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Unbounded operators, Friedrichs' extension theorem

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It is amazing that *resolvents* $R_\lambda = (T - \lambda)^{-1}$ exist, as *everywhere-defined*, continuous linear maps on a Hilbert space, even for T *unbounded*, and only densely-defined. Of course, some further hypotheses on T are needed, but these hypotheses are met in useful situations occurring in practice.

In particular, we will need that T is *symmetric*, in the sense that $\langle Tv, w \rangle = \langle v, Tw \rangle$ for v, w in the domain of T . And we will need to replace T by its *Friedrichs extension*, described explicitly below. For example, the Friedrichs extension replaces genuine differentiation by L^2 -differentiation. ^[1]

So-called **unbounded operators** on a Hilbert space V are not literally operators on V , being defined on *proper subspaces* of V . For unbounded operators on V , the actual *domain* is an essential part of a description: an unbounded operator T on V is a subspace D of V and a linear map $T : D \rightarrow V$. The interesting case is that the domain D is *dense* in V .

The linear map T is most likely *not* continuous when D is given the subspace topology from V , or it would extend by continuity to the closure of D , presumably V .

Explicit naming of the domain of an unbounded operator is often suppressed, instead writing $T_1 \subset T_2$ when T_2 is an **extension** of T_1 , in the sense that the domain of T_2 contains that of T_1 , and the restriction of T_2 to the domain of T_1 agrees with T_1 .

An operator T', D' is a **sub-adjoint** to an operator T, D when

$$\langle Tv, w \rangle = \langle v, T'w \rangle \quad (\text{for } v \in D, w \in D')$$

For D *dense*, for given D' there is *at most* one T' meeting the adjointness condition.

The **adjoint** T^* is the *unique maximal* element, in terms of domain, among all sub-adjoints to T . That there is a unique maximal sub-adjoint requires proof, given below.

An operator T is *symmetric* when $T \subset T^*$, and *self-adjoint* when $T = T^*$. These comparisons refer to the *domains* of these not-everywhere-defined operators. In the following claim and its proof, the domain of a map S on V is incorporated in a reference to its *graph*

$$\text{graph } S = \{v \oplus Sv : v \in \text{domain } S\} \subset V \oplus V$$

[0.0.1] **Remark:** In practice, anticipating that a given unbounded operator is self-adjoint *when extended suitably*, a simple version of the operator is defined on an easily described, small, dense domain, specifying a *symmetric* operator. Then a self-adjoint *extension* is shown to exist, as in Friedrichs' theorem below.

[0.0.2] **Remark:** A symmetric operator that *fails* to be self-adjoint is necessarily *unbounded*, since bounded symmetric operators are self-adjoint, because of the existence of orthogonal complements in Hilbert spaces. The latter idea is applied to not-necessarily-bounded operators in the following.

[1] [Friedrichs 1934] construction of suitable extensions predates [Sobolev 1937,1938], though the extensions use an abstracted version of what nowadays are usually called Sobolev spaces. The physical motivation for the construction is *energy estimates*. *Existence* results for self-adjoint extensions had been discussed in [Neumann 1929], [Stone 1929,30,34], but a useful description of a *natural* extension first occurred in [Friedrichs 1934]. Further, a Hilbert-space precursor of the Lax-Milgram theorem of [Lax-Milgram 1954] also appears in [Friedrichs 1934], following by the argument Friedrichs uses to prove that his construction gives an extension.

The direct sum $V \oplus V$ is a Hilbert space, with natural inner product

$$\langle v \oplus v', w \oplus w' \rangle = \langle v, v' \rangle + \langle w, w' \rangle$$

Define an isometry U of $V \oplus V$ by

$$U : V \oplus V \longrightarrow V \oplus V \quad \text{by} \quad v \oplus w \longrightarrow -w \oplus v$$

[0.0.3] **Claim:** Given T with *dense* domain D , there is a unique *maximal* T^* , D^* among all sub-adjoints to T, D . Further, the adjoint T^* is *closed*, in the sense that its *graph* is closed in $V \oplus V$. In fact, the adjoint is *characterized* by its graph, which is the orthogonal complement in $V \oplus V$ to an image of the graph of T , namely,

$$\text{graph } T^* = \text{orthogonal complement of } U(\text{graph } T)$$

Proof: The adjointness condition $\langle Tv, w \rangle = \langle v, T^*w \rangle$ for given $w \in V$ is an orthogonality condition

$$\langle w \oplus T^*w, U(v \oplus Tv) \rangle = 0 \quad (\text{for all } v \text{ in the domain of } T)$$

Thus, the graph of *any* sub-adjoint is a subset of

$$X = U(\text{graph } T)^\perp$$

Since T is densely-defined, for given $w \in V$ there is *at most* one possible value w' such that $w \oplus w' \in X$, so this orthogonality condition determines a well-defined function T^* on a subset of V , by

$$T^*w = w' \quad (\text{if there exists } w' \in V \text{ such that } w \oplus w' \in X)$$

The linearity of T^* is immediate. It is maximal among sub-adjoints to T because the graph of any sub-adjoint is a subset of the graph of T^* . Orthogonal complements are closed, so T^* has a closed graph. ///

[0.0.4] **Corollary:** For $T_1 \subset T_2$ with dense domains, $T_2^* \subset T_1^*$, and $T_1 \subset T_1^{**}$. ///

[0.0.5] **Corollary:** A self-adjoint operator has a closed graph. ///

[0.0.6] **Remark:** The closed-ness of the graph of a self-adjoint operator is essential in proving existence of *resolvents*, below.

[0.0.7] **Remark:** The use of the term *symmetric* in this context is potentially misleading, but standard. The notation $T = T^*$ allows an inattentive reader to forget non-trivial assumptions on the *domains* of the operators. The equality of domains of T and T^* is understandably essential for legitimate computations.

[0.0.8] **Proposition:** Eigenvalues for symmetric operators T, D are *real*.

Proof: Suppose $0 \neq v \in D$ and $Tv = \lambda v$. Then

$$\lambda \langle v, v \rangle = \langle \lambda v, v \rangle = \langle Tv, v \rangle = \langle v, T^*v \rangle \quad (\text{because } v \in D \subset D^*)$$

Further, because T^* agrees with T on D ,

$$\langle v, T^*v \rangle = \langle v, \lambda v \rangle = \bar{\lambda} \langle v, v \rangle$$

Thus, λ is real. ///

[0.0.9] **Definition:** A densely-defined symmetric operator T, D is *positive* (or *non-negative*) when

$$\langle Tv, v \rangle \geq 0 \quad (\text{for all } v \in D)$$

Certainly all the eigenvalues of a positive operator are non-negative real.

[0.0.10] **Theorem:** (Friedrichs) A *positive*, densely-defined, symmetric operator T, D has a positive *self-adjoint* extension.

Proof: [2] Define a new hermitian form $\langle \cdot, \cdot \rangle_1$ and corresponding norm $\| \cdot \|_1$ by

$$\langle v, w \rangle_1 = \langle v, w \rangle + \langle Tv, w \rangle \quad (\text{for } v, w \in D)$$

The symmetry and non-negativity of T make this positive-definite hermitian on D . Note that $\langle v, w \rangle_1$ makes sense whenever at least one of v, w is in D .

Let D_1 be the closure in V of D with respect to the metric d induced by $\| \cdot \|$. We claim that D_1 is also the d -completion of D . Indeed, for v_i a d -Cauchy sequence in D , v_i is Cauchy in V in the original topology, since

$$|v_i - v_j| \leq |v_i - v_j|_1$$

For two sequences v_i, w_j with the same d -limit v , the d -limit of $v_i - w_i$ is 0. Thus,

$$|v_i - w_i| \leq |v_i - w_i|_1 \longrightarrow 0$$

For $h \in V$ and $v \in D_1$, the functional $\lambda_h : v \rightarrow \langle v, h \rangle$ has a bound

$$|\lambda_h v| \leq |v| \cdot |h| \leq |v|_1 \cdot |h|$$

Thus, the norm of the functional λ_h on D_1 is at most $|h|$. By Riesz-Fischer, there is unique Bh in the Hilbert space D_1 with $|Bh|_1 \leq |h|$, such that

$$\lambda_h v = \langle Bh, v \rangle_1 \quad (\text{for } v \in D_1)$$

Thus,

$$|Bh| \leq |Bh|_1 \leq |h|$$

The map $B : V \rightarrow D_1$ is verifiably linear. There is an obvious *symmetry* of B :

$$\langle Bv, w \rangle = \lambda_w Bv = \langle Bv, Bw \rangle_1 = \overline{\langle Bw, Bv \rangle_1} = \overline{\lambda_v Bw} = \overline{\langle Bw, v \rangle} = \langle v, Bw \rangle \quad (\text{for } v, w \in V)$$

Positivity of B is similar:

$$\langle Bv, v \rangle = \lambda_v Bv = \langle Bv, Bv \rangle_1 \geq \langle Bv, Bv \rangle \geq 0$$

Finally B is *injective*: if $Bw = 0$, then for all $v \in D_1$

$$0 = \langle v, 0 \rangle_1 = \langle v, Bw \rangle_1 = \lambda_w v = \langle v, w \rangle$$

Since D_1 is dense in V , $w = 0$. Similarly, if $w \in D_1$ is such that $\lambda_v w = 0$ for all $v \in V$, then $0 = \lambda_w w = \langle w, w \rangle$ gives $w = 0$. Thus, $B : V \rightarrow D_1$ is bounded, symmetric, positive, injective, with dense image. In particular, B is self-adjoint.

[2] We essentially follow [Riesz-Nagy 1955], pages 329-334.

Thus, B has a possibly *unbounded* positive, symmetric inverse A . Since B injects V to a dense subset D_1 , necessarily A *surjects* from its domain (inside D_1) to V . We claim that A is *self-adjoint*. Let $S : V \oplus V \rightarrow V \oplus V$ by $S(v \oplus w) = w \oplus v$. Then

$$\text{graph } A = S(\text{graph } B)$$

Also, in computing orthogonal complements X^\perp , clearly

$$(SX)^\perp = S(X^\perp)$$

From the obvious $U \circ S = -S \circ U$, compute

$$\begin{aligned} \text{graph } A^* &= (U \text{ graph } A)^\perp = (U \circ S \text{ graph } B)^\perp = (-S \circ U \text{ graph } B)^\perp \\ &= -S((U \text{ graph } B)^\perp) = -\text{graph } A = \text{graph } A \end{aligned}$$

since the domain of B^* is the domain of B . Thus, A is self-adjoint.

We claim that for v in the domain of A , $\langle Av, v \rangle \geq \langle v, v \rangle$. Indeed, letting $v = Bw$,

$$\langle v, Av \rangle = \langle Bw, w \rangle = \lambda_w Bw = \langle Bw, Bw \rangle_1 \geq \langle Bw, Bw \rangle = \langle v, v \rangle$$

Similarly, with $v' = Bw'$, and $v \in D_1$,

$$\langle v, Av' \rangle = \langle v, w' \rangle = \lambda_{w'} v = \langle v, Bw' \rangle_1 = \langle v, v' \rangle_1 \quad (v \in D_1, v' \text{ in the domain of } A)$$

Since B maps V to D_1 , the domain of A is contained in D_1 . We claim that the domain of A is dense in D_1 in the d -topology, not merely in the subspace topology from V . Indeed, for $v \in D_1$ $\langle \cdot, v \rangle_1$ -orthogonal to the domain of A , for v' in the domain of A , using the previous identity,

$$0 = \langle v, v' \rangle_1 = \langle v, Av' \rangle$$

Since B injects V to D_1 , A surjects from its domain to V . Thus, $v = 0$.

Last, prove that A is an extension of $S = 1_V + T$. On one hand, as above,

$$\langle v, Sw \rangle = \lambda_{Sw} v = \langle v, BSw \rangle_1 \quad (\text{for } v, w \in D)$$

On the other hand, by definition of $\langle \cdot, \cdot \rangle_1$,

$$\langle v, Sw \rangle = \langle v, w \rangle_1 \quad (\text{for } v, w \in D)$$

Thus,

$$\langle v, w - BSw \rangle_1 = 0 \quad (\text{for all } v, w \in D)$$

Since D is d -dense in D_1 , $BSw = w$ for $w \in D$. Thus, $w \in D$ is in the range of B , so is in the domain of A , and

$$Aw = A(BSw) = Sw$$

Thus, the domain of A contains that of S and extends S . ///

Let $R_\lambda = (T - \lambda)^{-1}$ for $\lambda \in \mathbb{C}$ when this inverse exists as a linear operator defined at least on a dense subset of V .

[0.0.11] Theorem: Let T be self-adjoint and densely defined. For $\lambda \in \mathbb{C}$, $\lambda \notin \mathbb{R}$, the operator R_λ is everywhere defined on V , and the operator norm is estimated by

$$\|R_\lambda\| \leq \frac{1}{|\text{Im}\lambda|}$$

For T positive, for $\lambda \notin [0, +\infty)$, R_λ is everywhere defined on V , and the operator norm is estimated by

$$\|R_\lambda\| \leq \begin{cases} \frac{1}{|\operatorname{Im} \lambda|} & (\text{for } \operatorname{Re}(\lambda) \leq 0) \\ \frac{1}{|\lambda|} & (\text{for } \operatorname{Re}(\lambda) \geq 0) \end{cases}$$

Proof: For $\lambda = x + iy$ off the real line and v in the domain of T ,

$$\begin{aligned} |(T - \lambda)v|^2 &= |(T + x)v|^2 + \langle (T - x)v, iyv \rangle + \langle iyv, (T - x)v \rangle + y^2|v|^2 \\ &= |(T + x)v|^2 - iy\langle (T - x)v, v \rangle + iy\langle v, (T - x)v \rangle + y^2|v|^2 \end{aligned}$$

The symmetry of T , and the fact that the domain of T^* contains that of T , implies that

$$\langle v, Tv \rangle = \langle T^*v, v \rangle = \langle Tv, v \rangle$$

Thus,

$$|(T - \lambda)v|^2 = |(T - x)v|^2 + y^2|v|^2 \geq y^2|v|^2$$

Thus, for $y \neq 0$, $(T - \lambda)v \neq 0$. Let D be the domain of T . On $(T - \lambda)D$ there is an inverse R_λ of $T - \lambda$, and for $w = (T - \lambda)v$ with $v \in D$,

$$|w| = |(T - \lambda)v| \geq |y| \cdot |v| = |y| \cdot |R_\lambda(T - \lambda)v| = |y| \cdot |R_\lambda w|$$

which gives

$$|R_\lambda w| \leq \frac{1}{|\operatorname{Im} \lambda|} \cdot |w| \quad (\text{for } w = (T - \lambda)v, v \in D)$$

Thus, the operator norm on $(T - \lambda)D$ satisfies $\|R_\lambda\| \leq 1/|\operatorname{Im} \lambda|$ as claimed.

We must show that $(T - \lambda)D$ is the whole Hilbert space V . If

$$0 = \langle (T - \lambda)v, w \rangle \quad (\text{for all } v \in D)$$

then the adjoint of $T - \lambda$ can be defined on w simply as $(T - \lambda)^*w = 0$, since

$$\langle Tv, w \rangle = 0 = \langle v, 0 \rangle \quad (\text{for all } v \in D)$$

Thus, $T^* = T$ is defined on w , and $Tw = \bar{\lambda}w$. For λ not real, this implies $w = 0$. Thus, $(T - \lambda)D$ is *dense* in V .

Since T is self-adjoint, it is *closed*, so $T - \lambda$ is closed. The equality

$$|(T - \lambda)v|^2 = |(T - x)v|^2 + y^2|v|^2$$

gives

$$|(T - \lambda)v|^2 \ll_y |v|^2$$

Thus, for fixed $y \neq 0$, the map

$$v \oplus (T - \lambda)v \longrightarrow (T - \lambda)v$$

respects the metrics, in the sense that

$$|(T - \lambda)v|^2 \leq |(T - \lambda)v|^2 + |v|^2 \ll_y |(T - \lambda)v|^2 \quad (\text{for fixed } y \neq 0)$$

The graph of $T - \lambda$ is *closed*, so is a *complete* metric subspace of $V \oplus V$. Since F respects the metrics, it preserves completeness. Thus, the metric space $(T - \lambda)D$ is *complete*, so is a closed subspace of V . Since the

closed subspace $(T - \lambda)D$ is dense, it is V . Thus, for $\lambda \notin \mathbb{R}$, R_λ is everywhere-defined. Its norm is bounded by $1/|\operatorname{Im} \lambda|$, so it is a continuous linear operator on V .

Similarly, for T positive, for $\operatorname{Re}(\lambda) \leq 0$,

$$|(T - \lambda)v|^2 = |Tv|^2 - \lambda \langle Tv, v \rangle - \bar{\lambda} \langle v, Tv \rangle + |\lambda|^2 \cdot |v|^2 = |Tv|^2 + 2|\operatorname{Re} \lambda| \langle Tv, v \rangle + |\lambda|^2 \cdot |v|^2 \geq |\lambda|^2 \cdot |v|^2$$

Then the same argument proves the existence of an everywhere-defined inverse $R_\lambda = (T - \lambda)^{-1}$, with $\|R_\lambda\| \leq 1/|\lambda|$ for $\operatorname{Re} \lambda \leq 0$. ///

[0.0.12] **Theorem:** (Hilbert) For points λ, μ off the real line, or, for T positive, for λ, μ off $[0, +\infty)$,

$$R_\lambda - R_\mu = (\lambda - \mu)R_\lambda R_\mu$$

For the operator-norm topology, $\lambda \rightarrow R_\lambda$ is *holomorphic* at such points.

Proof: Applying R_λ to

$$1_V - (T - \lambda)R_\mu = ((T - \mu) - (T - \lambda))R_\mu = (\lambda - \mu)R_\mu$$

gives

$$R_\lambda(1_V - (T - \lambda)R_\mu) = R_\lambda((T - \mu) - (T - \lambda))R_\mu = R_\lambda(\lambda - \mu)R_\mu$$

Then

$$\frac{R_\lambda - R_\mu}{\lambda - \mu} = R_\lambda R_\mu$$

For holomorphy, with $\lambda \rightarrow \mu$,

$$\frac{R_\lambda - R_\mu}{\lambda - \mu} - R_\mu^2 = R_\lambda R_\mu - R_\mu^2 = (R_\lambda - R_\mu)R_\mu = (\lambda - \mu)R_\lambda R_\mu R_\mu$$

Taking operator norm, using $\|R_\lambda\| \leq 1/|\operatorname{Im} \lambda|$,

$$\left\| \frac{R_\lambda - R_\mu}{\lambda - \mu} - R_\mu^2 \right\| \leq \frac{|\lambda - \mu|}{|\operatorname{Im} \lambda| \cdot |\operatorname{Im} \mu|^2}$$

Thus, for $\mu \notin \mathbb{R}$, as $\lambda \rightarrow \mu$, this operator norm goes to 0, demonstrating the holomorphy.

For *positive* T , the estimate $\|R_\lambda\| \leq 1/|\lambda|$ for $\operatorname{Re} \lambda \leq 0$ yields holomorphy on the negative real axis by the same argument. ///

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