the same is true for all *complex* reductive groups. Thus, in the family of hyperbolic spaces and orthogonal groups acting on them, the true oddity is that only $SO(3,1)$, admitting a two-fold cover by $SL_2(\mathbb{C})$, is a complex Lie group.

or simply

$$\tfrac{1}{4}\big(\varphi'' \;-\; \varphi\big) \;=\; s(s-1)\cdot\varphi \qquad\qquad (\text{with } \lambda = s(s-1))$$

Apparently[8] *miraculously, the differential equation has constant coefficients.* Thus, with $\varphi(r) = e^{\pm(2s-1)r}$,

$$\tfrac{1}{4}\big(\varphi'' \;-\; \varphi\big) \;=\; \tfrac{1}{4}\big((2s-1)^2 - 1\big)\cdot e^{\pm(2s-1)r} \;=\; s(s-1)\cdot e^{\pm(2s-1)r}$$

Thus,

$$\tfrac{1}{8}\Omega\Big(\frac{e^{\pm(2s-1)r}}{\sinh r}\Big) \;=\; s(s-1)\cdot\frac{e^{\pm(2s-1)r}}{\sinh r}$$

The standard normalization requires that a *spherical function* take value 1 at $1\in G$, that is, at $r=0$. The $\sinh r$ in the denominators makes both the above functions blow up like $1/r$ at $r\to 0^+$. However, since the functions with parameters s and $1-s$ have the same eigenvalue, taking the difference kills the blow-up. Thus, to get value 1 at $r=0$, put

$$\varphi_s(r) \;=\; \frac{\sinh(2s-1)r}{(2s-1)\,\sinh r} \qquad\qquad (\text{spherical function, eigenvalue } s(s-1))$$

Note that the best decay, just failing to be square-integrable, occurs for $\mathrm{Re}\, s = \frac{1}{2}$. When $s = \frac{1}{2}+it$, the spherical function is

$$\varphi_{\frac{1}{2}+it}(r) \;=\; \frac{\sin 2tr}{2t\,\sinh r} \qquad\qquad (\text{spherical function, eigenvalue } -(\tfrac{1}{4}+t^2))$$

[1.9] Spherical transform on $SL_2(\mathbb{C})$

For f a left K-invariant function on G/K, with sufficient decay, the **spherical transform** $\widetilde{f}$ of f is

$$\widetilde{f}(\xi) \;=\; \int_G f\cdot\overline{\varphi}_{\frac{1}{2}+i\xi} \;=\; \int_G f\cdot\varphi_{\frac{1}{2}-i\xi} \qquad\qquad (\text{with real } \xi)$$

Spherical **inversion** is

$$f \;=\; \frac{16}{\pi}\int_{-\infty}^{\infty}\widetilde{f}(\tfrac{1}{2}+i\xi)\cdot\varphi_{\frac{1}{2}+i\xi}\cdot|\mathbf{c}(\tfrac{1}{2}+i\xi)|^{-2}\,d\xi \qquad \text{where} \qquad \mathbf{c}(\tfrac{1}{2}+i\xi) \;=\; \xi^{-1}$$

The leading constant is inessential, and in any case depends on the normalization of measures. The fact that for $SL_2(\mathbb{C})$ the Harish-Chandra $\mathbf{c}$-function $\mathbf{c}(s)$ is elementary is another happy coincidence. That is, spherical inversion is simply

$$f \;=\; \frac{16}{\pi}\int_{-\infty}^{\infty}\widetilde{f}(\tfrac{1}{2}+i\xi)\cdot\varphi_{\frac{1}{2}+i\xi}\cdot\xi^2\,d\xi$$

Indeed, for $SL_2(\mathbb{C})$, we can prove spherical inversion from classical Fourier inversion on the real line, as follows. First, normalize Fourier transform on $\mathbb{R}$ so that Fourier inversion is

$$F(x) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x}\int_{\infty}^{\infty} e^{-i\xi x}\,F(u)\,du\,d\xi \qquad\qquad (\text{suitable } F \text{ on } \mathbb{R})$$

For F an *odd* function, this can be folded up into an assertion about *sine* transforms

$$F^{\sin}(\xi) \;=\; \int_0^{\infty}\sin\xi x\;F(x)\,dx$$

---

[8] Again, there is no luck or miracle involved. Rather, spherical functions are *always* elementary, for *complex* Lie groups, as opposed to *real*. Gelfand-Naimark studied the complex case, as did Harish-Chandra later, prior to the much subtler consideration of the real case.

as

$$F(x) \;=\; \frac{2}{\pi}\int_0^\infty \sin\xi x \int_0^\infty \sin\xi u\; F(u)\,du\,d\xi$$

Then the spherical transform can be rewritten to fit into this:

$$\widetilde{f}(\xi) \;=\; \int_0^\infty \varphi_{\frac{1}{2}+i\xi}(a_r)\,f(a_r)\,\sinh^2 r\,dr \;=\; \int_0^\infty \frac{\sin 2\xi r}{2\xi\,\sinh r}\,f(a_r)\,\sinh^2 r\,dr$$

$$=\; \frac{1}{2\xi}\int_0^\infty \sin 2\xi r\,\big(f(a_r)\,\sinh r\big)\,dr \;=\; \frac{1}{4\xi}\int_0^\infty \sin\xi r\,\big(f(a_{r/2})\,\sinh\frac{r}{2}\big)\,dr$$

That is,

$$4\xi\cdot\widetilde{f}(\xi) \;=\; \Big(\sinh\frac{r}{2}\;f(a_{r/2})\Big)^{\sin}(\xi)$$

Then inversion gives

$$\sinh\frac{r}{2}\;f(a_{r/2}) \;=\; \frac{2}{\pi}\int_0^\infty \sin\xi r\cdot 4\xi\cdot\widetilde{f}(\xi)\,d\xi$$

and

$$f(a_{r/2}) \;=\; \frac{2}{\pi}\int_0^\infty \frac{\sin\xi r}{\sinh\frac{r}{2}}\cdot 4\xi\cdot\widetilde{f}(\xi)\,d\xi$$

and

$$f(a_r) \;=\; \frac{2}{\pi}\int_0^\infty \frac{\sin 2\xi r}{2\xi\,\sinh r}\cdot 8\xi^2\cdot\widetilde{f}(\xi)\,d\xi \;=\; \frac{16}{\pi}\int_0^\infty \varphi_{\frac{1}{2}+i\xi}(a_r)\,\widetilde{f}(\xi)\,\xi^2\,d\xi$$

There is a corresponding Plancherel theorem for f, F left-and-right K-invariant [9] functions in $L^2(G)$:

$$\int_G f\cdot F \;=\; \frac{16}{\pi}\int_0^\infty \widetilde{f}(\tfrac{1}{2}+i\xi)\cdot\widetilde{F}(\tfrac{1}{2}+i\xi)\cdot|\mathbf{c}(\tfrac{1}{2}+i\xi)|^{-2}\,d\xi$$

[1.10] Spherical expansion of δ Juxtaposing Plancherel and the idea that the spherical inversion formula should converge *uniformly pointwise* for *smooth* left-and-right K-invariant f, a spherical expansion for Dirac δ at $z_o = 1\cdot K$ on G/K can be *inferred*: noting that $\varphi_s(z_o) = 1$, the presumed pointwise convergence

$$f(z_o) \;=\; \int_{-\infty}^\infty \widetilde{f}(\tfrac{1}{2}+i\xi)\,\varphi_{\frac{1}{2}+i\xi}(z_o)\,|\mathbf{c}(\tfrac{1}{2}+i\xi)|^{-2}\,d\xi \;=\; \int_{-\infty}^\infty \widetilde{f}(\tfrac{1}{2}+i\xi)\cdot 1\cdot|\mathbf{c}(\tfrac{1}{2}+i\xi)|^{-2}\,d\xi$$

suggests that $\widetilde{\delta}(\frac{1}{2}+i\xi) = 1$.

[1.10.1] Remark: The legitimacy of the spherical-function expansion for δ on G/K depends upon knowing convergence of the spherical inversion for smooth left-and-right K-invariant functions *with suitable decay properties* in the C^∞ topology. This is greatly facilitated by the elementariness of the spherical functions here. The notion of **tempered** spherical distribution is fortunately *much* simpler than the full notion of tempered distribution on G without restrictions on the behavior under K. Even though δ is compactly supported, solution of natural differential equations requires a larger class of tempered distributions.

[1.11] Free-space fundamental solution We start from the idea that the δ function on G/K has spherical transform computed by

$$\widetilde{\delta}(s) \;=\; \varphi_s(1) \;=\; 1$$

[9] The Plancherel theorem for the whole $L^2(SL_2(\mathbb{C}))$ was proven in [Gelfand-Naimark 1950].

and that δ has a spherical-function expansion (normalizing-away constants)

$$\delta \;=\; \int_{-\infty}^{\infty} \widetilde{\delta}(\tfrac{1}{2}+i\xi)\,\varphi_{\frac{1}{2}+i\xi}\,\xi^2\,d\xi \;=\; \int_{-\infty}^{\infty} \varphi_{\frac{1}{2}+i\xi}\,\xi^2\,d\xi \qquad \text{(convergent as tempered distribution)}$$

To find a **free-space fundamental solution** u_s for $\big(\Delta - s(s-1)\big)^2$ on G/K, use the spherical transform on tempered left-and-right K-invariant distributions on G. That is, take the spherical transform of both sides of the equation $\big(\Delta - s(s-1)\big)^2 u_s = \delta$, obtaining

$$\Big((\tfrac{1}{2}+i\xi)((\tfrac{1}{2}+i\xi)-1) \;-\; s(s-1)\Big)^2\, \widetilde{u}_s(\tfrac{1}{2}+i\xi) \;=\; \widetilde{\delta}(\tfrac{1}{2}+i\xi) \;=\; 1$$

Thus,

$$\widetilde{u}_s(\tfrac{1}{2}+i\xi) \;=\; \Big((\tfrac{1}{2}+i\xi)((\tfrac{1}{2}+i\xi)-1) \;-\; s(s-1)\Big)^{-2}$$

By spherical inversion for tempered distributions,

$$u_s \;=\; \int_{-\infty}^{\infty} \varphi_{\frac{1}{2}+i\xi}\, \frac{\xi^2\,d\xi}{\Big((\tfrac{1}{2}+i\xi)((\tfrac{1}{2}+i\xi)-1) \;-\; s(s-1)\Big)^2}$$

A local Sobolev space argument and easy estimates on φ_s prove that this integral converges in C^o. In fact, the elementary nature of φ_s allows a computation of u_s by residues, expressing u_s in elementary terms, making subsequent estimates easier. Recalling from above that

$$\varphi_{\frac{1}{2}+i\xi}(Ka_rK) \;=\; \frac{\sin 2\xi r}{2\xi\,\sinh r}$$

the expression for u_s is

$$\begin{aligned} u_s(Ka_rK) \;&=\; \int_{-\infty}^{\infty} \frac{\sin 2\xi r)}{2\xi\,\sinh r}\, \frac{\xi^2\,d\xi}{\Big((\tfrac{1}{2}+i\xi)((\tfrac{1}{2}+i\xi)-1) \;-\; s(s-1)\Big)^2} \\ &=\; \frac{1}{2\sinh r}\int_{-\infty}^{\infty} \frac{\xi\,\sin(2\xi\cdot r)\,d\xi}{\Big((\tfrac{1}{2}+i\xi)((\tfrac{1}{2}+i\xi)-1) \;-\; s(s-1)\Big)^2} \end{aligned}$$

Use

$$\sin 2\xi r \;=\; \frac{e^{2ir\xi} - e^{-2ir\xi}}{2i}$$

to break the integral into two corresponding pieces. Temporarily dropping the denominator of $2i$, one integral is

$$\int_{-\infty}^{\infty} \frac{\xi\,e^{2ir\xi}\,d\xi}{\Big((\tfrac{1}{2}+i\xi)((\tfrac{1}{2}+i\xi)-1) \;-\; s(s-1)\Big)^2}$$

Since we take $r \ge 0$, the exponential is bounded for ξ in the *upper* half-plane $\mathfrak{H}$, so auxiliary contours in $\mathfrak{H}$ can be used to evaluate the integral by residues. Since the outcome will be holomorphic in s, we may take $\mathrm{Re}\,s >> 1$ for specificity. Also, conveniently,

$$\frac{1}{w(w-1)-s(s-1)} \;=\; \frac{1}{(w-s)(w-(1-s))}$$

and

$$\frac{1}{(\tfrac{1}{2}+i\xi)((\tfrac{1}{2}+i\xi)-1) \;-\; s(s-1)} \;=\; \frac{1}{\big((\tfrac{1}{2}+i\xi)-s\big)\,\big((\tfrac{1}{2}+i\xi)-(1-s)\big)}$$

$$= \frac{1}{\big(i\xi - (s-\frac{1}{2})\big)\big(i\xi - (\frac{1}{2}-s)\big)} = \frac{-1}{\big(\xi + i(s-\frac{1}{2})\big)\big(\xi - i(s-\frac{1}{2})\big)}$$

Squaring will eliminate the sign. Thus,

$$\int_{-\infty}^{\infty} \frac{\xi\, e^{2ir\xi}\, d\xi}{\Big((\frac{1}{2}+i\xi)((\frac{1}{2}+i\xi)-1) \;-\; s(s-1)\Big)^2} = 2\pi i \cdot \big(\text{residues at } \xi \text{ in } \mathfrak{H}\big) = 2\pi i \cdot \big(\text{residue at } \xi = i(s-\tfrac{1}{2})\big)$$

Dropping the factor of $2\pi i$ for a moment, this is

$$\Big(\frac{\partial}{\partial\xi}\Big)\Big|_{\xi=i(s-\frac{1}{2})} \left(\frac{\xi\, e^{2ir\xi}}{(\xi+i(s-\frac{1}{2}))^2}\right) = \left(\frac{2ir\xi\, e^{2ir\xi}}{(\xi+i(s-\frac{1}{2}))^2} + \frac{e^{2ir\xi}}{(\xi+i(s-\frac{1}{2}))^2} + \frac{-2\xi\, e^{2ir\xi}}{(\xi+i(s-\frac{1}{2}))^3}\right)\Big|_{\xi=i(s-\frac{1}{2})}$$

$$= \frac{2ir \cdot i(s-\frac{1}{2}) \cdot e^{2ir\cdot i(s-\frac{1}{2})}}{\big(2i(s-\frac{1}{2})\big)^2} = \frac{r\, e^{-r(2s-1)}}{2s-1}$$

That is, putting back the denominator of $2i$ and factor of $2\pi i$,

$$\int_{-\infty}^{\infty} \frac{\xi\, e^{2ir\xi}\, d\xi}{2i\Big((\frac{1}{2}+i\xi)((\frac{1}{2}+i\xi)-1) \;-\; s(s-1)\Big)^2} = \frac{\pi r\, e^{-r(2s-1)}}{2s-1}$$

Similarly, noting that the contour of integration will now be *clockwise*, thus contributing a sign,

$$\int_{-\infty}^{\infty} \frac{\xi\, e^{-2ir\xi}\, d\xi}{\Big((\frac{1}{2}+i\xi)((\frac{1}{2}+i\xi)-1) \;-\; s(s-1)\Big)^2} = -2\pi i \cdot \big(\text{residue at } \xi = -i(s-\tfrac{1}{2}) \in \mathfrak{H}\big)$$

Temporarily dropping the $-2\pi i$, this is

$$\Big(\frac{\partial}{\partial\xi}\Big)\Big|_{\xi=-i(s-\frac{1}{2})} \left(\frac{\xi\, e^{-2ir\xi}}{(\xi-i(s-\frac{1}{2}))^2}\right) = \frac{r\, e^{-r(2s-1)}}{2s-1}$$

Thus, putting back the denominator of $2i$ and factor of $-2\pi i$,

$$\int_{-\infty}^{\infty} \frac{\xi\, e^{-2ir\xi}\, d\xi}{2i\Big((\frac{1}{2}+i\xi)((\frac{1}{2}+i\xi)-1) \;-\; s(s-1)\Big)^2} = \frac{-\pi r\, e^{-r(2s-1)}}{2s-1}$$

Therefore,

$$\int_{-\infty}^{\infty} \frac{\xi\, \sin(2\xi\cdot r)\, d\xi}{\Big((\frac{1}{2}+i\xi)((\frac{1}{2}+i\xi)-1) \;-\; s(s-1)\Big)^2} = \frac{\pi r\, e^{-r(2s-1)}}{2s-1} - \frac{-\pi r\, e^{-r(2s-1)}}{2s-1} = \frac{2\pi r\, e^{-r(2s-1)}}{2s-1}$$

Putting back the denominator of $2\sinh r$, the **free-space fundamental solution** for $\big(\Delta - s(s-1)\big)^2$ is

$$u_s(Ka_rK) = \text{const} \times \frac{re^{-(2s-1)r}}{(2s-1)\,\sinh r}$$

Again, the normalization of the constant depends upon the normalization of the Laplacian.

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