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Buildings, Bruhat decompositions, unramified principal series

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EDIT: ... draft...

Buildings are special simplicial complexes on which interesting groups such as $GL_n(\mathbf{Q})$ and $SL_n(\mathbf{Q}_p)$ act in an illuminating fashion. The geometric structure is critical, and the geometric language is a powerful heuristic and mnemonic. This viewpoint was created by Jacques Tits as a way to subsume the *geometric algebra* that is nearly sufficient to treat basic properties of *classical groups* such as general linear groups, symplectic groups, orthogonal groups, unitary groups, while giving a stronger language and viewpoint allowing treatment of the *exceptional* groups. Further, Bruhat and Tits subsequently gave an *intrinsic* treatment of buildings attached to reductive groups. Here we will give an economical treatment that presumes few prerequisites, but hopefully does not degrade the central ideas. [1]

The first application is to *Bruhat decompositions*, whose first assertion is a decomposition into *cells*[2]

$$G = \bigsqcup_{w \in W} PwP$$

In the simplest case where $G = GL_n(k)$ (invertible n-by-n matrices with entries in a field k), the *minimal parabolic*[3] P can be taken to be upper-triangular (invertible) matrices, and the *Weyl group*[4] W can be taken to be all permutation matrices. It is true that it is possible to give a direct *ad hoc* proof of this fact for $GL_n(k)$, for example. However, the *ad hoc* argument is arduous and unilluminating, and, for example, does not easily give the *disjointness* of the union, for larger n. Further, refinements of the decomposition for non-minimal parabolic subgroups, are less accessible by seemingly elementary methods.

As a global application, refinements of the Bruhat decomposition are essential to the discussion of *constant terms* of *Eisenstein series*, and in understanding their meromorphic continuations. Such issues lie behind both Langlands-Shahidi and Rankin-Selberg integral representations of L-functions.

In the mid-1960s Iwahori and Matsumoto made the surprising discovery that the Bruhat-Tits buildings formalism, conceived as a device to study parabolic subgroups, was applicable to a very different issue, that of *compact open* subgroups in p-adic reductive groups, such as $SL_n(\mathbf{Q}_p)$. In particular, this brought to light the technical importance of some subtler items than considered immediately from classical motivations. For example, rather than *maximal* compact open subgroups as fundamental objects, somewhat smaller compact open subgroups (now called *Iwahori subgroups*) were shown to be the fundamental gadgets.

An application of critical importance for the representation theory of p-adic groups stemming from the Iwahori-Matsumoto discovery is the Borel-Matsumoto theorem (circa 1976) that asserts (among other things) that an irreducible representation of a p-adic group possessing an Iwahori-fixed vector has an imbedding into

[1] It turns out that it is possible to ignore almost completely the traditional combinatorial group theory usually taken as a prerequisite. That is, we find that the theory of *Coxeter groups* is not necessary for what we do here.

[2] Under various further hypotheses, the sets PwP are literal cells in the topological sense of being homeomorphic to open balls. We will not need such particular details, but nevertheless will refer to the sets PwP as *Bruhat cells*.

[3] We give a building-theoretic definition below. There is also a definition coming from the viewpoint of algebraic geometry. The latter is not immediately helpful to our present purposes, and even the *comparison* to the building-theoretic definition does not help us, so we neglect the definition from algebraic geometry.

[4] There are many forms of a general definition of *Weyl group*. We give a building-theoretic definition below. Comparison with other definitions is a job in itself, which we do not undertake.

an *unramified principal series* representation. [5] Further, intertwining operators among unramified principal series can be understood well enough from the Borel-Matsumoto viewpoint (Casselman 1980) so as to give very clear criteria for irreducibility of unramified principal series and even *degenerate* principal series [6]

The theme of *groups acting on things* is pervasive. Further, often as much interest resides in the proof technique as in the results themselves. Even in the simplest example, the Sylow theorems, the fact that the action of the group on p-subgroups by conjugation is productive is more interesting than the specific conclusions of the theorem.

In the case of buildings and groups acting on them the *things* on which the groups act are now more structured, and more subtly structured, than in simpler examples.

- Simplicial complexes, chamber complexes, buildings
- Example: spherical building for $GL(n)$
- Canonical retractions, uniqueness lemma
- Group actions, parabolic subgroups
- Weyl groups, Bruhat decompositions
- Reflections, foldings
- Bruhat cell multiplication

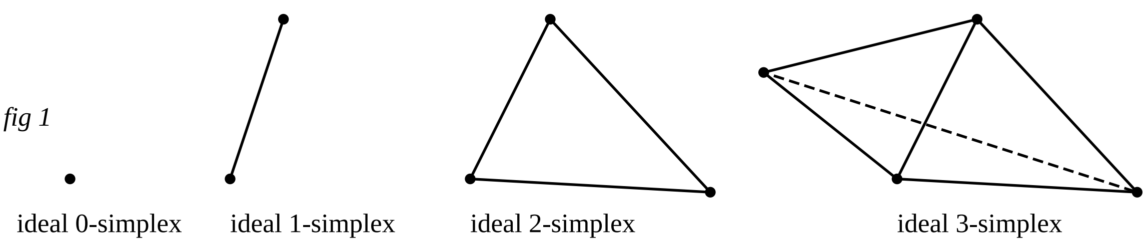
EDIT: ... more later...

1. Simplicial complexes, chamber complexes, buildings

This section introduces geometric language necessary to talk about buildings.

A **simplex** is a generic member of the family of geometric objects including *points*, *line segments*, *triangles*, *tetrahedrons*, and so on. A 0-simplex is a point, a 1-simplex is a line segment, a 2-simplex is a triangle, a 3-simplex is a solid tetrahedron.

fig 1



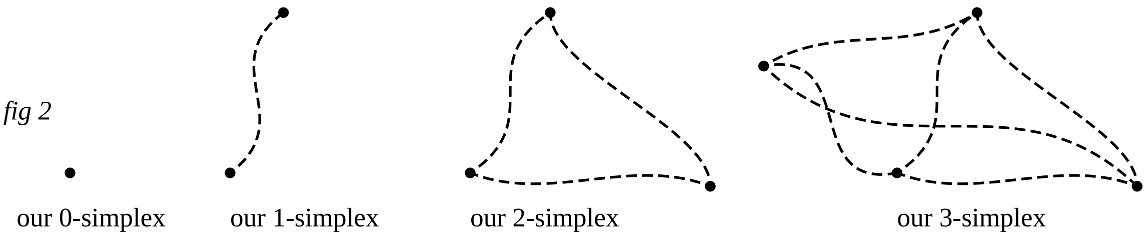
But for present purposes, we need only a shadow of geometry. The idea is that an n-simplex should be determined by its $n+1$ vertices, so a **simplex** is a *set* (of vertices). [7]

[5] We will define and illustrate these representations later. They are the most accessible and understandable of representations of p-adic groups.

[6] By definition, these are induced from one-dimensional characters on non-minimal parabolics. They are proper subrepresentations of unramified principal series, so it is not immediately clear how study of reducibility of unramified principal series helps us understand irreducibility of subrepresentations which only occur in reducible unramified principal series. More on this below.

[7] Thus, two simplices with the same set of vertices are identical.

fig 2



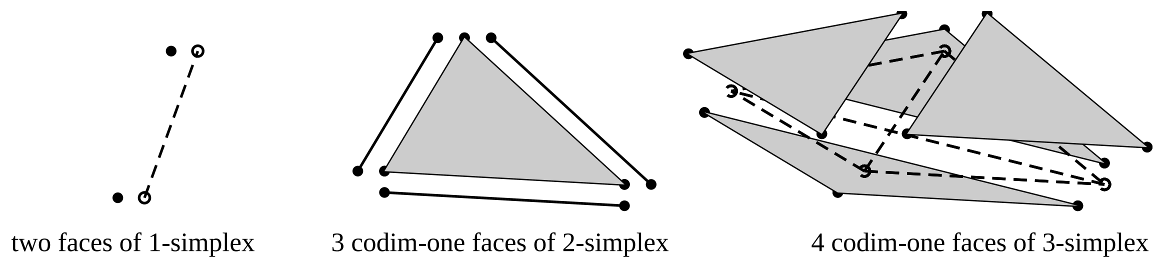
The **faces** τ of a simplex σ are the simplices appearing as non-empty subsets $\tau \subset \sigma$.[8] The **dimension** of a simplex σ is one less than the cardinality of the underlying set,[9] namely

$$\dim \sigma = |\sigma| - 1$$

The **codimension** of a face τ of a simplex σ is the difference of dimensions

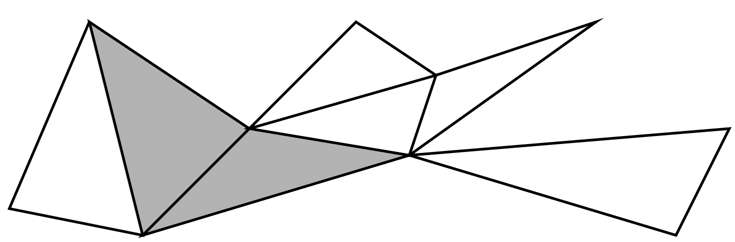
$$\text{codimension of } \tau \text{ in } \sigma \; = \dim \sigma - \dim \tau$$

fig 3



A **simplicial complex** X is a set V of vertices, and a distinguished set X of subsets of the vertices, with the property that $\sigma \in C$ and $\tau \subset \sigma$ implies $\tau \in C$. The sets of vertices[10] in X are the simplices in the complex, and that last requirement is that if a simplex σ is in X then all the *faces* of σ are in X as well. The **dimension** of a simplicial complex is the maximum of the dimensions of the simplices in it. A **subcomplex** Y of a simplicial complex X is a simplicial complex which has vertices which form a subset of the vertices of X, and has simplices which are a subset of those of X. In a simplicial complex, two simplices of the same dimension are **adjacent** if they have a common codimension-one face.

fig 4



simplicial complex
two adjacent simplices shaded

[8] Thus, the intersection of two simplices is a common face of the two. In this discretized geometry, 1-simplices (line segments) can only intersect at endpoints (or not at all), and two 1-simplices with both endpoints in common are identical. 2-simplices (triangles) can only intersect in vertices,

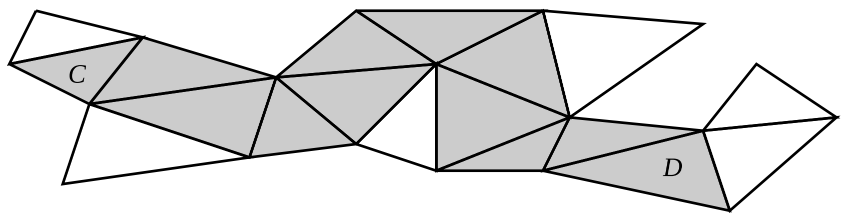
[9] A point is zero-dimensional, a one-dimensional line segment needs 2 points to specify it, a two-dimensional triangle needs 3 points, and so on.

[10] The 0-dimensional faces τ of a simplex σ are the singleton subsets $\{x\}$ for $x \in \sigma$. That is, the 0-dimensional faces are $\{x\}$ for *vertices* x of σ. The distinction between x and the singleton set $\{x\}$ is usually not important.

A **simplicial complex map** $f : X \longrightarrow Y$ is a set-map on the underlying sets of vertices, with the property that f is a *bijection* on every simplex in X, and for a simplex σ in X, the image $f(\sigma)$ is a simplex in Y. [11]

A **chamber** in a simplicial complex X is a simplex which is *maximal*, in the sense of not being a face of any other simplex in X. A **gallery** $C_0, \ldots, C_n$ **connecting** two chambers C_0 and C_n in a simplicial complex X is a sequence of chambers C_i such that C_i and C_{i+1} are *adjacent*. The length of that gallery is n. The **distance** between two chambers is the length of a shortest gallery connecting them. [12] A **chamber complex** is a simplicial complex in which any two chambers are connected by a gallery. [13] A chamber complex is **thin** if every codimension-one face is a face of *exactly two* chambers. [14]

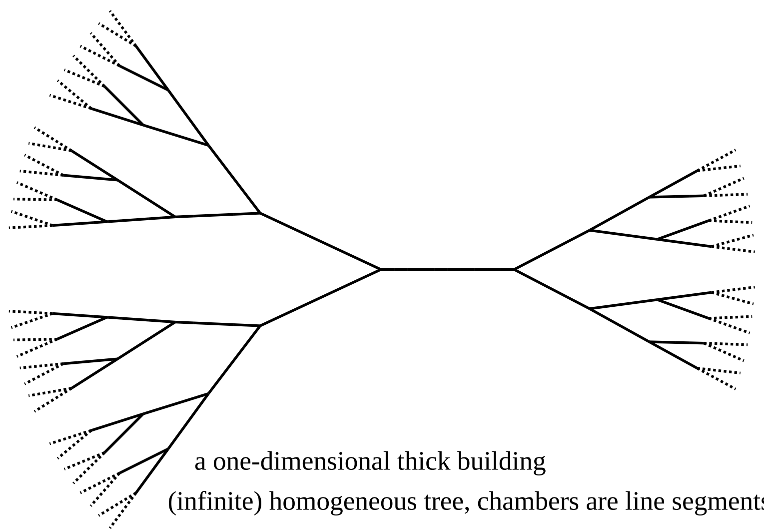
fig 5



shading shows gallery connecting chambers C and D in thin chamber complex

A chamber complex is **thick** if every codimension-one face is a face of *at least three* chambers. [15]

fig 6



a one-dimensional thick building
(infinite) homogeneous tree, chambers are line segments

A simplicial subcomplex Y of a chamber complex X is a **chamber subcomplex** if it is a chamber complex with chambers of the same dimension as those in X. A chamber-complex map is a simplicial complex map on chamber complexes of the same dimension.

[11] Thus, a simplicial complex map $f : X \longrightarrow Y$ need not be either injective or surjective on the sets of vertices of X and Y, but respects simplices in X in the sense that it neither reduces their dimension nor tears them apart (by mapping them to non-simplices in Y). We can also say that a simplicial complex map *preserves face relations*, since by its behavior on vertices, if τ is a face of σ in X, then $f(\tau)$ is a face of $f(\sigma)$ in Y.

[12] Thus, two adjacent chambers are at distance 1.

[13] This implicitly requires that any two chambers are of the same dimension.

[14] Thin chamber complexes are vaguely like manifolds, or manifolds with boundary.

[15] Vaguely, thick chamber complexes are like bunches of manifolds stuck together at various patches.

A simplicial complex X is a **(thick) building** if
• X is a thick chamber complex.
• There is a set of chamber subcomplexes (the **apartments**) of X such that
• Each apartment is a *thin* chamber complex.
• Any two chambers in X are contained in a common apartment.
• For two apartments a, a' containing chambers C and D in common, there is a chamber-complex isomorphism $f : a \to a'$ fixing the vertices of both C and D. [16]

Remark: Note that the chamber-complex property of the whole building X will follow from the two fact that the apartments are chamber complexes, and from the fact that any two chambers are contained in a common apartment. All that needs to be proven separately about the building is the *thickness*.

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2. Example: spherical building for $GL(n)$

Here we construct a family of buildings on which the k-linear automorphisms of a vector space V over a field k will act nicely. This construction will be used to study the group $GL(n)$. [17] Specifically, groups such as the subgroup of upper-triangular matrices in $GL(n)$ will arise, by design, as stabilizers of chambers. We construct the simplicial complex, then prove that it meets the requirements for a thick building. *There is non-trivial substance in the argument required to verify that this is a building.*

Let V be an n-dimensional vectorspace over a field k. A **flag** in V is a nested chain

$$V_{d_1} \subset \ldots \subset V_{d_t}$$

with proper inclusions, of vector subspaces[18] of V. Often the subscript denotes the dimension. The **length** of the flag is the number of subspaces in it. A flag is **maximal** if it cannot be made longer.

Let X be the simplicial complex whose vertices are proper vector subspaces of V, and whose maximal simplices are maximal flags C of proper subspaces of V

$$V_1 \subset V_2 \subset \ldots \subset V_{n-1}$$

where V_i is of dimension i. The *faces* of this simplex C are the (non-empty) subflags F of σ, namely all (non-empty) flags F

$$V_{i_1} \subset \ldots \subset V_{i_\ell} \quad \text{(necessarily } i_1 < \ldots < i_\ell\text{)}$$

The subcomplexes we propose as apartments consists of subcomplexes a specified by **frames**, that is, by unordered collections of n one-dimensional vector subspaces $L_1, \ldots, L_n$ which span the vectorspace. A frame

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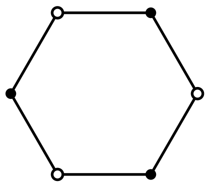
[16] In fact, each apartment in a building has the structure of a *Coxeter complex*, meaning the following. First, a Coxeter *group* is a group G with generators S and relations as follows. For all $s \in S$, $s^2 = e$. For every $s, t \in S$ there is $m = m(s,t)$ (possibly infinite, meaning no relation) such that $(st)^m = e$. There are no other relations. The *generalized reflections* in G are the conjugates of elements of S. Two group elements g, h are *adjacent* if there is a generalized reflection t in G such that $tg = h$. Beginning with this idea, one can construct a thin chamber complex, the associated *Coxeter complex* from a Coxeter group. The usual way of defining a building includes a requirement that apartments be Coxeter complexes. However, such a definition has disadvantages, and we proceed differently.

[17] There are several related but mutually incompatible notations. First, for positive integer n and field k, $GL_n(k)$ is the group of invertible n-by-n matrices with entries in k. This is sometimes denoted by $GL(n,k)$, or even $GL(n)$ when the reference to k is implicit or irrelevant. So far, this is fairly self-consistent. However, for a k-vectorspace V (without a choice of basis), one may also write $GL_k(V)$ for the group of k-linear automorphisms of V.

[18] For our purposes we prohibit the 0-subspace and the whole space in flags.

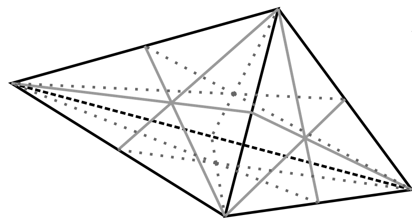
determines a subcomplex a by including in a all flags involving only subspaces expressible as sums of lines from the frame.

fig 7



vertices of
two types:
lines and planes

apartment for GL(3)



vertices of three types
extreme vertices are lines
edge midpoints are planes
face centers are 3-folds

apartment for GL(4) (barycentric subdivision of tetrahedron)

Claim: The simplicial complex X of flags in V, with apartments determined by frames, is a thick building.

Proof: We must verify four things, that the whole is *thick*,[19] that each apartment is a *thin*[20] chamber complex,[21] that any two simplices are contained in a common apartment, and, last, that for two apartments a and b containing common chambers C, D there is a chamber-complex isomorphism $\varphi : a \longrightarrow b$ fixing the vertices of both $\mathbf{C}$ and D.

Let C be a top-dimensional simplex in X, namely a maximal flag

$$V_1 \subset \ldots \subset V_{n-1}$$

where V_i is i-dimensional. The codimension-one faces of C are the simplices τ_i given by omitting the i^{th} subspace, that is,

$$V_1 \subset \ldots \subset V_{i-1} \subset V_{i+1} \subset \ldots \subset V_{n-1}$$

Consider the case that the index i is properly between the end values, that is, $1 < i < n$. Then the *other* maximal simplices with τ_i as face are obtained by replacing V_i by some *other* i-dimensional subspace lying between V_{i-1} and V_{i+1}. The collection of such subspaces is in bijection with the set of *lines* the quotient V_{i+1}/V_{i-1}. This quotient is a two-dimensional k-vectorspace. If k is infinite, there are infinitely-many such lines. If k is finite, with q elements, then there are $(q^2-1)/(q-1) = q+1$ lines, which is at least 3. This proves the thickness.[22] The cases $i = 1$ and $i = n$ are nearly identical, except for notation.

To prove that the apartments are thin chamber complexes, we need to prove that they are chamber complexes in the first place, and that they are *thin.* Fix a frame F. As in the discussion of the last paragraph, for a maximal simplex C in X given by a maximal flag

$$V_1 \subset \ldots \subset V_{n-1}$$

the other maximal simplices adjacent along the i^{th} face

$$V_1 \subset \ldots \subset V_{i-1} \subset V_{i+1} \subset \ldots \subset V_{n-1}$$

are obtained by replacing V_i by another i-dimensional subspace lying between V_{i-1} and V_{i+1}. In a fixed apartment, the choice of this i-dimensional subspace is constrained. In particular, for some pair of (distinct) lines L_1, L_2 in the frame,

$$V_{i+1} = V_{i-1} \oplus L_1 \oplus L_2$$

[19] Again, *thickness* is that each codimension-one face is the face of *at least three* chambers.

[20] Again, *thin-ness* is that each codimension-one face is the face of *exactly two* chambers.

[21] Again, a chamber complex is a simplicial complex in which any two maximal simplices are connected by a gallery.

[22] As noted earlier, since we will prove that in apartments any two chambers are connected by a gallery, and we will prove that any two chambers lie in a common apartment, we will have proven that in the building as a whole any two chambers are connected by a gallery. That is, the whole building is a chamber complex.

Thus, there are just two choices of i-dimensional subspace between between V_{i-1} and V_{i+1}, namely

$$V_{i-1} \oplus L_1 \qquad \text{and} \qquad V_{i-1} \oplus L_2$$

This proves the thin-ness of apartments.

Now prove that any two maximal simplices in an apartment are connected by a gallery. Let C be the chamber given by the maximal flag

$$V_1 \subset \ldots \subset V_{n-1}$$

where

$$V_i = L_1 \oplus \ldots \oplus L_i$$

with lines $L_1, \ldots, L_n$ specifying the frame. Thus, *choice of a chamber is equivalent to choice of an ordering of the lines in the frame.* From the previous paragraph, the chamber adjacent to C across the i^{th} codimension-one face

$$V_1 \subset \ldots \subset \widehat{V_i} \subset \ldots \subset V_{n-1}$$

is

$$V_1 \subset \ldots \subset V_{i-1} \subset V_{i-1} \oplus L_{i+1} \subset V_{i+1} \subset \ldots \subset V_{n-1}$$

That is, moving across the i^{th} face interchanges the i^{th} and $(i-1)^{\text{th}}$ lines in the ordering specifying the chamber. Since every permutation of n things is expressible as a product of adjacent transpositions,[23] every ordering of lines is obtained in such fashion. That is, any two chambers in the apartment are connected by a gallery.

Now prove that any two chambers C, D are contained in a common apartment. That is, we find a *frame* in terms of which all the subspaces occurring in C or in D are expressible.[24] Let C be the maximal flag

$$V_1 \subset \ldots \subset V_{n-1}$$

and let D be the maximal flag

$$W_1 \subset \ldots \subset W_{n-1}$$

In order to find a common frame for the two flags, we recall some standard but slightly technical points.

Lemma: For subspaces X, Y, η of a k-vectorspace V, with $Y \supset \eta$,

$$(X + \eta) \cap Y = (X \cap Y) + \eta$$

Proof: The proof is inevitable, once one realizes that this is what we'll need. (See proof of Zassenhaus' theorem just following.) First,

$$(X \cap Y) + \eta \subset Y \cap (X + \eta)$$

where or not $\eta \subset Y$. On the other hand, let $x + y' = y$ with $x \in X$, $y \in Y$, and $y' \in \eta$. Then

$$x = y - y' \in X \cap (Y - \eta) = X \cap Y$$

from which y is in $(X \cap Y) + \eta$. ///

[23] This is standard, proven by induction on n: Let π be a permutation of n things, that is, a bijection of $\{1, \ldots, n\}$ to itself. If $\pi(n) = n$, then π can be identified with a permutation of $n-1$ things, and we're done, by induction. For $\pi(n) = m < n$, do induction on $n - m$. Let s be the permutation which interchanges m and $m+1$ and does not move any other element. Then $(s \circ \pi)(n) = m + 1$, and induction finishes the argument.

[24] Finding the frame is an application of *Zassenhaus' theorem*, which is an idea preliminary to the proof of Jordan-Hölder-type theorems.

In a slightly different notational style, possible since we'll not need to refer to elements:

Theorem: *(Zassenhaus)*[25] Let $X \supset x$ and $Y \supset y$ be subspaces of a vector space V. Then there are natural isomorphisms

$$\frac{(X+y)\cap Y}{(x+y)\cap Y} \approx \frac{X\cap Y}{(x\cap Y)+(X\cap y)} \approx \frac{(x+Y)\cap X}{(x+y)\cap X}$$

Proof: The kernel of

$$X\cap Y \subset (X+y)\cap Y \longrightarrow \frac{(X+y)\cap Y}{(x+y)\cap Y}$$

is

$$(X\cap Y)\cap((x+y)\cap Y) = X\cap(x+y)\cap Y$$

Applying the previous lemma twice gives

$$X\cap(x+y)\cap Y = X\cap((x\cap Y)+y) = (X\cap\eta)+(x\cap Y)$$

This gives the left isomorphism. The right isomorphism follows by reversing the roles of X, x and Y, y. ///

Now we return to the proof that there is a common apartment for two given chambers. First, as a matter of notation, let $V_n = W_n = V$. For a given index i, $\dim_k V_i/V_{i-1} = 1$, so there is a smallest index j such that

$$\dim_k \frac{V_i}{V_{i-1}} = \dim_k \frac{(V_i\cap W_j)+V_{i-1}}{V_{i-1}} = 1$$

Then

$$\dim_k \frac{(V_i\cap W_{j-1})+V_{i-1}}{V_{i-1}} = 0$$

so $(V_i\cap W_{j-1})+V_{i-1} = V_{i-1}$, and, thus,

$$\frac{V_i}{V_{i-1}} \approx \frac{(V_i\cap W_j)+V_{i-1}}{V_{i-1}} \approx \frac{(V_i\cap W_j)+V_{i-1}}{(V_i\cap W_{j-1})+V_{i-1}}$$

For $\ell > j$, still

$$\dim_k \frac{(V_i\cap W_\ell)+V_{i-1}}{V_{i-1}} = 1$$

but also

$$\dim_k \frac{(V_i\cap W_{\ell-1})+V_{i-1}}{V_{i-1}} = 1$$

so

$$\dim_k \frac{(V_i\cap W_\ell)+V_{i-1}}{(V_i\cap W_{\ell-1})+V_{i-1}} = 0$$

That is, given i, there is a *unique* index j such that

$$\frac{V_i}{V_{i-1}} \approx \frac{(V_i\cap W_j)+V_{i-1}}{V_{i-1}} \approx \frac{(V_i\cap W_j)+V_{i-1}}{(V_i\cap W_{j-1})+V_{i-1}}$$

Then, via Zassenhaus' theorem, given i, there is exactly one index j such that

$$\frac{V_i}{V_{i-1}} \approx \frac{(V_i\cap W_j)+V_{i-1}}{(V_i\cap W_{j-1})+V_{i-1}} \approx \frac{V_i\cap W_j}{(V_i\cap W_{j-1})+(V_{i-1}\cap W_j)}$$

[25] This result is sometimes called the *Butterfly Lemma*, due to the fact that one can manage to draw a diagram indicating the relations among the various subspaces that resembles a butterfly.

Invoking Zassenhaus' theorem again,

$$\frac{V_i \cap W_j}{(V_i \cap W_{j-1}) + (V_{i-1} \cap W_j)} \approx \frac{(V_i \cap W_j) + W_{j-1}}{(V_{i-1} \cap W_j) + W_{j-1}}$$

By a symmetrical argument, for given j, the right-hand side of the last isomorphism is one-dimensional for exactly one index i. Thus, we have shown that there is a bijection $i \leftrightarrow j$ of $\{1, \ldots, n\}$ to itself such that all these quotients are one-dimensional. Given a pair i and j, let L_i be a one-dimensional subspace of V mapping surjectively to both V_i/V_{i-1} and W_j/W_{j-1}. Then sums of the lines $L_1, \ldots, L_n$ express all subspaces V_i and W_j, so give a frame specifying an apartment in which both the given chambers lie. That is, we have proven that there is an apartment containing any two given chambers.

Last, we verify that for chambers C, D lying in the intersection $a \cap b$ of two apartments, there is a simplicial complex isomorphism $f : a \longrightarrow b$ fixing C and D pointwise. In fact, letting $L_1, \ldots, L_n$ and $M_1, \ldots, M_n$ be the one-dimensional subspaces specifying the two apartments, we will give a bijection between these sets of lines which will yield the identity map on the two given chambers.

By renumbering the lines if necessary, we can suppose that the chamber C corresponds to the orderings $L_1, \ldots, L_n$ and $M_1, \ldots, M_n$, that is, to the flags

$$L_1 \subset L_1 \oplus L_2 \subset \ldots \subset L_1 \oplus \ldots \oplus L_{n-1}$$

$$M_1 \subset M_1 \oplus M_2 \subset \ldots \subset M_1 \oplus \ldots \oplus M_{n-1}$$

Define a map

$$f : a \longrightarrow b$$

on vertices by

$$f(L_{i_1} \oplus \ldots \oplus L_{i_m}) = M_{i_1} \oplus \ldots \oplus M_{i_m})$$

for any m-tuple of indices $i_1 < \ldots < i_m$. On the chamber C this is the identity. By the uniqueness lemma, if there *is* an isomorphism $a \longrightarrow b$, this map must be it.[26] It suffices to prove that f is the identity map on $a \cap b$, and to prove this it suffices to prove that f is the identity on *vertices* in that intersection. We claim that

$$L_{i_1} \oplus \ldots \oplus L_{i_m} = M_{j_1} \oplus \ldots \oplus M_{j_m}$$

with $i_1 < \ldots < i_m$ and $j_1 < \ldots < j_m$ implies that $i_\ell = j_\ell$ for all ℓ. We prove this by induction on m. The case $m = 1$ is trivial. For $m > 1$, let ℓ be the largest[27] index such that $i_\ell \neq j_\ell$. Without loss of generality, suppose that $i_\ell < j_\ell$. Our hypothesis about the numbers of the lines and the expressibility of C in terms of both gives

$$L_1 \oplus L_2 \oplus \ldots \oplus L_{j_\ell - 2} \oplus L_{j_\ell - 1} = M_1 \oplus M_2 \oplus \ldots \oplus M_{j_\ell - 2} \oplus M_{j_\ell - 1}$$

Adding these subspaces to the given one yields

$$L_1 \oplus L_2 \oplus \ldots \oplus L_{j_\ell - 2} \oplus L_{j_\ell - 1} \oplus L_{i_{\ell+1}} \oplus \ldots \oplus L_{i_m} = M_1 \oplus M_2 \oplus \ldots \oplus M_{j_\ell - 2} \oplus M_{j_\ell - 1} \oplus M_{j_\ell} \oplus \ldots \oplus M_{j_m}$$

For all $\mu > m$ we have $i_\mu = j_\mu$. Thus, taking dimensions, since $i_\ell < j_\ell$,

$$(j_\ell - 1) + (m - \ell) = (j_\ell - 1) + (m - \ell + 1)$$

which is impossible. Thus, $i_\ell = j_\ell$ for all ℓ, and $f : a \longrightarrow b$ is an isomorphism. This proves that our construction gives a building. ///

[26] The fact that the second chamber D played no role in the definition of f is less surprising by this point, in view of the uniqueness lemma.

[27] Yes, *largest*, not *smallest*.

Remark: The group $GL_k(V)$ of k-linear automorphisms of the vectorspace V acts by simplicial-complex maps on the building X just constructed, since $GL_k(V)$ preserves *dimension* and *containment* of subspaces. But we do not yet have sufficient information to do much with this yet.

Remark: These buildings are **spherical**, since, with some trouble, one can verify that the apartments are simplicial versions of *spheres*. We will make no use of this, so will not worry about justifying the terminology.

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3. Canonical retractions, uniqueness lemma

This section contains the first non-trivial abstract results on buildings.

A **retraction**[28] $r : X \longrightarrow Y$ of a simplicial complex X to a subcomplex Y of X is a simplicial complex map whose restriction to Y is the identity map. Two simplicial complex maps agree **pointwise** if they are equal on vertices, hence on all simplices and their faces.

A gallery $C_1, \ldots, C_n$ in a chamber complex **stutters** if a chamber appears twice or more consecutively, that is, if for some index $C_i = C_{i+1}$.

Theorem: Given an apartment a in a building X, there is retraction $X \longrightarrow a$. Indeed, given a chamber C in a, there is a *unique* retraction $X \longrightarrow a$ sending non-stuttering galleries starting at C to non-stuttering galleries in a (necessarily starting at C). Further, this retraction is an *isomorphism* $a' \longrightarrow a$ on any apartment a' containing C.

This retraction is the **canonical retraction** of the building to the given apartment, **centered** at the given chamber.

Proof: As a first step toward construction retractions, we prove a result important in its own right.

Lemma: *(Uniqueness)* Let X, Y be chamber complexes, with Y having the property[29] that each codimension-one face is a face of *at most* two chambers. Let $r : X \longrightarrow Y$, $g : X \longrightarrow Y$ be chamber complex maps which agree pointwise[30] on a chamber C in X, and both f and g send *non-stuttering* galleries starting at C to *non-stuttering* galleries. Then $f = g$.

Proof: (of lemma) Let $C = C_0, C_1, \ldots, C_n = D$ be a non-stuttering gallery. By hypothesis, its image under f and its image under g do not stutter. That is, $fC_i \neq fC_{i+1}$ for all i, and similarly for g. Suppose, inductively, that f agrees with g on C_i and all its faces. Certainly fC_i and fC_{i+1} are adjacent along the face

$$F = fC_i \cap fC_{i+1} = gC_i \cap gC_{i+1}$$

By the non-stuttering assumption, $fC_{i+1} \neq fC_i$ and $gC_{i+1} \neq gC_i$. Thus, by the hypothesis on Y, it must be that $fC_{i+1} = gC_{i+1}$, since there is no *third* chamber with facet F. Since there is a gallery from C to any other chamber, this proves that $f = g$ pointwise on all of X. ///

Remark: The uniqueness lemma allows formulation of a more memorable version of one of the defining conditions for a building. That is, rather than the original requirement that for any two apartments containing a chamber C and a simplex σ there is a simplicial complex isomorphism $f : a \longrightarrow a'$ fixing C and σ pointwise, we have the following.

---

[28] This notion of *retraction* is a discretized version of the usual notion in topology.

[29] The hypothesis on Y is certainly met for Y *thin*, but we need the slightly weaker hypothesis later.

[30] Again, *pointwise* agreement means agreement on vertices.

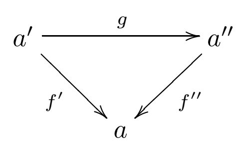
Corollary: For two apartments $a, a' \in A$ containing a common chamber C, there is be a chamber-complex isomorphism $f : a \longrightarrow a'$ fixing $a \cap a'$ *pointwise*.

Proof: This *implies* the original axiom. For a simplex $\sigma \in a \cap a'$, there is an isomorphism $f_\sigma : a \longrightarrow a'$ fixing σ and C pointwise, by the building axiom. The *Uniqueness Lemma* implies that there can be at most one such map which fixes C pointwise. Thus, $f_\sigma = f_\tau$ for all simplices σ, τ in the intersection. ///

Now we try to construct a retraction $r : X \longrightarrow a$ of X to a. For a chamber D not in a, let a' be an apartment containing both C and D, and $f' : a' \longrightarrow a$ an isomorphism which pointwise fixes $a \cap a'$. The existence of f' is assured by the last corollary. By the uniqueness lemma, there is just one such f'. For another apartment a'' containing both C and D, let $f'' : a'' \longrightarrow a$ be the unique isomorphism which fixes $a'' \cap a$ pointwise. We claim that $f'D = f''D$, so that we can define

$$rD = f'D = f''D$$

Let $g : a' \longrightarrow a''$ be the isomorphism fixing D pointwise, from the building axioms. Then by uniqueness $f'' \circ g = f'$, that is, the diagram



commutes. Then on $a' \cap a''$ the map $f'' \circ g$ is f. That is, these isomorphisms to a agree on overlaps, so give a well-defined retraction r to a. ///

Corollary: Let C and D be two chambers in X. Let a be an apartment containing both C and D. Then the length of a shortest gallery from C to D inside a is the same as the length of a shortest gallery from C to D inside the whole building X.

Proof: Let r be the retraction of X to a centered at C. Then the image under r of a gallery from C to D in X is no longer than the original gallery. ///

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4. Group actions, parabolic subgroups

We want simplices in buildings to have no non-trivial automorphisms, so that *fixing a simplex* will mean fixing it *pointwise*. To achieve this, we want to distinguish types of vertices, rather than seeing all vertices as the same.[31] Specifically, a **typing** or **labeling** of an n-dimensional chamber complex X is a simplicial complex map[32] $\lambda : X \longrightarrow \Delta$ where Δ is a simplex.[33] The **type** or **label** of a vertex is its image by λ in Δ. Given a labeling[34] $X \longrightarrow \Delta$, a simplicial complex map $f : X \longrightarrow X$ of X to itself is **label-preserving**

[31] A similar finer distinction is necessary in algebraic topology, where a notion of *orientation* of a simplex is introduced. This amounts to *ordering* the vertices, modulo *even* permutations. The initial confusion about this parity distinction in some cases motivated treatment of homology *modulo* 2, for no better reason than to avoid worry about signs.

[32] Recall that this means that dimensions of simplices are preserved, and implies that face relations are preserved.

[33] Necessarily Δ is of dimension at least that of X. We will only care about the case that Δ has the same dimension as X.

[34] In fact, it can be shown, with some effort, that every thick building has a labeling, but all our constructions will make a concrete labeling evident. Thus, we need not dally to prove the general fact, which would entail that we prove that each apartment is a Coxeter complex, etc.

or **type-preserving** if $\lambda \circ f = \lambda$, that is, if we have a commuting diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \lambda \downarrow & & \downarrow \lambda \\ \Delta & = & \Delta \end{array}$$

Example: In the spherical building X for an n-dimensional vectorspace V over a field k, the vertices are linear subspaces of V, and can be labeled by their *dimension*. That is, let $\Delta = \{1, 2, \ldots, n-1\}$ and

$$\lambda : X \longrightarrow \Delta \qquad \text{by} \qquad \lambda(x) = \dim_k x$$

for vertices x of X. The natural action of $G = GL_k(V)$ on subspaces certainly preserves dimension, so the action of $GL_k(V)$ is label-preserving.

Let X be a thick building with labeling $\lambda : X \longrightarrow \Delta$. Let G be a group acting on X[35] by simplicial complex maps *preserving labels*. The group action is said to be **strongly transitive** if it is transitive on *pairs* C, a, where C is a chamber in an apartment a. [36]

A **parabolic subgroup** P in G is a stabilizer of some simplex σ in the building, that is,

$$P = \{g \in G : g\sigma = \sigma\}$$

The **minimal parabolics** are stabilizers of *chambers*. **Maximal parabolics** are stabilizers of *vertices*. [37]

Example: In the action of $GL(n)$ on the $(n-1)$-dimensional spherical building attached to an n-dimensional vectorspace V over a field k, the parabolic subgroups admit visually memorable descriptions in terms of matrices. Let $e_1, \ldots, e_n$ be the standard basis for $V = k^n$, and identify $G = GL_k(V)$ with $GL_n(k)$, the group of n-by-n invertible matrices with entries in k. Take chamber C specified as maximal flag

$$ke_1 \subset ke_1 \oplus ke_2 \subset \ldots \subset ke_1 \oplus \ldots \oplus ke_{n-1}$$

The minimal parabolic stabilizing this flag is the so-called **standard** minimal parabolic consisting of upper-triangular matrices

$$\begin{pmatrix} * & * & \ldots & & * \\ 0 & * & & & \\ \vdots & & \ddots & & \vdots \\ & & & * & * \\ 0 & & \ldots & 0 & * \end{pmatrix}$$

The $n-1$ different maximal parabolics fixing faces of C are the fixers of length-one flags

$$ke_1 \oplus \ldots \oplus ke_i$$

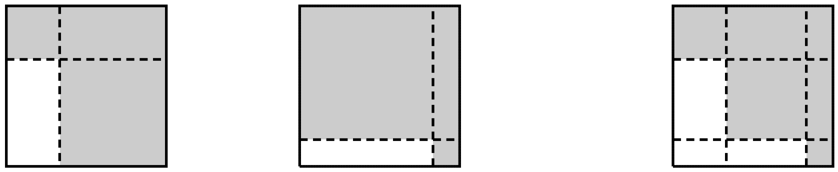
and consist of matrices where the two diagonal blocks are invertible. The general parabolic fixing some face of C has square blocks of varying sizes along the diagonal, zeros below, and anything above.

[35] A building always needs an implied system of apartments. We will not worry about possibly varying choices of such.

[36] Implicit in this is that the group maps apartments to apartments.

[37] As usual, smaller sets are stabilized by larger subgroups, and vice-versa.

fig 8



Transitivity gives some easy but important results that arise in all group actions on sets:

Proposition: All minimal parabolic subgroups are conjugate in G.

Proof: This is essentially because the group is postulated to act transitively on chambers. In more detail, as usual, let P be the stabilizer of a chamber C and Q the stabilizer of a chamber D. Let $g \in G$ be such that $gC = D$. For $q \in Q$,

$$q(gC) = qD = D = gC$$

so apply g^{-1} to obtain

$$(g^{-1}qg)C = C$$

so $g^{-1}qg \in P$. That is, $g^{-1}Qg \subset P$. The argument is clearly reversible, so we have equality. ///

Remark: Because of the labeling, with an n-dimensional building there are $n+1$ conjugacy classes of *maximal* parabolics, since G preserves labels.

Remark: For $GL(n)$, from our present viewpoint the causality will run the other way, that is, we will prove the strong transitivity of the group by looking at the explicit behavior of flags of subspaces.

Example: We claim that *the natural action of $GL_k(V)$ on the spherical building constructed earlier is strongly transitive.* Label the vertices of the spherical building by *dimension* of the subspace (which is the vertex). The natural action of $GL_k(V)$ on linear subspaces of V certainly preserves dimensions of subspaces, so preserves labels. Apartments are specified by frames, that is, unordered collections $\{L_1, \ldots, L_n$ of lines (one-dimensional subspaces) L_i whose direct sum is the whole space V. As in the earlier proof that this complex truly is a building, the chambers within the apartment specified by a frame F are in bijection with the *orderings* of the lines L_i. To prove strong transitivity is to prove that $GL_k(V)$ is transitive on sets of one-dimensional subspaces whose direct sums are the whole space V. Indeed, for another set $\mu_1, \ldots, \mu_n$, there is a unique invertible k-linear map which sends $L_i \longrightarrow \mu_i$ for all i. This proves the strong transitivity in this example.[38]

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5. Weyl groups, Bruhat decomposition

There are more subgroups of interest that can be nicely specified in terms of the action on the building. Again, let X be a thick building with a labeling $\lambda : X \longrightarrow \Delta$, and let G be a group acting on X strongly transitively, preserving labels. Let

$$\begin{array}{ccccccc} \mathcal{N} & = & \mathcal{N}(a) & = & & & \text{stabilizer of an apartment } a \\ A & = & A(a) & = & & & \text{pointwise fixer of an apartment } a \\ W & = & W(a) & = & \mathcal{N}/A & = & \text{Weyl group of an apartment } a \end{array}$$

[38] In this example, the strong transitivity has little content. By contrast, the proof that the alleged building is indeed a building is more serious.

Proposition: The Weyl group W of an apartment a has a well-defined action on a by invertible simplicial maps. The group W acts transitively on the chambers in the apartment, and is in *bijection* with the chambers by

$$w \leftrightarrow wC$$

for any fixed chamber C in a.

Proof: The strong transitivity immediately shows that $\mathcal{N}$ stabilizes a and is transitive on chambers in a. Since by definition A fixes a pointwise, the quotient $W = \mathcal{N}/A$ has a well-defined action on a. Let $w \in W$ fix a given chamber C. The *label-preserving* property of the group action implies that w fixes C *pointwise*. Since the elements of W give isomorphisms on a, they certainly send non-stuttering galleries to non-stuttering galleries. Thus, by the uniqueness lemma, there is at most one isomorphism $w : a \longrightarrow a$ to the thin chamber complex a fixing C. The identity map on a is one such, so w is necessarily the identity on a. Thus, the map $w \longrightarrow wC$ must be a bijection. ///

For a fixed chamber C in the apartment a in X, the corresponding minimal parabolic P is

$$P \quad = \quad \text{minimal parabolic} \quad = \quad \text{stabilizer of } \ C$$

The following is of central importance. [39]

Theorem: *(Bruhat decomposition)* There is a *disjoint* decomposition

$$G = \bigsqcup_{w \in W} PwP$$

In more detail: let $r : X \longrightarrow a$ be the retraction to a centered at C. Given $g \in G$ let $w \in W$ be the unique element such that $wC = r(gC)$. Then

$$g \in PwP$$

Remark: Note that although $W = \mathcal{N}/A$ is not a subgroup of G, since $A \subset P$ any double coset PwP for $w \in W$ is well-defined. And recall that W is in bijection with the chambers in a, from above.

Proof: By the building axioms, there is an apartment a' containing both gC and C. The retraction r restricted to a' is an isomorphism to a, fixing C pointwise. Likewise, the strong transitivity of G on X implies that there is an element $p \in P$ mapping a' to a. By *uniqueness*, these two maps $a' \longrightarrow a$ must be identical. That is,

$$pgC = r(gC) = wC$$

That is,

$$(w^{-1}pg)C = C$$

so $w^{-1}pg \in P$. ///

Remark: In fact, we could have done without the retraction $r : X \longrightarrow a$ entirely, but its existence exhibits the coherence among the various isomorphisms of other apartments to a given one.

The form of the Bruhat decomposition above suffices for many applications, but not all. For example, for the Borel-Matsumoto theorem on unramified principal series, we will need finer information on the Weyl group and on Bruhat-like decompositions. We begin the clarification of the nature of the Weyl group by the following discussion.

[39] Even in very simple situations, where instances of this result admit easy proofs, the fact seems to have only been discovered in the 1950s.

Proposition: Fix a chamber C in an apartment a in X. For each chamber D in a adjacent to C there is a unique element $s \in W$ (the **reflection** along $C \cap D$) such that $sC = D$ and $sD = C$. This reflection s has the property that $s^2 = 1$. The collection S of all reflections along codimension-one faces of C generates W.

Proof: First, we prove existence of a reflection along each codimension-one face of C. Given adjacent chambers C, D in a, by strong transitivity of the action of G there is an element $s \in G$ stabilizing a and sending C to D. Since s stabilizes a it lies in $\mathcal{N}$, and if we restrict our attention to the effect of s on a we may as well view s as lying in the quotient $W = \mathcal{N}/A$. Since G is label-preserving, s must fix pointwise $C \cap D$. Thus, the single vertex of C *not* lying in $C \cap D$ must be mapped by s to the single vertex of D *not* lying in $C \cap D$, and vice-versa. This proves existence.

Since $s : a \longrightarrow a$ is an isomorphism, it maps non-stuttering galleries to non-stuttering galleries, so by the uniqueness lemma the effect of s on a is completely determined by the fact that it maps $C \leftrightarrow D$ fixing $C \cap D$ pointwise. This proves uniqueness of the reflection along $C \cap D$.

Since s^2 is an automorphism of a fixing C pointwise, s^2 is the identity map on a, again by the uniqueness lemma.

To prove that the set S of reflections along codimension-one faces of C generates W, do induction on the length of a minimal gallery from C to a chamber wC for $w \in W$. Let $C, sC, \ldots, wC$ be a minimal gallery for some $s \in S$. Apply s to this gallery to obtain $sC, C, \ldots, swD$. That is, swD is strictly closer to C than is wD. By induction, S generates a subgroup of W transitive on chambers in a. Earlier we had shown that W is in bijection with chambers in a by the map $w \longrightarrow wC$, so S must generate all of W. ///

With fixed chamber C in an apartment a, the **length** $\ell(w)$ of an element $w \in W$ can be defined in two ways, that are not immediately identical:

gallery length C to wC word length of w with respect to S

where the **word length** of w is, by definition, the smallest n such that

$$w = s_1 s_2 \ldots s_n \quad (\text{with } s_i \in S)$$

The word length certainly depends on choice of set S of generators.

Proposition: Gallery length and word length are identical functions on W. In particular, for a shortest expression

$$w = s_1 \ldots s_n$$

for w in terms of s_i in S, the sequence of chambers

$$C,\ s_1 C,\ s_1 s_2 C,\ s_1 s_2 s_3,\ \ldots,\ s_1 \ldots s_n C$$

is a minimal gallery from C to wC.

Proof: First, we check that these chambers are successively adjacent. Indeed, each pair of chambers

$$s_1 \ldots s_i C,\ s_1 \ldots s_i s_{i+1} C$$

is the image under $s_1 \ldots s_i$ of the pair

$$C,\ s_{i+1} C$$

The chamber $s_{i+1}C$ is adjacent to C, and $s_1 \ldots s_i$ preserves adjacency, so the images are indeed adjacent. This shows that gallery length is less than or equal to word length.

We prove the opposite inclusion by induction on minimum gallery length. Let

$$C,\ C_1,\ C_2,\ \ldots,\ C_n = wC$$

be a minimal gallery from C to wC. There is some $s \in S$ such that $C_1 = sC$. Apply s to the gallery gives a gallery

$$sC,\ C,\ sC_2,\ \ldots .\ sC_n = swC$$

Thus, swC is strictly closer to C in gallery distance than was wC. That is,

$$1 + \text{gallery length } sw \leq \text{gallery length } w$$

Thus, by induction, the gallery length of sw is equal to the word length of sw. Visibly

$$\text{word length } w = \text{word length } s \cdot sw \leq 1 + \text{word length } sw$$

Thus, since we already know that gallery length is at most word length,

$$\text{gallery length } w \leq \text{word length } w \leq 1 + \text{word length } sw = 1 + \text{gallery length } sw \leq \text{gallery length } w$$

This proves the desired equality. ///

Remark: A critical point missing from this little discussion of generation of W by reflections S is clarification of length of sw versus length of w, for $w \in W$ and $s \in S$. For example, as it stands at the moment, it is hard to see why these lengths might not be the same. In fact, as we will see in the next section, things are as nice as we could hope:

$$\text{length } sw = \text{length } w \pm 1$$

This fact is critical for Hecke algebras and the Borel-Matsumoto theorem, and is non-trivial to prove. The technical discussion of *foldings* in the next section seems to be necessary to address this.

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6. Reflections, foldings

Still X is a thick building, but for the moment we can forget about groups acting upon it. The goal is to understand automorphism groups of apartments, without reference to any group that may be acting on the larger building. In particular, given two adjacent chambers in a building, we want a **reflection** automorphism which, by definition, should interchange the two adjacent chambers while fixing pointwise their common face. The existence of such a reflection on the subcomplex consisting just of the two chambers is immediate, but it is not at all obvious that this map extends to an automorphism of the apartment.

A further technical surprise is that the construction (or proof of existence) of reflections (following Jacques Tits) uses **foldings**, defined and discussed below. The foldings themselves are constructed via the retractions of the whole building to various apartments. It is at this point that the thickness of the building itself is used to prove things about the apartments.

As a by-product, we construct (many) retractions of an apartment to a given chamber within the apartment, thus proving that an apartment is label-able. Combining this with the (canonical) retraction of the building to an apartment, we prove that there always exist labelings.

A **folding** of an apartment a in X is a chamber-complex map $f : a \longrightarrow a$ which is a *retraction* to its image and is two-to-one on chambers.[40] The **opposite folding** f' (if it exists) to a given folding f reverses the roles in each pair C, C' of chambers that have the same image under f. That is, for $f(C) = f(C') = C$, the opposite folding is $f'(C) = f'(C') = C'$. A folding with an opposite is **reversible**.

Theorem: *(Existence)*Given two adjacent chambers C, D in an apartment a in a thick building X, there are (mutually opposite) foldings $f : a \longrightarrow a$ and $g : a \longrightarrow a$ such that $f(C) = C = f(D)$ and $g(C) = D = g(D)$.

[40] That is, the inverse image of any chamber consists of *two* chambers.

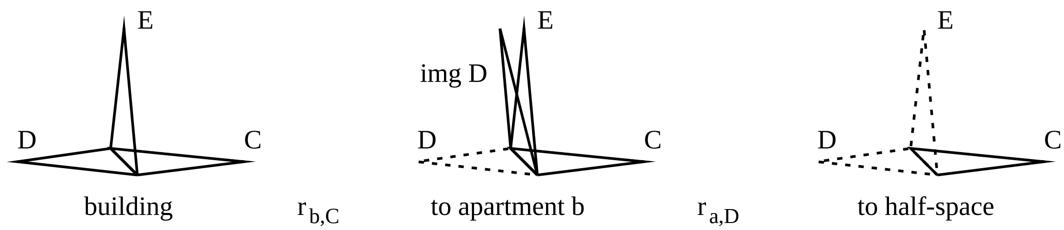
Letting $H = f(a)$ and $H' = g(a)$, the restriction to H' of f is an isomorphism $f : H' \longrightarrow H$. Similarly, the restriction to H of g is an isomorphism $g : H \longrightarrow H'$. We have $a = H \cup H'$, and on the other hand $H \cap H'$ contains no chamber.

Proof: Invoking the thickness, let B be a third chamber with face $C \cap D$. Invoking the building axioms, let b be an apartment containing C and B. Let $r_{b,C}$ be the canonical retraction of X to b centered at C, and let $r_{a,D}$ be the canonical retraction of X to a centered at D. We claim that

$$f = r_{a,D} \circ r_{b,C}$$

is the desired folding.

fig 9



First, $r_{b,C}(C) = C$. And $r_{a,D}$ is a retraction to a, so is the identity on a, so maps C to itself. On the other hand, $r_{b,C}$ must map D to a chamber in b sharing the face $C \cap D$ (but not C), which must be B, by the thin-ness of b. Then $r_{a,B}$ maps $r_{b,C}(D) = B$ to a chamber in a sharing the face $D \cap B = D \cap C$ (other than D), which must be C (by thin-ness). Thus, $f(C) = C = f(D)$.

Next, claim that f is an *isomorphism* on a minimal gallery $C_0, \ldots, C_n = E$ from *any* chamber C_0 with face $F = C \cap D$ to a chamber E in X. To prove this, we prove that the two canonical retractions $r_{b,C}$ and $r_{a,D}$ have this property. Let $r = r_{b,C}$. For E in b,

$$r(C_0), \ldots, r(E) = E$$

is a gallery wholly within b, so the gallery distance in X from F to $E \in a$ is equal to the gallery distance in b.[41] Denote gallery distance in X by $d_X(,)$ and gallery distance in an apartment b by $d_b(,)$. We know that the restriction $r : b \longrightarrow b$ of the retraction r is an isomorphism (from the uniqueness lemma). For arbitrary E in X, let b be an apartment containing both C and E. We have

$$d_X(F, rE) = d_b(F, rE) = d_b(F, E) = d_X(F, E)$$

This verifies that r preserves gallery distances from F. The argument for $r_{a,D}$ is the same. Since $F = C \cap D$ is the common face, the composite f preserves gallery distances from F. Thus, the alleged folding $f : a \longrightarrow a$ is an isomorphism on minimal galleries in a from $F = C \cap D$, since of course it cannot cause a minimal gallery to stutter without shortening it.

For a minimal gallery $C_0, \ldots, C_n = E$ in a from F (a face of C_0) to E (in a), C_0 is either C or D. For $C_0 = C$ we have $d_X(F, E) = d_X(C, E)$. Since f does not shorten this gallery, by the uniqueness lemma, $f(E) = E$ (pointwise). Motivated by this, let H be the subcomplex of a consisting of all chambers E (and their faces) such that

$$d_X(F, E) = d_X(C, E)$$

Then f is the identity on H. On the other hand, for $C_0 = D$, application of f gives $f(D) = C$. That is, the image of the gallery begins with C, and is still minimal. That is, $f(E) \subset H$, so f is indeed a retraction to H.

[41] Of course, *a priori* the gallery distance in b is greater than or equal to that in X.

Reversing the roles of C, D gives an analogous $g : a \longrightarrow a$ which is a retraction to the subcomplex H' consisting of all chambers E (and their faces) such that

$$d_X(F, E) = d_X(D, E)$$

Since C and D are the only two chambers in a with face F, we are assured that

$$a = H \cup H'$$

but the nature of the intersection $H \cap H'$ is not yet clear.

Now show that H and H' have no chamber in common. For E in $Y \cap Z$, both f and g fix E pointwise. Let γ be a minimal gallery from F (a face of C_0) to E, in a. Then the images $f(\gamma)$ and $g(\gamma)$ are galleries from F to E. Since γ was already minimal, these galleries cannot stutter. By the uniqueness lemma, $f = g$, which is false, since $f(C) = C \neq D = g(C)$. Thus, there are no chambers in common.

Last, we show that $f : H' \longrightarrow H$ is an isomorphism, and $g : H \longrightarrow H'$ is an isomorphism. This will also prove the two-to-one property of f and g.

The map $f \circ g : H \longrightarrow H$ maps C to itself pointwise. Galleries from F in H necessarily begin at C. Certainly $(f \circ g)(C) = C$. Let γ be a minimal gallery from F to a chamber E. The composite $f \circ g$ does not shorten this gallery, so does not make it stutter. Thus, by the uniqueness lemma applied to

$$f \circ g : \gamma \longrightarrow Y$$

the map $f \circ g$ is an isomorphism on γ. In particular, the map $f \circ g$ fixes every E in H pointwise, so is the identity on H. Similarly, $g \circ f$ is the identity on H'. Similarly, $g \circ f$ is the identity on H'. ///

The image of a folding is a **half-space** or **half-apartment**. Given adjacent chambers C, D in an apartment a, the folding f of a such that $fC = C = fD$ is a **folding along** the codimension-one face $C \cap D$ of C.

Several useful structural facts follow from the existence of reversible foldings.

Proposition: Let f be a reversible folding of a, with opposite folding g. Let C, D be adjacent chambers such that $f(C) = C = f(D)$. Then the half-space $H = f(a)$ is

$$H = \{\text{chambers } E \in a : d(C, E) < d(D, E)\}$$

where $d(,)$ denotes the length of shortest gallery connecting two chambers.

Proof: First, given a chamber E in a with $fE \neq E$, we claim that the image $fC, \ldots, fE$ of any gallery $C = C_0, \ldots, C_n = E$ in a connecting C and E *stutters*. Certainly there is an index i such that $fC_i = C_i$ and $fC_{i+1} \neq C_{i+1}$. Since a is thin, fC_{i+1} has no choice but to be C_i. That is, the image of the gallery stutters. In particular, fE is strictly closer to $fC = C$ than is E (in minimum-gallery distance).

Similarly, for $E \in H$, and γ a minimal gallery from D to E, $f\gamma$ is a *stuttering* gallery from $fD = C$ to $fE = E$, so can be shortened. Thus, $d(C, E) < d(D, E)$. The other half of the assertion follows from the presence of the opposite folding. ///

Corollary: Every thick building has a labeling. [42]

Proof: It suffices to prove that a building has a retraction to a given chamber C, since this gives a labeling. We already know that there is a (canonical) retraction of the whole building to an apartment, so it suffices

[42] Since every thick building has a labeling, we would not need to explicitly assume label-ability. However, in practice, we'll usually have an obvious labeling arising from external circumstances.

to exhibit a retraction of an apartment a to a given chamber within it. This last part is non-canonical. Let $f_0, f_1, \ldots, f_n$ be the foldings along the codimension-one faces of C, and let

$$\varphi = f_0 \circ f_1 \circ \ldots \circ f_n$$

Let D_i be the chamber adjacent to C such that f_i is a folding along $C \cap D_i$. From the previous proposition, any chamber closer to D_i than to C is moved closer by f_i, while chambers closer to C than to D_i are not moved at all by f_i. A given chamber E in a has a minimal gallery from C, and some D_i must be the second gallery in this chamber, so the corresponding f_i moves E strictly closer to C, and moves no chamber in a farther from C. Thus, the composition φ of all the foldings along the codimension-one faces of C is a simplicial complex map which moves every chamber in a strictly closer to C. Thus, chambers at gallery distance ℓ from C will be mapped to C by the ℓ-fold composite φ^ℓ. The function on vertices defined as

$$\varphi^\infty(x) = \varphi^\ell(x) \quad \text{(for } \ell \text{ sufficiently large)}$$

where *sufficiently large* means *such that* $\varphi^\ell(x)$ *is in* C is well-defined, since φ fixes C pointwise. This gives the retraction of a to C. ///

Corollary: *(Uniqueness)* Given adjacent chambers C, D in an apartment a, there is a *unique* reversible folding $f : a \longrightarrow a$ such that $fC = C = fD$.

Proof: *Existence* is the content of the theorem above. The proposition characterizes the half-space $H = f(a)$ intrinsically. Similarly, for g the opposite folding to f, the opposite half-space $H' = g(a)$ is similarly characterized. On H, the folding f is the identity. On H' the folding f is determined completely on D, and is an isomorphism on H', so is completely determined, by the uniqueness lemma. Thus, f is unique. ///

At last we can construct **reflections**, in terms of *foldings*, independently of any assumption of a group action on the building.

Corollary: Given adjacent chambers C, D in an apartment a in a thick building X, there is a unique automorphism $s : a \longrightarrow a$ such that s fixes $C \cap D$ pointwise, $sC = D$, and $sD = C$. This s is *the* reflection of a along $C \cap D$. It follows that $s^2 = 1$. This reflection is given explicitly as follows. Let f be the folding with $fC = C = fD$ and g its opposite, and half-spaces $f(a) = H$ and $g(a) = H'$. Then

$$s(x) = \begin{cases} fx & \text{(for } x \in H'\text{)} \\ gx & \text{(for } x \in H\text{)} \end{cases}$$

for vertices x in a.

Proof: Any automorphism s of a sends non-stuttering galleries to non-stuttering galleries. For s fixing $C \cap D$ pointwise and interchanging C and D, s is determined on C, so is completely determined, from the uniqueness lemma. Since $s^2C = C$ and s^2 fixes the codimension-one face $C \cap D$ of C, it must be that s^2 fixes C pointwise, so $s^2 = 1$ on a, again by uniqueness.

For existence, it remains to show that the indicated formula satisfies the conditions. First, the theorem showed that $f : H' \longrightarrow H$ is an isomorphism, and that $g : H \longrightarrow H'$ is an isomorphism. And f is a retraction to H, so is the identity on H, and similarly for g on H'. Thus, on $H \cap H'$ both f and g are the identity, which allows us to piece together s as in the indicated formula. [43] ///

[43] Implicit in this piecing-together is the fact that $H \cap H'$, H, and H' are simplicial subcomplexes of the apartment, and f, g are simplicial complex maps.

And, at last, we prove the critical fact about lengths. [44] In the following proposition, the **length** of an element $w \in W$ is the length n of a shortest gallery $C, C_1, C_2, \ldots, C_n = wC$ in a from C to wC.

Corollary: Let $w \in W$ and $s \in S$. Then

$$\text{length } ws = \text{length } w \pm 1$$

Proof: Let $d(E, F)$ be gallery distance between two chambers E, F. Since swC and wC are adjacent,

$$d(C, wC) - 1 \leq d(C, swC) \leq d(C, wC) + 1$$

That is, a minimal gallery from C to swC is at most as long as a minimal gallery from C to wC and *then* to swC, and, symmetrically, a minimal gallery from C to wC is at most as long as a minimal gallery from C to swC and *then* to wC. Thus, the issue is to prove that

$$d(C, wC) \neq d(C, swC)$$

Let f and g be the reversible foldings with half-spaces $H = f(a)$ and $H' = g(a)$, such that for a vertex x in a

$$s(x) = \begin{cases} fx & (\text{for } x \in H') \\ gx & (\text{for } x \in H) \end{cases}$$

We know from the theorem above that $H \cup H' = a$, and

$$H = \{wC : d(C, wC) < d(sC, wC)\} \qquad H' = \{wC : d(C, wC) > d(sC, wC)\}$$

Thus, $d(C, wC) \neq d(sC, wC)$. ///

Corollary: Let $w \in W$ and $s \in S$. Then

$$\text{length } sw = \text{length } w \pm 1$$

Proof: Now we use the fact proven in the previous secton that *gallery* length and *word* length (with respect to the generators S for W) are identical. We also use the fact that $s^{-1} = s$ for any $s \in S$. Thus, in terms of word length, observe that the length of w^{-1} is equal to that of w, since

$$(s_1 \ldots s_n)^{-1} = s_n s_{n-1} \ldots s_2 s_1$$

Using this fact and the previous corollary,

$$\text{length}(sw) = \text{length}((sw)^{-1}) = \text{length}(w^{-1}s) = \text{length}(w^{-1}) \pm 1 = \text{length}(w) \pm 1$$

as desired. ///

7. Bruhat cell multiplication

[44] In the previous section, with a group acting strongly transitively, we proved that *gallery* length from C to wC is identical to *word* length in the generating reflections S. The same proof applies in the present context. Indeed, the argument given earlier does not require any of the subtleties connected with foldings, so it was reasonable to give the argument at that earlier point.

The present discussion refines the Bruhat decomposition, aiming toward discussion of (Iwahori-)Hecke algebras.

Let X be a thick building with a strongly transitive label-preserving action of a group G. Fix a chamber C in an apartment a, and as usual let

$$\begin{array}{ccccccl} P & = & P(C) & = & & & \text{stabilizer of chamber } C \\ \mathcal{N} & = & \mathcal{N}(a) & = & & & \text{stabilizer of apartment } a \\ A & = & A(a) & = & & & \text{pointwise fixer of apartment } a \\ W & = & W(a) & = & \mathcal{N}/A & = & \text{Weyl group of apartment } a \end{array}$$

Let S be the set of **reflections** of a along codimension-one faces of C.

The **Bruhat cells**[45] are the sets PwP for $w \in W$.[46] Let $\ell(w)$ be the length of $w \in W$, either as *word* length in terms of the generators S of W, or as *gallery* length of a minimal gallery from C to wC. We have shown that these two notions of length coincide.

Corollary: For $w \in W$ and $s \in S$

$$PwP \cdot PsP = \begin{cases} PwsP & (\text{for } \ell(sw) > \ell(w)) \\ PwsP \cup PwP & (\text{for } \ell(sw) < \ell(w)) \end{cases}$$

Remark: From the previous section, we know that the two cases are the only cases that can occur. In particular, $\ell(sw) \neq \ell(w)$.

Proof: First, whatever else it may be, the set

$$PwP \cdot PsP = \{p_1 w p_2 \cdot p_3 s p_4 : p_i \in P\}$$

is stable under left and right multiplication by P, so is a union of double cosets $Pw'P$ for $w' \in W$ (using the first Bruhat decomposition). And

$$ws = w \cdot s \in PwP \cdot PsP$$

so *always*

$$PwsP \subset PwP \cdot PsP$$

What is not clear is what other double cosets, if any, appear.

Let r be the retraction of X to a centered at C. In the first Bruhat decomposition we showed that

$$g \in PwP \quad \text{for} \quad wC = r(gC)$$

Thus, evidently for $p \in P$

$$r(pg) = r(g)$$

Since P is the stabilizer of C, it is even more immediate that $r(g) = r(gp)$ for $p \in P$. Thus, to determine all $w' \in W$ such that

$$Pw'P \subset PwP \cdot PsP$$

[45] In several circumstances where topological ideas have their usual sense, these Bruhat cells really are *cells* in the standard topological sense, namely, that they are homeomorphic to open balls. We don't need this property here, but will use the terminology.

[46] Again, W is not a subset of G, being the quotient $\mathcal{N}/A$, but the double cosets PwP are well-defined.

we must determine all chambers in $r(wPsC)$.

For any $p \in P$, since sC is adjacent to C, the chamber psC is adjacent to $C = pC$. Then $wpsC$ is adjacent to wC. We know that the retraction $r : X \longrightarrow a$ is the identity on $a \cap b$ for any apartment b containing C. Let b be an apartment containing $wpsC$ and C. Then $wpsC \cap wC$ is a simplex in $a \cap b$, so is fixed pointwise by r. Thus, $r(wpsC)$ has face $wpsC \cap wC$. By thin-ness of a, the only possibilities for $r(wpsC)$ are wC and one other chamber in a. We claim that this other chamber is wsC. Indeed, since $p \in P$ fixes C pointwise,

$$C \cap sC = p(C \cap sC) = pC \cap psC = C \cap psC$$

and then

$$wC \cap wsC = w(C \cap sC) = w(C \cap psC) = wC \cap wpsC$$

proving the claim. (This is also clear from the observation that $w \cdot s$ always lies in $PwP \cdot PsP$.) Thus,

$$PwP \cdot PsP \subset PwsP \cup PwP$$

That is, the only possible Bruhat double cosets appearing are PwP and $PwsP$. Combining this with the first observation, for any $w \in W$ and $s \in S$

$$PwsP \subset PwP \cdot PsP \subset PwsP \cup PwP$$

Suppose that $\ell(ws) > \ell(w)$, and show that

$$PwP \cdot PsP = PwsP$$

The retraction $r' = r_{a,wC}$ sends a minimal gallery γ going from C to $wpsC$ to a gallery going from wsC to C, since r' is a retraction to a, and wsC is the only other chamber in a with face $wsC \cap wC$. Thus, whether or not this retraction r' causes the gallery γ to stutter,

$$d(wC, C) < d(wsC, C) = d(r'(wspC), C) \leq \text{length}\gamma = d(wspC, C)$$

Then, since $r = r_{a,C}$ is an isomorphism $r : b \longrightarrow a$ on an apartment b containing both C and $wpsC$, the image $r(\gamma)$ is a non-stuttering gallery from C to $r(wspC)$. Thus,

$$d(wC, C) < \text{length}\gamma = \text{length}r(\gamma) = d(r(wspC), C)$$

Thus, $r(wspC) \neq wC$, so, again, there is no choice but that $r(wspC) = wsC$. Thus,

$$\ell(ws) > \ell(w) \quad \text{implies} \quad PwP \cdot PsP = PwsP$$

Now suppose that $\ell(ws) < \ell(w)$. We want to show that $w \in PwP \cdot PsP$, that is, that for some $p \in P$

$$r(wpsC) = wC$$

First, with $w' = ws$, the assumption $\ell(ws) < \ell(w)$ gives $\ell(w') < \ell(w's)$, since $w's = (ws)s = w$. By the previous paragraph,

$$Pw'P \cdot PsP = Pw'sP$$

In terms of w, this is

$$PwsP \cdot PsP = PwP$$

Multiply on the right by PsP to obtain

$$PwsP \cdot PsP \cdot PsP = PwP \cdot PsP$$

We claim

$$PsP \cdot PsP = P \cup PsP$$

If we have this, then the previous identity gives

$$PwsP \cdot (P \cup PsP) = PwP \cdot PsP$$

and the left-hand side contains $ws \cdot s = w$, as desired. Thus, we will be done if we can prove that

$$PsP \cdot PsP = P \cup PsP$$

We have already seen that

$$P \subset PsP \cdot PsP \subset P \cup PsP$$

so we only need to show that $s \in PsP \cdot PsP$. In terms of the retraction $r = r_{a,C}$, we need to find $p \in P$ such that $r(spsC) = sC$.

Here we use the *thickness* of the building: there exists a chamber D with face $C \cap sC$, other than C or sC. By the building axioms, there is an apartment b containing both D and C. By strong transitivity, there is $p \in P$ such that $pa = b$. Since P fixes C pointwise, P fixes $C \cap sC$. But in b (by thin-ness) there is only one chamber other than C with face $C \cap sC$, which must be D. Thus, $psC = D$.

Then, since s (or, more correctly, an element in $\mathcal{N}$ which maps to s in the quotient $W = \mathcal{N}/A$) merely interchanges C and sC, the chamber $sD = spsC$ is neither C nor sC. Applying r, $r(sD)$ is not C, so $r(sD) = sC$. That is,

$$r(spsC) = sC$$

which is equivalent to

$$PspsP = PsP$$

In particular,

$$s \in PspsP \subset PsPsP = PsP \cdot PsP$$

As indicated, this fact yields the conclusion

$$PwP \cdot PsP = PwsP \cup PwP \quad (\text{for } \ell(ws) < \ell(s))$$

and the theorem is proven. ///

Remark: Note that the proof needed the fact that either $\ell(ws) > \ell(w)$ or $\ell(ws) < \ell(w)$, but never $\ell(ws) = \ell(w)$. The impossibility of this inequality seems to need the discussion of foldings.

Corollary: Let $w = s_1 \ldots s_n$ be a minimal expression (in word length in W, with respect to generators S). The smallest subgroup H of G containing PwP contains all the s_i.

Proof: From the cell multiplication rules,

$$PwP = Ps_1P \cdot Ps_2P \cdot \ldots \cdot Ps_nP$$

Thus, certainly H is contained in the subgroup of G generated by all the s_i together with P. For the opposite inclusion, we do induction on the length $\ell(w)$ of w.

Since $\ell(ws_n) < \ell(w)$, the cell multiplication rules give

$$PwP \cdot Ps_nP = PwP \cup Pws_nP$$

which implies that wPs_n meets PwP, so Ps_n meets $w^{-1}PwP$, and then

$$s_n \in Pw^{-1}PwP = (PwP)^{-1} \cdot PwP$$

Thus, since H contains inverses and is closed under multiplication, H contains s_n. Since H contains P, we have

$$H \supset PwP \cdot Ps_nP = Pws_nP \cup PwP$$

The element

$$ws_n = s_1 \ldots s_{n-1}s_n \cdot s_n = s_1 \ldots s_{n-1}$$

is shorter than w, so by induction the subgroup generated by Pws_nP contains all of $s_1, \ldots, s_{n-1}$.
///

Corollary: Let T be a subset of the set S of reflections, and $\langle T \rangle$ the subgroup of W generated by T. Then

$$Q = \bigsqcup_{w \in \langle T \rangle} PwP$$

is a *subgroup* of G. Conversely, every subgroup Q of G containing P is of this form with

$$T = \{s \in S : Q \supset PsP\} = S \cap Q$$

Proof: By the cell multiplication rules

$$Ps_1 \ldots s_nP \cdot Ps_{n+1}P \subset Ps_1 \ldots s_ns_{n+1}P \cup Ps_1 \ldots s_nP$$

so the indicated subset Q of G is closed under multiplication. Regarding inverses,

$$(Ps_1 \ldots s_nP)^{-1} = P(s_1 \ldots s_n)^{-1}P = Ps_n \ldots s_1P$$

since each element s_i is of order 2. Thus, Q is a subgroup.

On the other hand, given a subgroup Q containing P, Q is a union of double cosets PgP, and we may as well take g in W. [47] For $PwP \subset Q$, with minimal expression $w = s_1 \ldots s_n$, the previous corollary shows that all the s_i are in Q, and $Ps_iP \subset Q$ since Q contains P and is a group. Thus, letting

$$T = \{s \in S : Q \supset PsP\}$$

we have the desired expression for Q. ///

Remark: This last corollary shows that the only subgroups of G containing the stabilizer P of a chamber C are the stabilizers of faces of C. This is not obvious.

EDIT: to be continued...

[47] Or, more properly, take g among representatives $\mathcal{N}$ in G for the quotient $W = \mathcal{N}/A$. This detail is irrelevant.