

(January 6, 2011)

# Global automorphic Sobolev spaces

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The goal is legitimization of term-wise differentiation of  $L^2$  spectral expansions, so that computations producing a classical outcome are correct. We are fond of  $L^2$  expansions because they are what Plancherel gives.

Typically,  $L^2$  expansions are not continuous, much less differentiable, so the issue cannot be *proving* classical differentiability, which does *not* hold.

To say that  $L^2$  spectral expansions are term-wise differentiable in a *distributional* sense is often valid, but too weak, since it is difficult to return from the large world of distributions to the smaller world of  $L^2$  functions.

Further, already for Fourier transforms on  $\mathbb{R}^n$ , the integral expressing Fourier inversion is *not* a superposition of  $L^2$  functions, since the exponentials are not in  $L^2(\mathbb{R}^n)$ .

Notions of  $L^2$  Sobolev spaces are a balance of the simplicity of Hilbert space structures with extensions of notions of differentiability, insofar as solving elliptic partial differential equations of sufficiently high degree can move back to  $L^2$ . That is, Sobolev spaces are within *finite distance* of  $L^2$ , in terms of basic processes of analysis.

Especially with respect to *invariant* operators such as Casimir operators, especially in delicate situations such as *automorphic forms*, Plancherel theorems most naturally yield corollaries about  $L^2$ -differentiation, not about classical pointwise differentiation.

**[1.1] Spherical automorphic spectral expansion** For simplicity, take  $X = \Gamma \backslash G / K$  where  $\Gamma = SL_2(\mathbb{Z}[i])$  and  $G = SL_2(\mathbb{C})$  and  $K = SU(2)$ . Let  $\Delta$  be the  $G$ -invariant Laplacian on  $G/K$  descended from Casimir on  $G$ . Functions  $f$  in  $L^2(\Gamma \backslash G / K)$  decompose in an  $L^2$  sense

$$f = \sum_F \langle f, F \rangle \cdot F + \frac{\langle f, 1 \rangle \cdot 1}{\langle 1, 1 \rangle} + \int_{-\infty}^{\infty} \langle f, E_{\frac{1}{2}+it} \rangle \cdot E_{\frac{1}{2}+it} dt$$

where  $F$  runs over an orthonormal basis of *cusps*. Plancherel is

$$\|f\|^2 = \sum_F |\langle f, F \rangle|^2 + \frac{|\langle f, 1 \rangle|^2}{\langle 1, 1 \rangle} + \int_0^{\infty} |\langle f, E_{\frac{1}{2}+it} \rangle|^2 dt$$

Note that Plancherel does not need nor assert anything about pointwise values of cuspforms or Eisenstein series.

**[1.2] Spectral parametrization** Let

$$\Xi = \{\text{orthonormal basis of cuspforms}\} \cup \{1\} \cup \frac{1}{2} + i[0, \infty)$$

where the half-line parametrizes Eisenstein series  $E_{\frac{1}{2}+it}$ . The **spectral measure** on  $\Xi$  gives each cuspform and the constant function 1 point-mass measure, and gives the half-line  $\frac{1}{2} + i[0, \infty)$  a constant multiple of Lebesgue measure. Let

$$\Phi_{\xi} = \begin{cases} F & (\text{for } \xi = F) \\ 1/\langle 1, 1 \rangle^{1/2} & (\text{for } \xi = 1) \\ E_s & (\text{for } \xi = s = \frac{1}{2} + it) \end{cases} \quad (\text{for } \xi \in \Xi)$$

Then the spectral decomposition can be rewritten as

$$f = \int_{\Xi} \langle f, \Phi_{\xi} \rangle \cdot \Phi_{\xi} d\xi$$

and Plancherel is

$$\|f\|^2 = \int_{\Xi} |\langle f, \Phi_{\xi} \rangle|^2 d\xi$$

The integrals  $\langle f, \Phi_{\xi} \rangle$  are the **spectral components**, but the implied integrals do not converge for all  $f$  in  $L^2$ , especially integrals against Eisenstein series. Nevertheless, Plancherel asserts that the literal integrals on *test functions extend* to an isometry

$$\mathcal{F} : L^2(X) \longrightarrow L^2(\Xi)$$

The essential point is that on test functions  $\mathcal{F}$  is given by literal integrals against the preferred functions: cuspforms, constants, and Eisenstein series.

**[1.3] Spectral expansions of derivatives** For test functions, integration by parts is immediately legitimate to compute spectral projections. Let  $\lambda_{\xi}$  be the eigenvalue of  $\Delta$  on  $\Phi_{\xi}$ . Note that these eigenvalues are *real*. Then

$$\langle \Delta f, \Phi_{\xi} \rangle = \int_X f \Delta \bar{\Phi}_{\xi} = \lambda_{\xi} \cdot \int_X f \bar{\Phi}_{\xi} \quad (\text{for } f \in C_c^{\infty}(X))$$

Thus, for test functions,

$$\Delta f = \int_{\Xi} \langle \Delta f, \Phi_{\xi} \rangle \Phi_{\xi} d\xi = \int_{\Xi} \lambda_{\xi} \cdot \langle f, \Phi_{\xi} \rangle \Phi_{\xi} d\xi$$

That is, the differential operator  $\Delta$  differentiates *term-wise*, in the sense of moving inside the integration over  $\Xi$  giving the *spectral synthesis*. More succinctly,

$$\Delta f = \mathcal{F}^{-1} \mathcal{F} \Delta f = \mathcal{F}^{-1} \lambda_{\xi} \mathcal{F} f$$

**[1.4] Small and smaller Sobolev spaces** To discuss general *differentiability* of functions on  $X = \Gamma \backslash G / K$  it is sufficient for us to use the right action of the Lie algebra  $\mathfrak{g}$  of  $G$  on  $\Gamma \backslash G$ . Most of these derivatives will no longer be right  $K$ -invariant, but that harms nothing. Indeed, up to a constant, the measure and integral on  $\Gamma \backslash G / K$  are simply that of  $\Gamma \backslash G$  restricted to right  $K$ -invariant functions. So it is consistent with the above to write

$$\|f\|^2 = \int_{\Gamma \backslash G} |f|^2 \quad \langle f, \varphi \rangle = \int_{\Gamma \backslash G} f \bar{\varphi} \quad (\text{for } f, \varphi \in C_c^{\infty}(\Gamma \backslash G))$$

Let  $U_{\mathfrak{g}}^{\leq \ell}$  be the finite-dimensional subspace of elements of the universal enveloping algebra  $U_{\mathfrak{g}}$  of  $\mathfrak{g}$  consisting of elements of degree  $\leq \ell$ . This is stable under  $K$ . Each  $\alpha \in U_{\mathfrak{g}}^{\leq \ell}$  gives a seminorm  $\nu_{\alpha}$  on  $C_c^{\infty}(\Gamma \backslash G)$  by

$$\nu_{\alpha}(f) = \|\alpha f\|^2 \quad (\text{right action of } U_{\mathfrak{g}})$$

Supers of finite linear combinations of these seminorms give the same topology as seminorms given by *bounded subsets*  $B$  of  $U_{\mathfrak{g}}^{\leq \ell}$ , by

$$\nu_B(f) = \sup_{\beta \in B} \nu_{\beta}(f)$$

For  $\ell \geq 0$ , a *smaller* Sobolev space  $H_o^{\ell}(\Gamma \backslash G)$  is the completion of *test functions*  $C_c^{\infty}(\Gamma \backslash G)$  in the topology given by all  $\nu_{\alpha}$  with  $\alpha \in U_{\mathfrak{g}}^{\leq \ell}$ , or, equivalently, by all  $\nu_B$  with  $B$  bounded subsets of  $U_{\mathfrak{g}}^{\leq \ell}$ . The Sobolev space of interest is

$$H_o^{\ell}(X) = H_o^{\ell}(\Gamma \backslash G)^K = \{\text{right } K\text{-fixed elements of } H_o^{\ell}(\Gamma \backslash G)\}$$

Since  $U_{\mathfrak{g}}^{\leq \ell}$  is finite-dimensional, the topology on  $H_o^{\ell}(\Gamma \backslash G)$  can be given by finitely-many Hilbert space norms, so is *isomorphic to* a Hilbert space. However, at this point there is no *canonical* Hilbert space structure:

only the *topology* is canonical. Nevertheless, the collection of *bounded sets* in a topological vector space sense depends only upon the *topology*, and the *strong dual*  $H_o^{\ell*}$  is canonically defined as a topological vector space. Then the fact that the limit has a finite cofinal subset implies that the strong dual is a Hilbert space, invoking Riesz-Fischer on the limitands.

The *small* Sobolev space  $H^\ell(\Gamma \backslash G)$  is the completion of the space of *smooth* functions having finite  $\nu_\alpha$ -norm for all  $\alpha \in U\mathfrak{g}^{\leq \ell}$ . And  $H^\ell(X)$  is the right  $K$ -invariant elements of  $H^\ell(\Gamma \backslash G)$ . These also have the canonical topologies.

From Urysohn's lemma, the group  $G$  acts continuously on  $L^2(\Gamma \backslash G)$ . Thus, since the Adjoint action of  $G$  stabilizes  $U\mathfrak{g}^{\leq \ell}$ , the group  $G$  acts continuously on  $H^\ell(\Gamma \backslash G)$ .

The differential operator  $\Delta$  on test functions gives a map

$$\Delta : H_o^\ell \cap C_c^\infty(\Gamma \backslash G) \longrightarrow H_o^{\ell-2} \cap C_c^\infty(\Gamma \backslash G)$$

*continuous by design*. Thus,  $\Delta$  extends by continuity to a continuous linear map

$$\tilde{\Delta}_\ell : H_o^\ell(\Gamma \backslash G) \longrightarrow H_o^{\ell-2}(\Gamma \backslash G) \quad (\text{for } \ell \geq 2)$$

Similarly,  $\Delta$  extends by continuity to

$$\tilde{\Delta}_\ell : H^\ell(\Gamma \backslash G) \longrightarrow H^{\ell-2}(\Gamma \backslash G) \quad (\text{for } \ell \geq 2)$$

These extensions are  $L^2$  **differentiation** by  $\Delta$  on  $\Gamma \backslash G$ .

Since  $\Delta$  commutes with  $K$ , the  $L^2$  differentiation by  $\Delta$  descends to  $H_o^\ell(X) \rightarrow H_o^{\ell-2}(X)$  and  $H^\ell(X) \rightarrow H^{\ell-2}(X)$ .

[1.4.1] **Remark:** We will prove just below that these two definitions yield the same completion, that is,  $H_o^\ell(\Gamma \backslash G) = H^\ell(\Gamma \backslash G)$ .

[1.4.2] **Remark:** We will prove below that, in fact, the semi-norms attached to polynomials in  $\Delta$  suffice to give the topology on Sobolev spaces, returning us to the straightforward Hilbert space setting.

[1.5] **Smooth cut-offs** To prove that test functions are dense in  $H^\ell$ , smooth cut-off functions are required. Some mild specifics about the geometry of  $\Gamma \backslash G$  are necessary. Use Iwasawa decomposition coordinates  $z = (x, y)$  on  $G/K$ , from

$$n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \quad m_y = \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} \quad (\text{with } x \in \mathbb{C} \text{ and } y > 0)$$

With  $N = \{n_x\}$  and  $M = \{m_y\}$ , the Iwasawa decomposition  $G = NMK$  shows that  $NM$  maps surjectively to  $G/K$ . By reduction theory, for sufficiently small  $t > 0$  the Siegel set

$$\mathfrak{S}_t = \{z \in G/K : y \geq t\} \subset \mathfrak{S}$$

covers the quotient  $\Gamma \backslash G/K$ . Thus, the inverse image of this in  $G$  covers  $\Gamma \backslash G$ . Fix a smooth function  $\eta_o$  on  $\mathbb{R}$  such that  $0 \leq \eta_o \leq 1$  everywhere,  $\eta_o$  is identically 1 on  $(-\infty, 0]$ , and  $\eta_o$  is identically 0 on  $[1, \infty)$ . Let  $q : \mathfrak{S}_t \rightarrow \Gamma \backslash G/K$  be the quotient map  $G/K \rightarrow \Gamma \backslash G/K$  restricted to  $\mathfrak{S}_t$ . Define a smooth cut-off function on  $\Gamma \backslash G/K$  for large  $R > 0$ , by defining it on a Siegel set  $\mathfrak{S}_t$ , by

$$\eta(z) = \eta_R(z) = \eta_o(\log y - R) \quad (\text{for } z = n_x m_y K \in \mathfrak{S}_t)$$

The obvious issue is well-definedness.

Reduction theory assures existence of sufficiently large  $t_1 > t$  such that  $\gamma \cdot z = z'$  with  $\gamma \in \Gamma$ ,  $z \in \mathfrak{S}_t$  and  $z' \in \mathfrak{S}_{t_1}$  implies  $z \in \mathfrak{S}_{t_1}$ . Thus, for  $R > t_1$ , since  $\eta = 1$  on  $\mathfrak{S}_t - \mathfrak{S}_{t_1}$ , it is certainly *well defined* on  $q(\mathfrak{S}_t - \mathfrak{S}_{t_1})$ .

Meanwhile, note that  $\mathfrak{S}_{t_1}$  is stable under

$$\Gamma_\infty = \{\gamma \in \Gamma : \gamma\mathfrak{S}_{t_1} \cap \mathfrak{S}_{t_1} \neq \emptyset\}$$

By its definition  $\eta$  is invariant under  $\Gamma_\infty$ , which in this example is simply translation by  $\mathbb{Z}$ . Thus,  $\eta$  is well-defined on  $\Gamma \backslash (\Gamma \cdot \mathfrak{S}_{t_1})$ .

Then  $\eta$  lifts to  $\Gamma \backslash G$ .

The derivatives by  $U\mathfrak{g}$  of  $\eta$  on the right are easy to estimate: from the definition, in the usual Iwasawa coordinates  $NMK$ ,  $\eta(nmk) = \eta(m)$ . The relevant derivative is from the Lie algebra of  $M$ , namely,  $y \frac{\partial}{\partial y}$  in Iwasawa coordinates on  $G/K$ . By design, this derivative has a bound independent of  $R$ , because

$$y \frac{\partial}{\partial y} \eta(\log y - R) = \eta'(\log y - R)$$

[1.6]  $H_o^\ell = H^\ell$  The idea is to prove that test functions are dense in  $H^\ell(\Gamma \backslash G/K)$ , by showing that *smooth cut-offs* approach a given smooth function.

Let  $\eta = \eta_R$  be as above. We claim that the *smooth cut-offs* (point-wise products, not mollifications)  $\eta f$  approach  $f$  in  $H^\ell$ . For fixed  $g = nmk$ , from Leibniz' rule, the derivative of  $\eta f$  by an element of  $\alpha \in U\mathfrak{g}^{\leq \ell}$  is a finite sum of derivatives  $\alpha \eta \cdot \beta f$  where  $\alpha \in U\mathfrak{g}^{\leq \ell_1}$  and  $\beta \in U\mathfrak{g}^{\leq \ell_2}$  with  $\ell_1 + \ell_2 = \ell$ .

Further, we only need consider differential operators from a *compact* subset of  $U\mathfrak{g}^{\leq \ell}$ . The Adjoint action by the compact  $K$  produces another compact subset of  $U\mathfrak{g}^{\leq \ell}$ , for fixed  $k$  acting on the right on  $\eta(nm)$ . The differential operators coming from the Lie algebras of  $K$  and  $N$  act trivially on the right on  $\eta(nm)$ . The only non-trivial differential operators are from the Lie algebra of  $M$ , namely, iterates of  $y \frac{\partial}{\partial y}$ , and

$$\left(y \frac{\partial}{\partial y}\right)^\ell \eta_R(nm) = \eta^{(\ell)}(\log y - R)$$

By design, this is bounded, independent of  $R$ .

Any extreme term where  $\eta$  is *not* differentiated appears only as  $\eta \cdot \beta f - \beta f = (\eta - 1)\beta f$ .

Thus, all terms vanish on  $\{t \leq \log y \leq R\}$ , and the  $L^2$  norms of the rest are estimated uniformly in  $R$  by the tails of the integrals of (norms-squared of) derivatives of  $f$ , which go to 0 by square-integrability. Thus,  $H_o^{2\ell} = H^{2\ell}$  for  $2\ell \geq 0$ .

[1.7] **Reduction to Laplacian** The topologies on Sobolev spaces  $H^\ell(X)$  can be given more simply by the seminorms from a smaller class of differential operators, namely, the polynomials in  $\Delta$  or Casimir  $\Omega$ . This is fortunate, because the preferred functions in spectral expansions are eigenfunctions for  $\Delta$  and/or  $\Omega$ .

Let  $V$  be a Hilbert-space representation of  $G$ . Suppose that the subspace  $V^K$  of  $K$ -fixed (that is, *spherical*) vectors is not trivial. Let  $U\mathfrak{g}^{\leq n}$  be the usual filtration on the universal enveloping algebra. We claim that, given  $n$ , there is a constant  $C$  such for  $\alpha \in U\mathfrak{g}^{\leq n}$  and  $f \in V^K$

$$\langle \alpha f, \alpha f \rangle \leq \langle (-\Omega + C)^n f, f \rangle$$

Let  $\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$  be the Cartan decomposition with  $\mathfrak{k}$  the Lie algebra of  $K$ . By Poincaré-Birkhoff-Witt,  $U\mathfrak{g}$  is spanned by monomials of the form  $x_1 \dots x_m y_1 \dots y_n$  with  $x_i \in \mathfrak{p}$  and  $y_i \in \mathfrak{k}$ . Since  $\mathfrak{k}$  annihilates spherical vectors  $f$ , the effect of  $U\mathfrak{g}$  on spherical vectors  $f$  is that of linear combinations of monomials from  $\mathfrak{p}$ .

Let  $\Omega_{\mathfrak{p}}$  and  $\Omega_{\mathfrak{k}}$  be the corresponding components of Casimir  $\Omega$ . The Killing form is positive-definite on  $\mathfrak{p}$ , and negative definite on  $\mathfrak{k}$ , so  $\Omega_{\mathfrak{p}}$  is a non-positive symmetric operator in any Hilbert-space representation  $V$  of the group  $G$ , and  $\Omega_{\mathfrak{k}}$  is a non-negative symmetric operator. An element  $x \in \mathfrak{p}$  of length 1 can be extended to a self-dual basis  $\{x_i\}$  for  $\mathfrak{p}$ , and

$$\|xf\|^2 = \int xf \cdot x\bar{f} \leq \sum_i \int x_i f \cdot x_i \bar{f} = \int \sum_i -x_i^2 f \cdot \bar{f} = \int -\Omega_{\mathfrak{p}} f \cdot \bar{f}$$

For fixed  $n$ , let  $C$  be a large positive constant, strictly larger than the largest eigenvalue of  $\pm\Omega_{\mathfrak{k}}$  on  $\mathcal{U}\mathfrak{g}^{\leq n}$ . Then  $-\Omega_{\mathfrak{k}} + C$  is a non-negative, symmetric operator, and

$$T = -\Omega_{\mathfrak{p}} - \Omega_{\mathfrak{k}} + C = -\Omega + C$$

is a *strictly* positive symmetric (unbounded) operator on the Hilbert-space closure of  $\mathcal{U}\mathfrak{g}^{\leq n} \cdot V^K$ .

By Friedrichs,  $T$  has an (everywhere-defined) *inverse*  $R$ , a positive, symmetric *bounded* operator. By spectral theory for bounded, symmetric operators, there is a positive, symmetric  $\sqrt{R}$  in the closure of the polynomial algebra  $\mathbb{C}[R]$ . Thus,  $1\sqrt{R}$  is a symmetric positive  $\sqrt{T}$  commuting with all operators commuting with  $T$ . Thus,

$$\begin{aligned} \langle -\Omega_{\mathfrak{p}} \alpha f, \alpha f \rangle &\leq \langle (-\Omega_{\mathfrak{p}} - \Omega_{\mathfrak{k}} + C) \alpha f, \alpha f \rangle = \langle (-\Omega + C) \alpha f, \alpha f \rangle = \langle \sqrt{T} \sqrt{T} \alpha f, \alpha f \rangle \\ &= \langle \sqrt{T} \alpha f, \sqrt{T} \alpha f \rangle = \langle \alpha \sqrt{T} f, \alpha \sqrt{T} f \rangle \end{aligned}$$

By induction on the degree of  $\alpha$ ,

$$\langle \alpha \sqrt{T} f, \alpha \sqrt{T} f \rangle \leq \langle T^{\deg \alpha} \sqrt{T} f, \sqrt{T} f \rangle = \langle T^{1+\deg \alpha} f, f \rangle$$

That is,

$$\langle x \alpha f, x \alpha f \rangle \leq \langle (-\Omega + C)^{1+\deg \alpha} f, f \rangle$$

Since we want a comparison of *topologies*, not *metrics*, the salient feature is that, for  $\alpha \in \mathcal{U}\mathfrak{g}^n$ ,

$$\langle \alpha f, \alpha f \rangle \leq \langle (-\Omega + C)^{\deg \alpha} f, f \rangle \ll \langle (-\Omega + 1)^{\deg \alpha} f, f \rangle$$

Since test functions are dense in  $H^\ell(X)$ , the same relation holds for all vectors in  $H^\ell(X)$ .

In particular, this gives  $H^\ell(X)$  a Hilbert-space structure, with respect to the norm

$$\|f\|_\ell^2 = \langle (-\Omega + 1)^\ell f, f \rangle \quad (\text{for } f \in C_c^\infty(X))$$

and then completing.

**[1.7.1] Remark:** The integration-by-parts argument does *not* immediately apply to merely *smooth* functions, rather than *test* functions. Thus, we do *not* immediately obtain the dominance of all the Sobolev seminorms by the special sub-family involving only the Laplacian for *smooth* functions.

**[1.7.2] Remark:** In the Euclidean case, invocation of standard facts about Schwartz spaces and tempered distributions resolves the discrepancy mentioned in the previous remark. However, the analogous considerations for automorphic forms are less standard, less easily applicable, and less well understood.

**[1.8] Large Sobolev space** For  $\ell \geq 0$  the *large* Sobolev space  $W^{2,\ell} = W^{2,\ell}(X)$  is the collection of  $u \in L^2(X)$  such that  $u$  and its distributional derivatives  $\alpha u$  with  $\alpha \in \mathcal{U}\mathfrak{g}^{\leq \ell}$  are in  $L^2(X)$ . Give  $W^{2,\ell}$  the topology from the Sobolev norm  $\|\cdot\|_\ell$ . We claim that

$$W^{2,\ell}(X) = H^\ell(X) \quad (\text{for } \ell \geq 0)$$

In fact, proving  $W^{2,\ell}(\Gamma \backslash G) = H^\ell(\Gamma \backslash G)$  is convenient, because differentiations on the right by  $U\mathfrak{g}$  disrupt the right  $K$ -invariance. Taking right  $K$ -fixed elements at the end suffices.

Urysohn's lemma and the definition of integral show that  $C_0^0(\Gamma \backslash G)$  is dense in  $L^2(\Gamma \backslash G)$ . From this, the group  $G$  acts continuously on  $L^2(\Gamma \backslash G)$  by translation

$$(g \cdot f)(x) = f(xg) \quad (\text{for } f \in L^2(\Gamma \backslash G), g \in G)$$

The right action of  $G$  does *not* commute with the right action of  $U\mathfrak{g}$ , but  $G$  *does* map bounded subsets of  $U\mathfrak{g}^{\leq \ell}$  to bounded subsets, so the action of  $G$  *does* preserve the topology given by seminorms  $\nu_B$  for  $B$  ranging over bounded subsets of  $U\mathfrak{g}^{\leq \ell}$ . Then the continuity of  $G$  on  $L^2$  gives the continuity of  $G$  on  $W^{2,\ell}$ .

From the continuity of the group action, a typical application of Gelfand-Pettis integrals proves density of smooth functions in  $W^{2,\ell}$ , as follows. Let  $\{\eta\}$  be a smooth *approximate identity*. For  $u \in W^{2,\ell}$ , the basic estimate on Gelfand-Pettis integrals<sup>[1]</sup> shows that the averaged translations

$$\eta \cdot u = \int_G \eta(g) g \cdot u dg$$

often called *packets*, approach  $f$  in the topology on the representation space  $W^{2,\ell}$ .

At the same time, taking a genuine measurable function  $u$  as a representative for an  $L^2$  function,

$$(\eta \cdot u)(x) = \int_G \eta(g) u(xg) dg = \int_G \eta(x^{-1}g) u(g) dg$$

The literal definition of derivative can be applied: for a fixed vector  $v \in G$  and  $t$  real, the differentiability

$$|\eta(g \exp(tv)) - \eta(g) - t \nabla \eta(g) \cdot v| = o(t)$$

gives

$$\left| \int_G \eta(g \exp(tv)) u(g) dg - \int_G \eta(g) u(g) dg - \int_G t \nabla \eta(g) \cdot v u(g) dg \right| = o(t)$$

Thus,

$$\frac{\partial}{\partial t} \Big|_{t=0} = \frac{1}{t} \int_G \eta(g \exp(tv)) u(g) dg - \int_G \eta(g) u(g) dg = \int_G \nabla \eta(g) \cdot v u(g) dg$$

Since  $\nabla \eta \cdot v$  is again a test function, we see that  $\eta \cdot u$  is *smooth*.

Thus,  $H_o^\ell = H^\ell = W^{2,\ell}$  for  $\ell \geq 0$ .

**[1.9] Differentiation and spectral expansion** Since  $\mathcal{F} : L^2(X) \rightarrow L^2(\Xi)$  is an isometric isomorphism obtained by extension by continuity from  $\mathcal{F}$  on  $C_c^\infty(X)$ , defining  $\Delta$  by extension by continuity, the computation for test functions

$$\mathcal{F} \Delta f = \int_X \Delta f \Phi_\xi = \int_X f \Delta \bar{\Phi}_\xi = \lambda_\xi \int_X f \bar{\Phi}_\xi = \lambda_\xi \mathcal{F} f \quad (\text{for } f \in C_c^\infty(X))$$

gives the analogous assertion for  $f$  in  $H_o^\ell = H^\ell = W^{2,\ell}$  with  $\ell \geq 2$ . This is  **$L^2$ -differentiation** for non-negative index Sobolev spaces. *Term-wise differentiation* in the  $L^2$  sense is valid:

$$\Delta f = \mathcal{F}^{-1} \mathcal{F} \Delta f = \mathcal{F}^{-1} \lambda_\xi \mathcal{F} f \quad (\text{for } f \in H^\ell = W^{2,\ell} \text{ with } \ell \geq 2)$$

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[1] The basic estimate on Gelfand-Pettis integrals is that an integral  $\int_X f$  over a probability space  $X$  of a continuous  $V$ -valued function  $f$  lies in the closure of the convex hull of the image  $f(X)$ . This is valid for  $V$  quasi-complete and locally convex.

This differentiation is in a non-classical sense, and the maps  $\mathcal{F}$  and  $\mathcal{F}^{-1}$  are *not* literal integrals.

**[1.10] Characterization of Sobolev spaces by spectral transforms** By expressing Sobolev norm and differentiation via spectral transforms  $\mathcal{F}$ , certainly  $\mathcal{F}W^{2,\ell}$  is *contained in*

$$V^\ell = \{v : (1 - \lambda_\xi)^{\ell/2} v \in L^2(\Xi)\} \quad (\text{for } \ell \geq 0)$$

Give  $V^\ell$  the Hilbert-space structure from the expected norm

$$\|v\|_{V^\ell}^2 = \int_{\Xi} (1 - \lambda_\xi)^\ell |v|^2$$

inherited from the Sobolev space. We claim that Fourier transform gives a Hilbert space *isomorphism*

$$\mathcal{F} : W^{2,\ell} \longrightarrow V^\ell \quad (\text{an isomorphism})$$

The key point is to show that, for  $(1 - \lambda_\xi)v \in L^2(\Xi)$ ,

$$(1 - \Delta)\mathcal{F}^{-1}v = \mathcal{F}^{-1}((1 - \lambda_\xi)v) \quad (\text{distributional derivative})$$

To verify this equality, evaluate on test functions  $\varphi$ :

$$\left((1 - \Delta)\mathcal{F}^{-1}v\right)(\varphi) = \mathcal{F}^{-1}v((1 - \Delta)\varphi) = v\left(\mathcal{F}(1 - \Delta)\varphi\right)$$

by Plancherel for  $L^2(X)$  and  $L^2(\Xi)$ . Using the identities for test functions, this is

$$v\left((1 - \lambda_\xi)\mathcal{F}\varphi\right) = (1 - \lambda_\xi)v(\mathcal{F}\varphi) = \left(\mathcal{F}^{-1}((1 - \lambda_\xi)v)\right)(\varphi)$$

again by  $L^2$  Plancherel. This proves the above equality, namely, that the distribution  $(1 - \Delta)\mathcal{F}^{-1}v$  is given by integration against the  $L^2$  function  $\mathcal{F}^{-1}((1 - \lambda_\xi)v)$ .

This proves the necessary intertwining of multiplication and differentiation by inverse Fourier transform on  $V^\ell$ .

Since the seminorms  $\|(1 - \Delta)^\ell f\|$  give the topology on  $H_o^{2\ell} = H^{2\ell} = W^{2,2\ell}$ , the above intertwining yields the desired equality  $\mathcal{F}W^{2,2\ell} = V^{2\ell}$ , as follows. The case  $2\ell = 0$  is Plancherel. For arbitrary  $2\ell \geq 0$ , for  $v \in V^{2\ell}$ , for  $k \leq \ell$ ,

$$(1 - \Delta)^k \mathcal{F}^{-1}v = \mathcal{F}^{-1}(1 - \lambda_\xi)^k v \in F^{-1}V^{2\ell-2k} \subset F^{-1}L^2 = L^2$$

That is, all these distributional derivatives  $(1 - \Delta)^k \mathcal{F}^{-1}v$  are in  $L^2$ , so  $\mathcal{F}^{-1}v$  is in  $W^{2,2\ell}$ .

Thus, we have identified the precise image of the Sobolev space under the spectral transform  $\mathcal{F}$ .

**[1.11] Negative-index Sobolev spaces** Obviously we cannot easily describe negative-index Sobolev spaces in terms of negative-order differential operators. Instead, characterize negative-index Sobolev spaces as Hilbert-space duals

$$H^{-\ell} = \text{Hilbert-space dual to } H^\ell \quad (\text{for } \ell \geq 0)$$

The continuous  $L^2$ -differentiation  $\Delta : H^\ell \rightarrow H^{\ell-2}$  for  $\ell \geq 2$  on positive-index Sobolev spaces gives an adjoint, still denoted  $\Delta$ ,

$$\Delta : H^{-(\ell-2)} \longrightarrow H^{-\ell} \quad (\text{for } \ell \geq 2)$$

For  $\ell \geq 0$  the spectral transform  $\mathcal{F}$  is an isomorphism  $\mathcal{F} : H^\ell \rightarrow V^\ell$ , where, as above,

$$V^\ell = \{\text{measurable } v : (1 - \lambda_\xi)^{\ell/2} \cdot v \in L^2(\Xi)\} \quad (\text{for any } \ell \in \mathbb{Z})$$

with corresponding Hilbert space structure.

The space  $V^{-\ell}$  is naturally the Hilbert space dual  $V^{\ell*}$  to  $V^\ell$ , by integration:

$$\langle v, w \rangle = \int_X v(\xi) w(\xi) d\xi \quad (\text{complex-bilinear, for } v \in V^\ell \text{ and } w \in V^{-\ell})$$

The adjoint  $\mathcal{F}^*$  of the restriction of  $\mathcal{F}$  to  $H^\ell$  gives an isomorphism

$$\mathcal{F}^* : V^{-\ell} \longrightarrow H^{-\ell} \quad (\text{for } \ell \geq 0)$$

The density of  $V^{\ell+1}$  in  $V^\ell$  and the auto-duality of  $V^0 = L^2$  implies that the chain of *inclusions*

$$\dots \subset V^2 \subset V^1 \subset V^0 \approx V^{0*} \subset V^{-1} \subset V^{-2} \subset \dots$$

really does give the maps  $V^{-\ell} \rightarrow V^{-\ell-1}$  adjoint to the inclusions  $V^{\ell+1} \rightarrow V^\ell$  for  $\ell \geq 0$ . That is, these adjoint maps can be identified with the literal inclusions of spaces of functions. Visibly,  $V^{-\ell}$  is dense in  $V^{-\ell-1}$ .

Further, these inclusions and their adjoints are compatible with the multiplication operator(s)  $\mu : V^\ell \rightarrow V^{\ell-2}$  multiplying by  $1 - \lambda_\xi$ . Indeed, each  $\mu : V^\ell \rightarrow V^{\ell-2}$  is obviously an isomorphism compatible with the inclusions.

Plancherel asserts that the complex-linear adjoint of  $\mathcal{F} : L^2(X) \rightarrow L^2(\Xi)$  is simply  $\mathcal{F}^{-1}$ . This allows gluing together the commutative diagram

$$\begin{array}{ccccccc} \dots & \longrightarrow & H^2 & \xrightarrow{\text{inc}} & H^1 & \xrightarrow{\text{inc}} & H^0 \\ & & \mathcal{F} \downarrow \approx & & \mathcal{F} \downarrow \approx & & \mathcal{F} \downarrow \approx \\ \dots & \longrightarrow & V^2 & \xrightarrow{\text{inc}} & V^1 & \xrightarrow{\text{inc}} & V^0 \end{array}$$

and the adjoint diagram

$$\begin{array}{ccccccc} H^0 & \xrightarrow{\text{inc}^*} & H^{-1} & \xrightarrow{\text{inc}^*} & H^{-2} & \longrightarrow & \dots \\ \mathcal{F}^* \uparrow \approx & & \mathcal{F}^* \uparrow \approx & & \mathcal{F}^* \uparrow \approx & & \\ V^0 & \xrightarrow{\text{inc}^*} & V^{-1} & \xrightarrow{\text{inc}^*} & V^{-2} & \longrightarrow & \dots \end{array}$$

Using the observation that the adjoints of inclusions of the  $V^\ell$  are again inclusions, we have a commutative diagram

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & H^2 & \xrightarrow{\text{inc}} & H^1 & \xrightarrow{\text{inc}} & H^0 & \xrightarrow{\text{inc}^*} & H^{-1} & \xrightarrow{\text{inc}^*} & H^{-2} & \xrightarrow{\text{inc}^*} & \dots \\ & & \mathcal{F} \downarrow \approx & & \mathcal{F} \downarrow \approx & & \mathcal{F}^* \uparrow \approx & & \mathcal{F}^* \uparrow \approx & & \mathcal{F}^* \uparrow \approx & & \\ \dots & \longrightarrow & V^2 & \xrightarrow{\text{inc}} & V^1 & \xrightarrow{\text{inc}} & V^0 & \xrightarrow{\text{inc}} & V^{-1} & \xrightarrow{\text{inc}} & V^{-2} & \xrightarrow{\text{inc}} & \dots \end{array}$$

The density of  $V^\ell$  in  $V^{\ell-1}$  for every  $\ell \in \mathbb{Z}$  is converted by Fourier transform to density of  $H^\ell$  in  $H^{\ell-1}$  for every  $\ell$ . Thus, the adjoint maps  $\text{inc}^*$  on the top row are genuine inclusions. On the negative-index spaces, *define* spectral inversions  $\mathcal{F}^{-1}$  as the inverses of the isomorphism  $\mathcal{F}^*$ . Thus, we have a diagram

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & H^2 & \xrightarrow{\text{inc}} & H^1 & \xrightarrow{\text{inc}} & H^0 & \xrightarrow{\text{inc}} & H^{-1} & \xrightarrow{\text{inc}} & H^{-2} & \longrightarrow & \dots \\ & & \mathcal{F} \downarrow \approx & & \mathcal{F} \downarrow \approx & & \mathcal{F} \downarrow \approx & & \mathcal{F} \downarrow \approx & & & & \\ \dots & \longrightarrow & V^2 & \xrightarrow{\text{inc}} & V^1 & \xrightarrow{\text{inc}} & V^0 & \xrightarrow{\text{inc}} & V^{-1} & \xrightarrow{\text{inc}} & V^{-2} & \longrightarrow & \dots \end{array}$$

The adjoint  $\Delta^* : H^{-\ell+2} \rightarrow H^{-\ell}$  of the  $L^2$  differentiation  $\Delta : H^\ell \rightarrow H^{\ell-2}$  is *still* converted by Fourier transforms to multiplication by  $\lambda_\xi$ , and the adjoint of this multiplication map is multiplication by the same scalar. Thus, **define** the  $L^2$  **differentiation** on negative-index spaces to be these adjoints. This gives a commutative diagram

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & H^4 & \xrightarrow{\Delta} & H^2 & \xrightarrow{\Delta} & H^0 & \xrightarrow{\Delta} & H^{-2} & \xrightarrow{\Delta} & H^{-4} & \longrightarrow & \cdots \\ & & \mathcal{F} \downarrow \approx & & \mathcal{F} \downarrow \approx & & \mathcal{F} \downarrow \approx & & \mathcal{F} \downarrow \approx & & & & \\ \cdots & \longrightarrow & V^4 & \xrightarrow{\times \lambda_\xi} & V^2 & \xrightarrow{\times \lambda_\xi} & V^0 & \xrightarrow{\times \lambda_\xi} & V^{-2} & \xrightarrow{\times \lambda_\xi} & V^{-4} & \longrightarrow & \cdots \end{array}$$

and a similar diagram for odd-index spaces.

Since Fourier transform converts  $1 - \Delta$  to multiplication by  $1 - \lambda_\xi$ , and this multiplication gives isomorphisms  $V^\ell \rightarrow V^{\ell-2}$  for all  $\ell \in \mathbb{Z}$ ,  $1 - \Delta$  is an isomorphism  $H^\ell \rightarrow H^{\ell-2}$  for all  $\ell \in \mathbb{Z}$ .

The density of test functions in negative-index Sobolev spaces follows from the density of  $V^0$  in every  $V^{-\ell}$ , and the density of test functions in  $H^0$ .

This density implies the extension by continuity to *all* Sobolev spaces of the integration by parts adjoint relation

$$\langle \Delta f, v \rangle_{H^{\ell-2} \times H^{-\ell+2}} = \langle f, \Delta v \rangle_{H^\ell \times H^{-\ell}} \quad (\text{for } f \in H^\ell \text{ and } v \in H^{-\ell})$$

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