

(April 3, 2011)

# Bernstein's analytic continuation of complex powers

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Let  $f$  be a polynomial in  $x_1, \dots, x_n$  with real coefficients. For complex  $s$ , let  $f_+^s$  be the function defined by

$$f_+^s(x) = \begin{cases} f(x)^s & (\text{for } f(x) \geq 0) \\ 0 & (\text{for } f(x) \leq 0) \end{cases}$$

Certainly for  $\operatorname{Re}(s) \geq 0$  the function  $f_+^s$  is *locally integrable*. For  $s$  in this range, define a *distribution*, also denoted  $f_+^s$ , by

$$f_+^s(\phi) = \int_{\mathbb{R}^n} f_+^s(x) \phi(x) dx \quad (\text{for } \phi \in C_c^\infty(\mathbb{R}^n))$$

The object is to analytically continue the distribution  $f_+^s$ , as a meromorphic (distribution-valued) function of  $s$ . One should also ask about analytic continuation as a *tempered* distribution. Several provocative examples appear in [Gelfand-Shilov 1964]. In a lecture at the 1963 Amsterdam Congress, I.M. Gelfand refined this question to require further that one show that the poles lie in a finite number of arithmetic progressions.

[Bernstein 1968] gave a *first* proof, under a regularity hypothesis on the zero-set of the polynomial  $f$ . We essentially reproduce that argument here, with supporting material from complex function theory and from the theory of distributions.

[Bernstein-Gelfand 1969] proved the meromorphic continuation without any hypotheses, but assuming desingularization [Hironaka 1966]. [Atiyah 1970] gave a similar argument. Although this resolved the original question, invocation of desingularization was unsatisfactory.

[Bernstein 1971] created the theory of  $D$ -modules, and [Bernstein 1972] used it to prove the existence of the *Bernstein polynomial*  $F_f(s)$  for any  $f$ , and so on. Invocation of desingularization was avoided, and much more was done, besides.

Thanks to Bill Casselman for suggesting revisions, especially bibliographic, to an old version of this note.

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## 1. Analytic continuation of distributions

We recall the nature of the topologies on test functions and on distributions. Let  $C_c^\infty(U)$  be the collection of compactly-supported smooth functions with support inside a set  $U \subset \mathbb{R}^n$ . As usual, for  $U$  *compact*, we have a countable family of seminorms on  $C_c^\infty(U)$ :

$$\mu_\nu(f) = \sup_x |D^\nu f|$$

where for  $\nu = (\nu_1, \dots, \nu_n)$ , as usual,

$$D^\nu = \left( \frac{\partial}{\partial x_1} \right)^{\nu_1} \cdots \left( \frac{\partial}{\partial x_n} \right)^{\nu_n}$$

It is elementary that (for  $U$  compact)  $C_c^\infty(U)$  is a Fréchet space. To treat  $U$  not necessarily compact (e.g.,  $\mathbb{R}^n$  itself) let

$$U_1 \subset U_2 \subset \dots$$

be compact subsets of  $U$  whose union is  $U$ . Then  $C_c^\infty(U)$  is the union of the spaces  $C_c^\infty(U_i)$ , with the locally convex *colimit (direct limit) topology*. There is a *countable* cofinal sub-colimit, and each Fréchet space is closed in the next, so the colimit is called an *LF-space* (strict-(co-)limit-of-Fréchet). LF-spaces are not complete metrizable, but are *quasi-complete*, meaning that *bounded Cauchy nets converge*.

The spaces  $\mathcal{D}^*(U)$  and  $\mathcal{D}^*(\mathbb{R}^n)$  of *distributions* on  $U$  and on  $\mathbb{R}^n$ , respectively, are the continuous duals of  $\mathcal{D}(U) = C_c^\infty(U)$  and  $\mathcal{D}(\mathbb{R}^n) = C_c^\infty(\mathbb{R}^n)$ . The topology on the continuous dual space  $V^*$  of a topological vector space  $V$  is the *weak-\* topology*: a sub-basis near 0 in  $V^*$  consists of sets

$$U_{v,\epsilon} = \{\lambda \in V^* : |\lambda(v)| < \epsilon\}$$

A  $V^*$ -valued function  $f$  on an open subset  $\Omega$  of  $\mathbb{C}$  is *holomorphic* on  $\Omega$  when, for every  $v \in V$ , the  $\mathbb{C}$ -valued function

$$z \rightarrow f(z)(v)$$

on  $\Omega$  is holomorphic in the usual sense. More precisely, this is *weak-\* holomorphy*, referring to the topology.

Let  $z_o \in \Omega$  for an open subset  $\Omega$  of  $\mathbb{C}$  and  $f$  a holomorphic  $V^*$ -valued function on  $\Omega - z_o$ . Say that  $f$  is *weakly meromorphic* at  $z_o$  when, for every  $v \in V$ , the  $\mathbb{C}$ -valued function  $z \rightarrow f(z)(v)$  has a *pole* (as opposed to essential singularity) at  $z_o$ . Say that  $f$  is *strongly meromorphic* at  $z_o$  if the orders of these poles are *bounded independently of  $v$* . That is,  $f$  is strongly meromorphic at  $z_o$  if there is an integer  $n$  and an open set  $\Omega$  containing  $z_o$  so that, for all  $v \in V$  the  $\mathbb{C}$ -valued function

$$z \rightarrow (z - z_o)^n f(z)(v)$$

is holomorphic on  $\Omega$ . If  $n$  is the least integer  $f$  so that  $(z - z_o)^n f$  is holomorphic at  $z_o$ , then  $f$  is *of order  $-n$  at  $z_o$* , etc.

To say that  $f$  is *strongly meromorphic* on an open set  $\Omega$  is to require that there be a set  $S$  of points of  $\Omega$  *with no accumulation point in  $\Omega$*  so that  $f$  is holomorphic on  $\Omega - S$ , and so that  $f$  is strongly meromorphic at each point of  $S$ .

For brevity, but risking some confusion, we will often say *meromorphic* rather than *strongly meromorphic*.

## 2. Statements of theorems on analytic continuation

Let  $\mathcal{O}$  be the polynomial ring  $\mathbb{R}[x_1, \dots, x_n]$ . For  $z \in \mathbb{R}^n$ , let  $\mathcal{O}_z$  be the *local ring* at  $z$ , i.e., the ring of ratios  $P/Q$  of polynomials where the denominator does not vanish at  $z$ . Let  $\mathfrak{m}_z$  be the maximal ideal of  $\mathcal{O}_z$  consisting of elements of  $\mathcal{O}_z$  whose numerator vanishes at  $z$ . Let  $I_z$  (depending upon  $f$ ) be the ideal in  $\mathcal{O}_z$  generated by

$$\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}$$

A point  $z \in \mathbb{R}^n$  is said to be *simple* with respect to the polynomial  $f$  when

- $f(z) = 0$
- $I_z \supset \mathfrak{m}_z^N$  for some  $N$
- There are  $\alpha_i \in \mathfrak{m}_z$  so that  $f = \sum_i \alpha_i \frac{\partial f}{\partial x_i}$

[2.1.1] **Remark:** The second condition is equivalent to the  $\mathcal{O}_z/I_z$  being finite-dimensional. The simplest situation in which the second condition holds is when  $I_z = \mathcal{O}_z$ , i.e., some partial derivative of  $f$  is non-zero

at  $z$ . The third condition does not follow from the first two: Bernstein notes that the first two conditions hold but the third does not for

$$f(x, y) = x^5 + y^5 + x^2y^2$$

[2.1.2] **Theorem:** (local version) For  $z$  a simple point with respect to  $f$ , there is a neighborhood  $U$  of  $x$  so that the distribution

$$f_{+,U}^s(\phi) = \int f_+^s(x) \phi(x) dx$$

on test functions  $\phi \in C_c^\infty(U)$  on  $U$  has an analytic continuation to a meromorphic  $C_c^\infty(U)^*$ -valued function.

[2.1.3] **Theorem:** (global version) If *all* real zeros of  $f(x)$  are *simple* (with respect to  $f$ ), then  $f_+^s$  is a meromorphic (distribution-valued) function of  $s \in \mathbb{C}$ .

### 3. Bernstein's proof

Let  $R_z$  be the ring of linear differential operators with coefficients in  $\mathcal{O}_z$ . Note that  $R_z$  is both a left and a right  $\mathcal{O}$ -module: for  $D \in R_z$ , for  $f, g \in \mathcal{O}$  and  $\phi$  a smooth function near  $z$ , the definition is

$$(fDg)(\phi) = fD(g\phi)$$

[3.1.1] **Lemma:** There is a differential operator  $D \in R_z$  and a non-zero *Bernstein polynomial*  $H$  in a single variable so that

$$D(f^{n+1}) = H(n)f^n$$

for any natural number  $n$ . (*Proof below.*)

*Proof of Local Theorem from Lemma:* Let  $U$  be a small-enough neighborhood of  $z$  so that on it all coefficients of  $D$  are holomorphic on  $U$ . For sufficiently large  $n$  the function  $f_+^{n+1}$  is continuously differentiable, so

$$Df_+^{n+1} = H(n)f_+^n$$

For each fixed  $\phi \in C_c^\infty(U)$  consider the function

$$g(s) = (Df_+^{s+1} - H(s)f_+^s)(\phi)$$

The hypotheses of the proposition below are satisfied, so the equality for all large-enough natural numbers implies equality everywhere:

$$Df_+^{s+1} = H(s)f_+^s$$

This gives

$$f_+^s = \frac{Df_+^{s+1}}{H(s)}$$

Now claim that for any  $0 \leq n \in \mathbb{Z}$  the distribution  $f_+^s$  on  $C_c^\infty(U)$  is meromorphic for  $\operatorname{Re}(s) > -n$ . For  $n = 0$  this is clear. The formula just derived gives the induction step. Further, this argument shows that the poles of  $f_+^s$  restricted to  $C_c^\infty(U)$  are concentrated on the finite collection of arithmetic progressions

$$\lambda_i, \lambda_i - 1, \lambda_i - 2, \dots$$

where the  $\lambda_i$  are the roots of  $H(s)$ . In particular, the *order* of the pole of  $f_+^s$  at a point  $s_o$  is equal to the number of roots  $\lambda_i$  so that  $s_o$  lies among

$$\lambda_i, \lambda_i - 1, \lambda_i - 2, \dots$$

In particular, the distribution  $f_+^s$  really is (strongly) meromorphic. This proves the Theorem, granting the Lemma and granting the Proposition. ///

[3.1.2] **Proposition:** (attributed by Bernstein to Carlson) For  $g$  analytic for  $\operatorname{Re} s > 0$  and  $|g(s)| < be^{c\operatorname{Re} s}$  and  $g(n) = 0$  for all sufficiently large natural numbers  $n$ , we have  $g \equiv 0$ .

*Proof:* (of Global Theorem) Invoking the Local Theorem *and its proof above*, for each  $z \in \mathbb{R}^n$  choose a neighborhood  $U_z$  of  $z$  in which  $f_+^s$  is meromorphic, so that  $U_z$  is *Zariski-open*, i.e., is the complement of a finite union of zero sets of polynomials. Indeed, writing

$$f(x) = \sum \alpha_i \frac{\partial f}{\partial x_i}$$

with  $\alpha_i \in \mathcal{O}_z$ , as in the proof of the Local Theorem, let  $\alpha_i = g_i/h_i$  with polynomials  $g_i$  and  $h_i$ , and take  $U_z$  to be the complement of the union of the zero-sets of the denominators  $h_i$ .

By Hilbert's Basis Theorem,  $\mathbb{R}^n$  is covered by finitely-many  $U_{z_1}, \dots, U_{z_n}$ . Make a partition of unity subordinate to this finite cover, i.e., take  $\psi_1, \dots, \psi_n$  so that  $\psi_i \geq 0$ ,  $\sum \psi_i \equiv 1$ , and  $\operatorname{spt} \psi_i \subset U_{z_i}$ . Then

$$f_+^s = \sum_i \psi_i f_+^s$$

By choice of the  $U_{z_i}$ , the right-hand side is a finite sum of meromorphic (distribution-valued) functions.

## 4. Proof of the Lemma: the Bernstein polynomial

Now we prove existence of the differential operator  $D$  and the *Bernstein polynomial*  $H$ . This is the most serious part of the argument. (The complex function theory proposition is not trivial, but is standard).

*Proof:* (of Lemma) Let

$$P = \sum \alpha_i \frac{\partial}{\partial x_i} \in R_z$$

where the  $\alpha_i \in \mathfrak{m}_z$  are so that

$$f = \sum \alpha_i \frac{\partial f}{\partial x_i} \in R_z$$

Put

$$S_i = \frac{\partial f}{\partial x_i} P - f \frac{\partial}{\partial x_i} = \frac{\partial f}{\partial x_i} (P + 1) - \frac{\partial}{\partial x_i} Q$$

Then

$$P(f) = \sum_i \alpha_i \frac{\partial f}{\partial x_i} = f \quad S_i f = \frac{\partial f}{\partial x_i} f - f \frac{\partial f}{\partial x_i} = 0$$

By Leibniz' formula,

$$P(f^n) = n f^n \quad S_i(f^n) = 0$$

[4.1.1] **Sublemma:** There is a non-zero polynomial  $M$  in one variable so that

$$M(P) = \sum_i J_i \frac{\partial f}{\partial x_i} \quad (\text{for some } J_i \in R_z)$$

*Proof:* (of Sublemma) As usual, write  $|\nu| = \nu_1 + \dots + \nu_n$ . For a natural number  $m$ , move all the coefficients to the right of the differential operators, that is, write

$$P^m = \sum_{|\nu| \leq m} D^\nu \gamma_{m,\nu} \quad (\text{where } \gamma_{m,\nu} \in \mathcal{O}_z)$$

That this is possible is easy to see:

$$x_i \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j} x_i = \begin{cases} 0 & (\text{for } i \neq j) \\ -1 & (\text{for } i = j) \end{cases}$$

The coefficients  $\gamma_{m,\nu}$  are *polynomials* in the  $\alpha_i$ , so  $\gamma_{m,\nu} \in \mathfrak{m}_z^{|\nu|}$ . Taking  $M(P)$  of the form

$$M(P) = \sum_{m \leq q} b_m P^m = \sum_{m,\nu} D^\nu b_m \gamma_{m,\nu} \quad (\text{with } b_m \in \mathbb{R})$$

the condition of the sublemma will be met if

$$\sum_m b_m \gamma_{m,\nu} \in I_z \quad (\text{for all indices } \nu)$$

If  $|\nu| \geq N$ , where  $I \supset \mathfrak{m}^N$ , then this condition is automatically fulfilled. Thus, there are finitely-many conditions

$$\sum_m b_m \bar{\gamma}_{m,\nu} = 0$$

where  $\bar{\gamma}_{m,\nu}$  is the image of  $\gamma_{m,\nu}$  in  $\mathcal{O}_z/\mathfrak{m}_z^N$ . Since the latter quotient is, by hypothesis, a finite-dimensional vector space, the collection of such conditions gives a finite collection of homogeneous equations in the coefficients  $b_m$ . More specifically, the number of such conditions is

$$\dim \mathcal{O}_z/\mathfrak{m}_z^N \times \text{card}\{\nu : |\nu| < N\}$$

Taking  $q$  large enough assures the existence of a non-trivial solution  $\{b_m\}$ , proving the sublemma. ///

Returning to the proof of the lemma: as an equation in  $R_z$

$$M(P)(P+1) = \sum J_i \frac{\partial}{\partial x_i} (P+1) = \sum J_i S_i + \sum J_i \frac{\partial}{\partial x_i} f$$

Put

$$D = \sum J_i \frac{\partial}{\partial x_i} \quad H(P) = M(P)(P+1)$$

Then, as desired,

$$D(f^{n+1}) = \left( \sum J_i \frac{\partial}{\partial x_i} f \right) (f^n) = H(P)(f^n) = H(n)f^n$$

This proves the Lemma, constructing the differential operator  $D$ . ///

## 5. *Proof of the Proposition: estimates on zeros*

The necessary result is standard, although not so elementary as to be an immediate corollary of Cauchy's Theorem:

**[5.1.1] Proposition:** For  $g$  an analytic function for  $\text{Re } s > 0$  and  $|g(s)| < be^{c\text{Re } s}$ , if  $g(n) = 0$  for all sufficiently large natural numbers  $n$ , then  $g \equiv 0$ .

*Proof:* Consider

$$G(z) = e^{-c} g\left(\frac{z+1}{z-1}\right)$$

Then  $g$  is turned into a bounded function  $G$  on the disc, with zeros at points  $(n-1)/(n+1)$  for sufficiently large natural numbers  $n$ .

We claim that, for a bounded function  $G$  on the unit disc with zeros  $\rho_i$ , either  $G \equiv 0$  or

$$\sum_i (1 - |\rho_i|) < +\infty$$

If we prove this, then in the situation at hand the natural numbers are mapped to

$$\rho_n = (n-1)/(n+1) = 1 - \frac{1}{n+1}$$

so here

$$\sum_n (1 - |\rho_n|) = \sum_n \frac{1}{n+1} = +\infty$$

Thus, we would conclude  $G \equiv 0$  as desired.

Recall *Jensen's formula*: for any holomorphic function  $G$  on the unit disc with  $G(0) \neq 0$  and with zeros  $\rho_1, \dots$ , for  $0 < r < 1$  we have

$$|G(0)| \prod_{|\rho_i| \leq r} \frac{r}{|\rho_i|} = \exp \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |G(re^{i\theta})| d\theta \right\}$$

Granting this, the assumed boundedness of  $G$  on the disc gives an absolute constant  $C$  so that for all  $N$

$$|G(0)| \prod_{|\rho_i| \leq r} \frac{r}{|\rho_i|} \leq C$$

(We can harmlessly divide by a suitable power of  $z$  to guarantee that  $G(0) \neq 0$ .) Then, letting  $r \rightarrow 1$ ,

$$\prod |\rho_i| \leq |G(0)|^{-1} C^{-1}$$

For an infinite product of positive real numbers  $|\rho_i|$  less than 1 to have a value  $> 0$ , it is elementary that we must have

$$\sum_i (1 - |\rho_i|) < +\infty$$

as claimed. This proves the proposition. ///

While we're here, let's recall the proof of Jensen's formula (e.g., as in [Rudin 1966], for example. Fix  $0 < r < 1$  and let

$$H(z) = G(z) \prod \frac{r^2 - \bar{\rho}z}{r(\rho - z)} \prod \frac{\rho}{\rho - z}$$

where the first product is over roots  $\rho$  with  $|\rho| < r$  and the second is over roots with  $|\rho| = r$ . Then  $H$  is holomorphic and non-zero in an open disk of radius  $r + \epsilon$  for some  $\epsilon > 0$ . Thus,  $\log |H|$  is *harmonic* in this disk, and has the mean value property

$$\log |H(0)| = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |H(re^{i\theta})| d\theta$$

On one hand,

$$|H(0)| = |G(0)| \prod \frac{r}{|\rho|}$$

On the other hand, if  $|z| = r$  the factors

$$\frac{r^2 - \bar{\rho}z}{r(\rho - z)}$$

have absolute value 1. Thus,

$$\log |H(re^{i\theta})| = \log |G(re^{i\theta})| - \sum_{|\rho|=r} \log |1 - e^{i(\theta - \arg \rho)}| \quad (\text{where } e^{i \arg \rho} = \rho)$$

As noted in Rudin (see below),

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log |1 - e^{i\theta}| d\theta = 0$$

Therefore, the integral in the assertion of the mean value property is unchanged upon replacing  $H$  by  $G$ . Putting this all together gives Jensen's formula. ///

We execute the integral computation, following Rudin. Since the disk is simply-connected, there is a function  $\lambda(z)$  on the open unit disc so that

$$\exp(\lambda(z)) = 1 - z$$

We completely specify this  $\lambda$  by requiring  $\lambda(0) = 0$ . We have  $\operatorname{Re} \lambda(z) = \log |1 - z|$  and  $|\operatorname{Im} \lambda(z)| < \frac{\pi}{2}$ . Let  $\delta > 0$  be small. Let  $\Gamma = \Gamma_\delta$  be the (counterclockwise) path around the unit circle from  $e^{i\delta}$  to  $e^{(2\pi-\delta)i}$  and let  $\gamma = \gamma_\delta$  be the (clockwise) path around a small circle centered at 1 from  $e^{(2\pi-\delta)i}$  to  $e^{i\delta}$ . Then

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \log |1 - e^{i\theta}| d\theta &= \lim_{\delta \rightarrow 0} \frac{1}{2\pi} \int_\delta^{2\pi-\delta} \log |1 - e^{i\theta}| d\theta \\ &= \operatorname{Re} \left[ \frac{1}{2\pi i} \int_\Gamma \lambda(z) \frac{dz}{z} \right] = \operatorname{Re} \left[ \frac{1}{2\pi i} \int_\gamma \lambda(z) \frac{dz}{z} \right] \end{aligned}$$

by Cauchy's theorem. Elementary estimates show that the latter integral has a bound of the form  $C\delta \log(1/\delta)$ , which goes to 0 as  $\delta \rightarrow 0$ .

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