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# Proof of a simple case of the Siegel-Weil formula

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First, I must confess that I never felt that I understood Siegel's arguments for his "mass formula" relating positive definite quadratic forms and Eisenstein series. Of course, at that time things could not be readily separated into *local* and *global* arguments.

On the other hand, while our current technique is arguably much more sophisticated, the questions addressed are commensurately more complicated, so that the potential simplification of a proof of the a basic Siegel-Weil formula may get lost in more difficult issues. For example, the work of Kudla-Rallis on *regularization* addresses much more delicate questions than the simple question of equality of holomorphic Eisenstein series and linear combinations of theta series in the region of convergence.

Here I will use by-now-standard methods to prove a modern form of a Siegel-Weil formula for  $SL(2)$  over totally real number fields  $k$ .

**Theorem:** (*Vague version of Siegel-Weil*) Certain holomorphic Eisenstein series of weights  $4\ell$  (with  $\ell = 1, 2, 3, \dots$ ) are certain linear combinations of holomorphic theta series attached to certain totally positive definite quadratic forms of dimension  $8\ell$ .

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## 1. Weil/oscillator representations

We must define a Weil representation for  $O(Q) \times SL(2)$ , where for simplicity we only use certain  $8\ell$ -dimensional quadratic forms.

Fix a non-degenerate quadratic form  $Q$  over a number field  $k$ , with dimension  $8\ell$  divisible by 8, and so that  $Q = Q_1 \oplus Q_1$  where  $Q_1$  is a non-degenerate quadratic form with dimension divisible by 4, and **discriminant** a square. (The discriminant of a  $4\ell$ -dimensional quadratic form is just the determinant of the matrix of the form with respect to a choice of basis.) This hypothesis on the quadratic form simplifies the following discussion.

Fix a non-trivial (continuous) character  $\psi$  on the adèles  $\mathbf{A}$  of  $k$ , trivial on  $k$  itself viewed as sitting inside  $\mathbf{A}$ . Let  $G = SL(2, k)$  and  $H = O(Q)$ . Let  $v$  be a place of  $k$ . The local **Weil** or **oscillator representation**  $\rho$  of  $G_v \times H_v$  is defined on the vector space  $\mathbf{S}(8\ell \times 1)$  of Schwartz functions on the space of  $8\ell$ -by-1 matrices with entries in  $k_v$ . For  $h \in H$ , the action is

$$\rho(h)f(t) = f(h^{-1}t)$$

for Schwartz function  $f$  and  $8\ell$ -by-1 matrix  $t$ . (The inverse is just to assure associativity.)

The definition of  $\rho_v$  on  $G_v$  must be done in pieces. For

$$n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in N_v$$

the unipotent radical of the standard parabolic subgroup

$$P = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$$

the action is

$$\rho(n)f(t) = \psi\left(\frac{x}{2} Q(t)\right) f(t)$$

For

$$m = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in M_v$$

where  $M_v$  is the standard Levi component of  $P_v$  (consisting of all such elements  $m$ )

$$\rho(m)f(t) = |a|_v^{4\ell} f(ta)$$

(Note that the exponent of  $|a|$  is half the dimension.) Finally, for the Weyl element

$$w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

define

$$\rho(w)f(t) = \hat{f}(t)$$

where  $\hat{f}$  is the Fourier transform

$$\hat{f}(t) = \int_{k^{8\ell \times 1}} \psi(-\langle \xi, t \rangle) f(\xi) d\xi$$

where the integral is over the  $8\ell$ -by-1 matrices with entries in  $k$ , and the measure is an additive Haar measure (normalized as usual, described a bit more below). Then  $\rho$  is defined on arbitrary group elements by using the (spherical) Bruhat decomposition of  $G_v$ .

**Remark:** This is indeed a peculiar way to define a representation. It is not at all clear from the definition that  $\rho$  is a group homomorphism on  $G_v$ . Indeed, a little thought is required to verify that saying ‘extend by Bruhat decomposition’ gives a well-defined thing in the first place. But it does. The verification that  $\rho$  is a group homomorphism is not entirely trivial, but can be done directly.

**Remark:** Without the simplifying hypotheses on the quadratic form, the definition of the Weil representation on the Levi component would have to be adjusted by a finite-order character, and the definition on the Weyl element would have to be adjusted by an eighth root of unity.

For each place  $v$  of  $k$  we define a local Weil representation  $\rho_v$  as sketched just above (where we suppressed the subscript). Then the **global Weil representation** is on the space of Schwartz functions on the space of  $8\ell$ -by-1 matrices with entries in the adèles  $\mathbf{A}$  of  $k$ . This Schwartz space is defined as a **restricted tensor product** of the local Schwartz spaces, so almost everywhere we need to choose a distinguished local Schwartz function  $f_{v\mathfrak{o}}$ . Further, so that the global/adelic Fourier transform is well-defined on the global/adelic Schwartz space, almost everywhere these distinguished vectors must be their own Fourier transform. A fairly canonical way to do this is as follows: at non-archimedean places  $v$  of  $k$  which are absolutely unramified, take  $f_{v\mathfrak{o}}$  to be the characteristic function of  $8\ell$ -by-1 matrices with entries in the local integers  $\mathfrak{o}_v$ .

Further, we must be sure that the character  $\psi$  and the measures  $d\xi$  are normalized properly so that these  $f_{v\mathfrak{o}}$  are their own Fourier transforms almost everywhere. This can be carried out by mimicking Tate’s thesis.

**Remark:** We must also check, for continuity on the restricted tensor product, that almost everywhere locally at places  $v$  we have

$$\rho(h_{\mathfrak{o}})f_{v\mathfrak{o}} = f_{v\mathfrak{o}}$$

where  $h_{\mathfrak{o}}$  lies in a specified (maximal) compact subgroup  $H_{v\mathfrak{o}}$  of  $H_v$ . We may as well take  $H_{v\mathfrak{o}}$  to be the elements of  $H_v$  with entries in the local integers. Due to our simplifying hypotheses on the quadratic form, at almost all primes  $v$  the local Witt index of  $Q$  is maximal, meaning that there is a  $4\ell$ -dimensional maximal isotropic subspace, and this choice of  $H_{v\mathfrak{o}}$  works out.

**Remark:** For subsequent applications, we would take the number field  $k$  to be totally real, and the quadratic form  $Q$  to be positive definite at every archimedean place of  $k$ . But for the basic set-up of the Weil representation this is not at all necessary.

## 2. Theta correspondences

Let  $\rho$  be the Weil representation as above on the product  $G_{\mathbf{A}} \times H_{\mathbf{A}}$  of the adelic  $SL(2)$  and orthogonal group  $H = O(Q)$ . Let  $\varphi$  be an adelic Schwartz function, and define the **theta kernel**

$$\Theta_{\varphi}(g, h) = \sum_x \rho(g, h)\varphi(x)$$

where  $x$  ranges over  $8\ell$ -by-1 matrices with entries in  $k$ . It is a little exercise to see that in general we can at least say that this theta kernel is continuous in both  $g$  and  $h$  (in the respective adèle groups), and is left  $SL(2)_k \times O(Q)_k$ -invariant.

We can use the theta kernel as we might *any* kernel function to define two maps, called **theta liftings** or **theta correspondences**. For  $f$  a compactly-supported continuous function on  $O(Q)_k \backslash O(Q)_{\mathbf{A}}$ , we can define the **theta lift**

$$\Theta_{\varphi}(f)(g) = \int_{O(Q)_k \backslash O(Q)_{\mathbf{A}}} f(h) \Theta_{\varphi}(g, h) dh$$

Similarly, for  $f$  a compactly-supported continuous function on  $SL(2)_k \backslash SL(2)_{\mathbf{A}}$ , we can define the **theta lift in the other direction** by

$$\Theta_{\varphi}(f)(h) = \int_{SL(2)_k \backslash SL(2)_{\mathbf{A}}} f(g) \Theta_{\varphi}(g, h) dg$$

In both cases the image is a continuous function on the target adèle group, and is left-invariant under  $SL(2)_k$ , respectively  $O(Q)_k$ .

*That is, at least in this crude sense, the theta liftings (which depend on the test function  $\varphi$ ) maps automorphic forms on  $O(Q)$  to automorphic forms on  $SL(2)$ , and vice-versa. We'll be concerned only with the first case, which roughly maps automorphic forms on the orthogonal group to automorphic forms on  $SL(2)$ .*

**Remark:** The left  $O(Q)_k$ -invariance is clear, but the  $SL(2)_k$ -invariance is subtler, requiring invocation of Poisson summation. Further, without the simplifying hypotheses on the quadratic form, some further details about the Weil representation would be necessary.

**Remark:** The general restriction to compactly-supported continuous functions is a non-trivial and too-constrictive condition for general application, since automorphic forms have compact support only on compact arithmetic quotients. However, in the case we care about for the Siegel-Weil formula this restrictive hypothesis will be met.

In particular, for orthogonal groups, by **reduction theory**, the arithmetic quotient  $O(Q)_k \backslash O(Q)_{\mathbf{A}}$  is compact if and only if the quadratic form is  **$k$ -anisotropic**. Thus, for globally anisotropic quadratic forms the associated arithmetic quotient of the orthogonal group is indeed compact, and the theta lifting assuredly makes sense as an operator sending automorphic forms on the orthogonal group to automorphic forms on  $SL(2)$ .

Specifically, for the example of the Siegel-Weil formula we address here, assume that the number field  $k$  is **totally real**, and that the quadratic form is **positive definite** at all archimedean places. This assures that the quadratic form is anisotropic at at least one place, so is surely globally isotropic.

**Remark:** Recall that the general form of the Hasse-Minkowski theorem asserts that a quadratic form is globally anisotropic if and only if it is locally anisotropic *everywhere*. Of course, one direction of this implication is trivial.

In particular, we are interested in precisely identifying the image  $\Theta_{\varphi}(1)$  of the identically-one function '1' on the orthogonal group. As noted just above, this function is compactly-supported when the quadratic form is anisotropic.

### 3. Siegel's Eisenstein series

Let  $P$  be the standard parabolic subgroup of  $SL(2)$  consisting of upper-triangular matrices, as usual. Define characters  $\chi_s$  for  $s \in \mathbf{C}$  on  $P_{\mathbf{A}}$  by

$$\chi_s \left( \begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix} \right) = |a|^{2s}$$

A continuous  $\mathbf{C}$ -valued right- $K$ -finite function  $\varepsilon(g)$  on the adèle group  $SL(2, \mathbf{A})$  with the left equivariance property

$$\varepsilon(pg) = \chi_s(p) \varepsilon(g)$$

for  $p \in P(\mathbf{A})$  is in the (adelic) **principal series**  $I_s$ . Consider the **Eisenstein series** attached to the **kernel**  $\varepsilon$  by

$$E(g) = \sum_{\gamma \in P_k \backslash SL(2)_k} \varepsilon(\gamma g)$$

This series is nicely convergent for  $\operatorname{Re}(s) > 1$ , so defines an automorphic form.

**Remark:** This convergence is not so hard to verify here in this special case of  $SL(2)$ , but is not elementary in general.

By the definition of continuity and  $K$ -finiteness on the adèle group the kernel  $\varepsilon$  is a finite sum of products

$$\varepsilon(\{g_v\}) = \bigotimes_v \varepsilon_v(g_v)$$

For a totally real number field  $k$ , when  $2 < s = \ell \in \mathbf{Z}$ , and when the infinite-prime vectors  $\varepsilon_v$  are (constant multiples of) the special form

$$\varepsilon_v \left( \begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \right) = |a|^\ell e^{i\ell\theta}$$

the Eisenstein series is **holomorphic** of weight  $\ell$  in two (equivalent) senses: First, the classical automorphic forms attached to this Eisenstein series are all literally holomorphic of weight  $\ell$ . Second, the representation of  $SL(2, k_v)$  generated by such  $\varepsilon_v$  is a **holomorphic discrete series** representation of weight  $\ell$ ; it is irreducible.

**Remark:** By definition of  $K$ -finiteness, for almost all finite primes  $v$ ,  $\varepsilon_v$  is right  $K_v$ -invariant. For such  $v$ ,  $\varepsilon_v$  is the standard **spherical vector** in the **principal series representations** of  $SL(2, k_v)$  consisting of all left  $P_v, \chi_s$ -equivariant (right  $K_v$ -finite) functions on  $SL(2, k_v)$ .

**Remark:** For  $2 < s = k \in \mathbf{Z}$  the finite-prime principal series are *irreducible*. (And, as noted above, the holomorphic discrete series subrepresentation of these infinite-prime principal series are also irreducible). Neither of these assertions is trivial to verify, but both assertions are standard. (*Insert some discussion of this later?*)

**Remark:** Also, for  $1 < \operatorname{Re}(s)$  the finite-prime principal series representations are definitely *not unitarizable*. This is also not trivial to prove, but appears to be 'standard'. (*Insert some discussion of this later?*)

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### 4. Eisenstein series from the Weil representation

One way to make an Eisenstein series from the Weil representation is as follows. This still does not directly make any connection with theta series nor the theta correspondence, but sets things up for that comparison shortly.

Let  $\varphi$  be in the Schwartz space  $\mathbf{S}(8\ell \times 1)$  on  $8\ell$ -by-1 matrices over the adèles  $\mathbf{A}$  of the number field  $k$ . Keep the earlier hypotheses on the quadratic form  $Q$  (and hence on the Weil representation). Then

$$\varepsilon(g) = (\rho(g)\varphi)(0)$$

is readily seen to be in the adelic principal series  $I_s$  with  $s = 4\ell$ . Define an Eisenstein series by

$$E_\varphi(g) = \sum_{\gamma \in P_k \backslash SL(2, k)} (\rho(\gamma g)\varphi)(0)$$

**Remark:** The Schwartz function  $\varphi$  is a finite sum of tensor products of local Schwartz functions  $\varphi_v$ , almost all of which are the characteristic function of the set of  $8\ell$ -by-1 matrices with entries in the local integers  $\mathfrak{o}_v$ . From this, a little computation with the Weil representation shows that the corresponding  $\varepsilon_v$  really is right  $K_v$ -invariant.

## 5. Holomorphic discrete series

At least for the most classical applications, we want all our automorphic forms to be ‘holomorphic’. That is, we want all our automorphic forms to generate *holomorphic discrete series* locally at all archimedean primes.

So we should assume that the global field  $k$  is *totally real* (that is, has only real archimedean completions), since  $SL(2, \mathbf{R})$  has holomorphic discrete series (discussed momentarily) while  $SL(2, \mathbf{C})$  has no representations which deserve to be called that. But for present purposes it is not necessary to know why  $SL(2, \mathbf{C})$  fails to have holomorphic discrete series, only to understand that  $SL(2, \mathbf{R})$  does have such.

For less obvious reasons, we also will assume that the quadratic form  $Q$  is *totally positive definite* (that is, is positive definite locally at all the archimedean places of  $k$ ). Otherwise, it turns out that we won’t have created holomorphic automorphic forms. Again, it is not easy to see why this positive-definiteness is *necessary*, but it is relatively easy to verify that taking  $Q$  totally positive definite is *sufficient*: the discussion below succeeds.

For the present discussion, let

$$c = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} / \sqrt{2}$$

be the usual **Cayley element**. The significance of this element can be understood in a variety of ways, but we choose a relatively elementary viewpoint here. The linear fractional transformation

$$z \longrightarrow c(z) = \frac{z + i}{iz + 1}$$

maps the complex unit disk to the upper half-plane. And for a real archimedean prime  $v$

$$K_v = SO(2) = c \cdot \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \right\} \cdot c^{-1} = \left\{ \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \right\}$$

Thus, on the complexified Lie algebra  $\mathfrak{g}_{\mathbf{C}}$  of  $SL(2, \mathbf{R})$ , we can check that  $K_v$  has three eigenspaces

$$\mathfrak{k}_{\mathbf{C}} = \text{trivial eigenspace} = c \cdot \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\} \cdot c^{-1}$$

$$\mathfrak{p}^+ = c \cdot \left\{ \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \right\} \cdot c^{-1}$$

$$\mathbf{p}^- = c \cdot \left\{ \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix} \right\} \cdot c^{-1}$$

We choose

$$R = c \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot c^{-1} = \text{raising operator}$$

$$L = c \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \cdot c^{-1} = \text{lowering operator}$$

(The justification for the names will be clear shortly.) We let any element  $\alpha$  in the Lie algebra  $\mathfrak{g}$  of the group  $SL(2, \mathbf{R})$  act by the standard infinitesimal action on any (*smooth*) vector  $x$  in a representation  $\pi$ :

$$d\pi(\alpha)x = \left. \frac{d}{dt} \right|_{t=0} \pi(e^{t\alpha})x$$

The ‘smoothness’ hypothesis on  $x$  is as usual that the function

$$g \longrightarrow \pi(g)x$$

is an infinitely-differentiable function on the real Lie group  $SL(2, \mathbf{R})$ . And we let  $R$  and  $L$  or any element in the *complexification* of the Lie algebra act on a (*smooth*) vector  $x$  in a by the complexification: suppose that the element is  $\alpha + i\beta$  with  $\alpha$  and  $\beta$  in the (real) Lie algebra, and put

$$d\pi(\alpha + i\beta)x = d\pi(\alpha)x + i d\pi(\beta)x$$

And of course we may often suppress the  $d\pi$  notation.

Let  $x$  be a vector in a representation  $\pi$  of  $SL(2, \mathbf{R})$  with the **weight- $\ell$   $K_v$ -equivariance property**

$$\pi \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} x = e^{i\ell} \cdot x$$

**Proposition:** With a smooth weight- $\ell$  vector  $x$ ,  $Rx$  has weight  $\ell + 2$  and  $Lx$  has weight  $\ell - 2$ .

*Proof:* Both assertions are immediate from the hypothesis and from the fact that

$$\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} R \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}^{-1} = e^{2i\theta} \cdot R$$

$$\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} L \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}^{-1} = e^{-2i\theta} \cdot L$$

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**Remark:** Thus,  $R$  raises weights and  $L$  lowers weights.

**Definition:** A (smooth) vector  $x$  in a representation is **holomorphic** if it is annihilated by  $L$ , and is an  $SO(2) = K_v$  eigenvector.

**Remark:** The condition that a holomorphic vector be an eigenvector for  $SL(2)$  avoids some technical complications.

**Remark:** It is supposedly well known by now that a classical automorphic form living on a complex domain is holomorphic in the elementary sense if and only if when converted to an automorphic form on a Lie group (or on an adèle group) it is holomorphic in this sense.

The defining property of the **weight  $\ell$  holomorphic discrete series** representation  $\pi_\ell$  of  $SL(2, \mathbf{R})$  (with  $\ell \geq 2$ ) is that it is generated by a **holomorphic vector**  $x_o$ . The **lowest  $K$ -type** of  $\pi_\ell$  is  $\ell$  (or, really, the representation of  $K_v$  indexed by  $\ell$ ). (The  $K$ -type is not quite so easily indexed for larger groups, of course.) In a typical abuse of language, the complex-linear span of  $x_o$  in  $\pi_\ell$  is also called the **lowest  $K$ -type** of  $\pi_\ell$ .

**Theorem:** For  $\ell \geq 2$  there exists a weight- $\ell$  holomorphic discrete series representation of  $SL(2, \mathbf{R})$ , and it is unique up to isomorphism. (We won't prove this here, but it's not so terribly difficult.) ///

**Proposition:** For totally real fields  $k$ , at infinite primes  $v$ , for (totally) positive-definite quadratic form  $Q$ , if  $\varphi_v$  is taken to be the Gaussian (attached to  $Q$ )

$$\varphi_v(x) = e^{-\pi Q[x]}$$

then for any  $8\ell$ -by-1 matrix  $x$  the function

$$g \longrightarrow \rho(g)\varphi(x)$$

is *holomorphic*, and generates a *holomorphic discrete series* representation of weight  $4\ell$ . It lies in the *lowest  $K$ -type*  $4\ell$  of the holomorphic discrete series  $\pi_{4\ell}$ .

**Corollary:** In this situation, the corresponding  $\varepsilon_v$  is *holomorphic*, and generates a *holomorphic discrete series* representation of weight  $4\ell$ . It lies in the *lowest  $K$ -type*  $4\ell$  of that holomorphic discrete series.

**Remark:** That is, with such choice of Schwartz functions at infinite primes, the Eisenstein series arising from  $\varphi$  via the Weil representation really is a holomorphic automorphic form.

**Corollary:** In this situation, the corresponding

$$g \longrightarrow \Theta_\varphi(g, h) = \sum_x \rho(g, h)\varphi(x)$$

is holomorphic (for any  $h \in H_v$ ).

**Remark:** That is, with such choice of Schwartz functions at infinite primes, the theta series arising from  $\varphi$  really are holomorphic automorphic forms.

## 6. Outline of proof of Siegel-Weil formula

We suppose that the number field  $k$  is totally real, and that the quadratic form  $Q$  is  $8\ell$ -dimensional, with further specifics as above at archimedean primes. In particular,  $Q$  is positive-definite at all infinite primes. Choose a Schwartz function  $\varphi$  in  $\mathbf{S}(8\ell \times 1)$  with the infinite-prime factors  $\varphi_v$  being the standard Gaussians. Let  $\mathbf{1}$  denote the identically-one function on the adelicized orthogonal group  $O(Q)$ . Then

**Theorem:** (*Siegel-Weil*)

$$E_\varphi = \Theta_\varphi(\mathbf{1})$$

That is, the theta lift of the constant function  $\mathbf{1}$  on the orthogonal group is ‘the’ Eisenstein series on  $SL(2)$ .

*Proof:* (*Sketch*) Of course the goal is to prove that the difference

$$E_\varphi - \Theta_\varphi(\mathbf{1})$$

is the identically-zero function.

Compute directly that, due to the holomorphy, the two Bruhat-cell parts of the constant term of  $E_\varphi$  simplify: the big-cell term is simply 0. The little-cell term is directly computed to be equal to the constant term of the automorphic form  $\Theta_\varphi(\mathbf{1})$ . And both  $E_\varphi$  and  $\Theta_\varphi(\mathbf{1})$  are holomorphic. Thus, the difference

$$E_\varphi - \Theta_\varphi(\mathbf{1})$$

is holomorphic, of moderate growth, and with constant term 0. By the simple way that holomorphic automorphic forms behave, this assures that  $E_\varphi - \Theta_\varphi(\mathbf{1})$  is square-integrable, so is a cuspform.

If it were the case that at some finite prime  $v$  the Eisenstein series  $E_\varphi$  and the theta-lift  $\Theta_\varphi(\mathbf{1})$  locally generated isomorphic irreducible representations of  $SL(2, k_v)$ , then the difference  $E_\varphi - \Theta_\varphi(\mathbf{1})$  would also generate an isomorphic copy of that irreducible representation of  $SL(2, k_v)$ . We know that almost everywhere the Eisenstein series  $E_\varphi$  generates the irreducible principal series  $I_{4\ell}$ . Below we will sketch a proof that almost everywhere  $\Theta_\varphi(\mathbf{1})$  generates that same irreducible. Granting this, we find that the difference  $E_\varphi - \Theta_\varphi(\mathbf{1})$  generates  $I_{4\ell}$  almost everywhere.

Since the difference  $E_\varphi - \Theta_\varphi(\mathbf{1})$  is square-integrable and locally generates the irreducible representation  $I_{4\ell}$  of  $SL(2, k_v)$  for at least one finite prime  $v$ , if this difference were not identically zero then the local representation  $I_{4\ell}$  would be unitarizable. We have noted that it is not, so we must conclude that the difference  $E_\varphi - \Theta_\varphi(\mathbf{1})$  is 0. This proves the Siegel-Weil formula. ///

**Remark:** We don't get any contradiction to unitariness by looking at archimedean primes, because the holomorphic discrete series *are* unitary.

## 7. Computing some Jacquet modules

To complete a reasonable sketch of a proof of Siegel-Weil, even in our simple case, we should verify that at almost all primes the theta-lift  $\Theta_\varphi(\mathbf{1})$  generates the principal series  $I_{4\ell}$  at almost all finite primes. This is true regardless of  $\varphi$ , although the finite set of exceptional primes of course changes depending upon the choice of  $\varphi$ .

To verify this, we compute the Jacquet module of the representation of  $SL(2, k_v)$  generated by the trivial-representation co-isotype for  $O(Q)_v$  of the Weil representation  $\rho$ . (Note that the Jacquet module is really the trivial-representation co-isotype for the standard unipotent radical  $N_v$  in  $SL(2, k_v)$ .)

**Lemma:** Almost everywhere locally, the trivial-representation co-isotype for  $O(Q)_v$  of the local Weil representation  $\rho_v$  is  $I_{4\ell}$ -isotypic.

*Proof:* We may suppose that the residue field characteristic is not 2. By Frobenius reciprocity, the assertion of the lemma is equivalent to the assertion that, as a  $P_v$ -representation, the trivial-representation co-isotype for  $O(Q)_v \times N_v$  of  $\rho_v$  is a sum of copies of the one-dimensional representation

$$\begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix} \rightarrow |a|^{4\ell}$$

That is, the trivial-representation co-isotype for  $O(Q)_v \times N_v$  of  $\rho_v$  consists entirely of vectors which are left  $P_v$ -equivariant by the character  $\chi_{4\ell}$ .

This terminology is standard, if not common: the trivial-representation co-isotype of a representation space  $(\sigma, V)$  of a group  $G$  is the smallest quotient space

$$q : (\sigma, V) \longrightarrow (\tau, W)$$

through which every intertwining operator from  $(\sigma, V)$  to the trivial  $G$ -representation factors. If all representations at hand were unitary, this quotient would be isomorphic to a subrepresentation, but in

general this is not so. The Jacquet module is the trivial-representation co-isotype for the unipotent radical  $N_v$ , and the theta-lift of  $\mathbf{1}$  generates the trivial representation of  $O(Q)_v$ , so the intertwining operator

$$\rho \rightarrow \Theta_\varphi(\mathbf{1})$$

must indeed factor through the trivial-representation co-isotype for  $O(Q)_v \times N_v$ .

Also we must recall that we have an exact sequence of Schwartz spaces

$$0 \rightarrow S(Y) \rightarrow S(X) \rightarrow S(X - Y) \rightarrow 0$$

for any totally disconnected space  $X$  and open subset  $Y \subset X$ . In this context,  $S(X)$  is the collection of locally constant compactly-supported complex-valued functions on a totally-disconnected space  $X$ . The proof of this assertion is not difficult, and goes back at least to Bruhat's early work circa 1960.

Let

$$q : \rho \rightarrow \text{trivial co-isotype of representation gen'd by } \Theta_\varphi(\mathbf{1})$$

be the map to the trivial representation co-isotype of the  $O(Q)_v \times N_v$  representation generated by  $\Theta_\varphi(\mathbf{1})$ .

First, we claim that any Schwartz function  $f$  with support on non-isotropic vectors in  $Q$  maps to 0 under  $q$ . Specifically, on the support of  $f$  the function  $x \rightarrow Q[x]$  is certainly continuous, being polynomial, and non-zero by assumption, so *uniformly* bounded away from 0 by the compactness of support of  $f$ . Thus,  $\text{ord}_v Q[x]$  is bounded from above for  $x$  in the support of  $f$ . Say  $\text{ord}_v Q[x] \leq n$ . Thus, for any fixed  $x$  in the support of  $f$ , the function  $t \rightarrow \psi(t \cdot Q[x]/2)$  on  $\varpi^{-n-1}\mathfrak{o}_v$  is a non-trivial character on  $\varpi^{-n-1}\mathfrak{o}_v$ , hence has integral 0 over  $\varpi^{-n-1}\mathfrak{o}_v$ . By standard apparatus for Jacquet modules, this implies that  $f$  maps to 0 in the quotient map to the Jacquet module.

Let  $Z$  be the set of isotropic vectors in  $Q$ . By the previous claim, the map  $q$  to the co-isotype factors through  $S(Z)$ . That is, the co-isotype is a quotient of  $S(Z)$ .

By Witt's theorem, the collection of isotropic vectors for  $Q$  falls into exactly two  $O(Q)_v$ -orbits:  $\{0\}$  and all non-zero isotropic vectors. (By the local results for the Hasse-Minkowski theorem, since  $8\ell > 4$  there *do* exist non-zero  $Q$ -isotropic vectors locally at all finite primes  $v$ .)

From this, invoking the exact sequence above, (and using Frobenius Reciprocity) we see that the trivial-representation co-isotype  $\pi$  fits into a short exact sequence

$$0 \rightarrow S(Z - \{0\})_1 \rightarrow \pi \rightarrow S(\{0\}) \rightarrow 0$$

of  $P_v$ -representations (trivial representation of  $O(Q)_v$ ), where  $S(Z)_1$  is shorthand for the  $O(Q)_v$  trivial co-isotype of the Schwartz space  $S(Z - \{0\})$  on the open subset  $Z - \{0\}$  of  $Z$ .

In particular, we **claim** that every  $O(Q)_v$ -invariant distribution on  $S(Z)$  is a linear combination of the functional

$$u_o(f) = f(0)$$

and the functional

$$u_1(f) = \int_Q f(x) dx$$

where we lift  $f$  back to a Schwartz function on the whole vector space  $Q$  of the quadratic form, over  $k_v$ . Certainly these two functionals are both  $O(Q)_v$ -invariant and are linearly independent. Thus, from the previous computation, it must be the the trivial-representation  $O(Q)_v$  co-isotype is the *direct sum* of these two trivial representations of  $O(Q)_v$ .

It remains to identify the representation of the Levi component  $M_v$  of the parabolic  $P_v$  of  $SL(2, k_v)$  which occurs in these two spaces. On the subspace in the co-isotype corresponding to the functional  $u_o$ , the Levi

component of  $SL(2, k_v)$  acts by the character  $\chi_{4\ell}$  coming from the definition of the Weil representation: with  $m = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ ,

$$u_o(mf) = u_o(|a|^{4\ell} f(*a)) = |a|^{4\ell} u_o(f(*a)) = |a|^{4\ell} f(0 \cdot a) = |a|^{4\ell} f(0) = |a|^{4\ell} u_o(f)$$

On the subspace corresponding to the functional  $u_1$ , the effect of such  $m$  is a little messier to verify but turns out to be  $|a|^{1-4\ell}$ .

Thus, finally, we see that the Jacquet module for  $SL(2, k_v)$  of the theta kernel is

$$\chi_{4\ell} \oplus \chi_{1-4\ell}$$

Noting that we hadn't done any normalizing, this is none other than the Jacquet module of the principal series. ///