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Most-continuous automorphic spectrum for GL_n

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We obtain the most-continuous part of the spectral decomposition of $L^2(PGL_n(\mathbb{Z})\backslash PGL_n(\mathbb{R})/O_n)$, expressed as $(n-1)$-fold integrals of minimal-parabolic Eisenstein series. The harmonic analysis reduces to Fourier transform on Euclidean spaces. All the $n!$ functional equations of the Eisenstein series are needed to obtain the expression of pseudo-Eisenstein series as integrals of Eisenstein series.

We show that the map from minimal-parabolic pseudo-Eisenstein series to their decomposition coefficients against minimal-parabolic Eisenstein series is an *isometry to its image*. The argument characterizing the image is sketched, but is incomplete without characterizing the other, less-continuous parts of the spectral decomposition. Thus, we do not quite prove Plancherel for this part of the spectrum.

As supporting material, we prove the meromorphic continuation of the minimal-parabolic Eisenstein series via Bochner's Lemma, the latter being reviewed in an appendix, compute the constant term, and prove the functional equations. Some relevant *reduction theory* is also recalled in an appendix.

As usual, *in principle* everything here has been known for forty or fifty years. [1] See the bibliography for indications.

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1. *Minimal-parabolic Eisenstein series*

Granting the meromorphic continuation of minimal-parabolic Eisenstein series, the functional equations are determined by the constant term.

[1.1] Spherical Eisenstein series Let $G = PGL_n(\mathbb{R})$ and $\Gamma = PGL_n(\mathbb{Z})$. We have the standard minimal parabolic B, standard Levi component A, unipotent radical N, Weyl group W, the latter represented by permutation matrices. Let A^+ be the image in G of *positive* diagonal matrices. Consider characters on B of the form

$$\chi = \chi_s \;:\; \begin{pmatrix} a_1 & & * \\ & \ddots & \\ & & a_n \end{pmatrix} \longrightarrow |a_1|^{s_1} \ldots |a_n|^{s_n} \qquad (\text{for } s = (s_1, \ldots, s_n) \in \mathbb{C}^n)$$

For the character to descend to PGL_n, necessarily $s_1 + \ldots + s_n = 0$. The modular function of B is $\chi_{2\rho}$, with the half-sum ρ of positive roots:

$$\rho \;=\; (\rho_1, \ldots, \rho_n) \;=\; \Big(\frac{n-1}{2}, \frac{n-3}{2}, \ldots, \frac{-n+3}{2}, \frac{-n+1}{2}\Big) \;\in\; \mathbb{C}^n$$

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[1] Thus, in principle, this discussion is merely an example computation slightly extending the clichéd example of GL_2. In practice, typical treatments of GL_2 do not clearly suggest what happens in any other example. At the same time, it is non-trivial to see what very general intrinsic treatments say about GL_n and other particular examples.

The standard spherical vector is

$$\varphi_s^{\rm sph}(pk) \;=\; \chi_s(p) \qquad\qquad \text{(for } p\in B \text{ and } k\in K\text{)}$$

and the spherical Eisenstein series is

$$E_s(g) \;=\; \sum_{\gamma\in B\cap\Gamma\backslash\Gamma} \varphi_s^{\rm sph}(\gamma\cdot g)$$

[1.2] Rewriting the Eisenstein series

For computations concerning the *constant term* just below, it is very convenient to rewrite the Eisenstein series.

The Bruhat decomposition $G=\bigcup_{w\in W} BwB$ is valid for the *rational* points of the groups, but not for *integer* points, motivating revising the presentation of the Eisenstein series: let $G_v = GL_n(\mathbb{Q}_v)$ for all completions $\mathbb{Q}_v$ of $\mathbb{Q}$, and similarly B_v. Let $K_v = GL_n(\mathbb{Z}_v)$ for finite places v, and $K_v = O_n$ for $\mathbb{Q}_v\approx\mathbb{R}$. The s^{th} character $\chi_{s,v}$ on B_v is

$$\chi_{s,v} \;:\; \begin{pmatrix} a_1 & & * \\ & \ddots & \\ & & a_n \end{pmatrix} \longrightarrow |a_1|_v^{s_1}\ldots|a_n|_v^{s_n} \qquad\qquad \text{(for } s=(s_1,\ldots,s_n)\in\mathbb{C}^n\text{)}$$

For the character to descend to PGL_n, necessarily $s_1+\ldots+s_n=0$. The standard v-adic spherical vector is

$$\varphi_{s,v}^{\rm sph}(pk) \;=\; \chi_{s,v}(p) \qquad\qquad \text{(for } p\in B_v \text{ and } k\in K_v\text{)}$$

and the global spherical vector is $\varphi_s^{\rm sph}(g) = \bigotimes_v \varphi_{v,s}^{\rm sph}(g_v)$, for $g=\{g_v\}\in PGL_n(\mathbb{A})$. Over an arbitrary number field k, not merely over $\mathbb{Q}$, the spherical Eisenstein series can be rewritten as

$$E_s(g) \;=\; \sum_{\gamma\in B_k\backslash G_k} \varphi_s^{\rm sph}(\gamma\cdot g) \qquad\qquad \text{(for } g\in PGL_n(\mathbb{R}) \text{ or } PGL_n(\mathbb{A})\text{)}$$

This expression is valid for g in the archimedean factor $G_\infty = PGL_n(\mathbb{R})$, and also extends the definition of E_s to a left G_k-invariant, right $\prod_v K_v$-invariant function of $g\in PGL_n(\mathbb{A})$.

[1.3] Shape of constant term

We will see below that the constant term

$$c_BE_s(g) \;=\; \int_{N\cap\Gamma\backslash N} E_s(ng)\,dn$$

determines the functional equations of E_s. We do not immediately need all the details of the constant term, only the general shape of it, as follows. Use the adelic reformulation of the Eisenstein series. From the Bruhat decomposition over a number field k, $G_k=\bigcup_{w\in W} B_kwB_k = \bigcup_{w\in W} B_kwN_k$, for $a\in A^+$

$$c_BE_s(a) \;=\; \int_{N_k\backslash N_{\mathbb{A}}} E_s(na)\,dn \;=\; \sum_{w\in W}\int_{N_k\backslash N_{\mathbb{A}}} \sum_{\gamma\in B_k\backslash B_kwN_k} \varphi_s^{\rm sph}(\gamma na)\,dn$$

$$=\; \sum_{w\in W}\int_{N_k\backslash N_{\mathbb{A}}} \sum_{\gamma\in(w^{-1}N_kw\cap N_k)\backslash N_k} \varphi_s^{\rm sph}(w\gamma na)\,dn$$

For fixed w, let $N'=w^{-1}Nw\cap N$. The Lie algebra of N' is a sum of positive root spaces. Let N'' be the *complement* of N' in N, in the sense that its Lie algebra is the sum of the positive root spaces *not* in N'. Note that $n\to\varphi_s^{\rm sph}(wng)$ is left $N''_{\mathbb{A}}$-invariant. Then the w^{th} summand above partly unwinds, and is

$$\int_{N''_k\backslash N''_{\mathbb{A}}}\int_{N'_{\mathbb{A}}} \varphi_s^{\rm sph}(wn''n'a)\,dn'\,dn'' \;=\; \Big(\int_{N''_k\backslash N''_{\mathbb{A}}} dn''\Big)\cdot\int_{N'_{\mathbb{A}}} \varphi_s^{\rm sph}(wn'a)\,dn'\,dn'' \;=\; \int_{N'_{\mathbb{A}}} \varphi_s^{\rm sph}(wn'a)\,dn'$$

Replacing n' by $an'a^{-1}$ gives and using the left A^+-equivariance of $\varphi_s^{\rm sph}$ yields

$$\delta_w(a) \cdot \chi_s(waw^{-1}) \cdot \int_{N'_{\mathbb{A}}} \varphi_s^{\rm sph}(wn')\, dn'$$

where δ_w is the modular function of N'. With $a = \exp(x)$ for x in the Lie algebra $\mathfrak{a}$ of A^+, write $\delta_w(a) = e^{\langle \lambda, x\rangle} = a^\lambda$ for suitable λ in the dual $\mathfrak{a}^*$. Write $\alpha > 0$ for positive roots α and $\alpha < 0$ for negative roots. Elements $w \in W$ act on roots by the adjoint action on $\mathfrak{a}^*$. To express λ, observe that

$$\lambda \;=\; \sum_{\alpha>0\,:\, w\alpha>0} \alpha$$

Letting ρ be the half-sum $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$ of positive roots, λ may be expressed more succinctly as

$$\sum_{\alpha>0\,:\, w\alpha>0} \alpha \;=\; \tfrac{1}{2}\sum_{\alpha>0}\alpha \;-\; \tfrac{1}{2}\sum_{\alpha>0} w\alpha \;=\; \rho - w\rho$$

Thus, the w^{th} summand of $c_B E_s(a)$ is $a^{\rho+w(s-\rho)} \cdot c_w(s)$ where

$$c_w(s) \;=\; \int_{N'_{\mathbb{A}}} \varphi_s^{\rm sph}(wn')\, dn'$$

Thus,

$$c_B E_s(a) \;=\; \sum_{w\in W} c_w(s) \cdot a^{\rho+w(s-\rho)} \;=\; \sum_{w\in W} c_w(s)\cdot \chi_{\rho+w(s-\rho)}(a) \qquad (\text{for } a \in A^+)$$

with W acting linearly on $s \in \mathbb{C}^n$. One sees directly from above that $c_1(s) = 1$. Note that the affine action $s \to \rho + w(s-\rho)$ is indeed associative:

$$\rho + (ww')(s-\rho) \;=\; \rho + w\big((\rho + w'(s-\rho)) - \rho\big)$$

For brevity and clarity, write

$$w\cdot s \;=\; \rho + w(s-\rho) \qquad (s \in \mathbb{C}^n \text{ and } w \in W)$$

[1.4] Functional equations

All functional equations

$$E_{\tau\cdot s} \;=\; A(\tau, s)\cdot E_s \qquad (\text{with } \tau \in W)$$

are determined by the constant term $c_B E_s(a)$, as follows. For $\tau \in W$,

$$c_B\, E_{\tau\cdot s}\Big|_{A^+} \;=\; \sum_{w\in W} c_w(\tau\cdot s)\cdot \chi_{w\cdot(\tau\cdot s)} \;=\; \sum_{w\in W} c_{w\tau^{-1}}(\tau\cdot s)\cdot \chi_{w\cdot s}$$

by replacing w by $w\tau^{-1}$. A relation $E_{\tau\cdot s} = A(\tau,\chi)\, E_s$ gives

$$\sum_{w\in W} c_{w\tau^{-1}}(\tau\cdot s)\cdot \chi_{w\cdot s} \;=\; A(\tau,\chi)\cdot \sum_{w\in W} c_w(s)\cdot \chi_{w\cdot s}$$

For typical $s \in \mathbb{C}^n$, the $w\cdot s$ give distinct characters, so

$$c_{w\tau^{-1}}(\tau\cdot s) \;=\; A(\tau,\chi)\cdot c_w(s) \qquad (\text{for all } w \in W)$$

Thus,

$$A(\tau, s) \;=\; \frac{c_{w\tau^{-1}}(\tau \cdot s)}{c_w(s)} \qquad\qquad \text{(for all } w \in W\text{)}$$

Using $c_1(s) = 1$, take $w = \tau$ and $w = 1$, successively, to find

$$A(\tau, s) \;=\; \frac{1}{c_\tau(s)} \;=\; c_{\tau^{-1}}(\tau \cdot s)$$

which gives the form of the τ^{th} functional equation expressed in terms of the τ^{th} summand of the constant term:

$$E_{\tau \cdot s} \;=\; \frac{E_s}{c_\tau(s)} \;=\; c_{\tau^{-1}}(\tau \cdot s)\; E_s \qquad\qquad \text{(where } \tau \cdot s = \rho + \tau(s - \rho)\text{)}$$

Also, the *cocycle relations* among the summands of the constant term are obtained: $c_{w\tau^{-1}}(\tau \cdot s) = c_w(s)/c_\tau(s)$, or, equivalently,

$$c_w(\tau \cdot s) \;=\; \frac{c_{w\tau}(s)}{c_\tau(s)} \qquad\qquad \text{(where } \tau \cdot s = \rho + \tau(s - \rho)\text{)}$$

[1.5] On the unitary hyperplane, $|c_w(s)| = 1$

With $\mathfrak{a}$ the Lie algebra of A^+ and $\mathfrak{a}^*$ its dual, the *unitary hyperplane* is

$$\rho + i\mathfrak{a}^* \;=\; \{s \in \mathbb{C}^n \;:\; s_1 + \ldots + s_n = 0, \text{ and } \mathrm{Re}(s_j) = \rho_j\} \qquad\qquad \text{(with } \rho = (\rho_1, \ldots, \rho_n)\text{)}$$

For $s \in \rho + i\mathfrak{a}^*$, we claim $|c_w(s)| = 1$ for all $w \in W$. This property does *not* quite follow from the cocycle relations. Rather, these relations reduce to the case of $c_\tau(s)$ with a *reflection* $\tau \in W$, as follows. The permutation group on $\{1, 2, \ldots, n\}$ is generated by *adjacent transpositions* τ_j mapping $j \leftrightarrow j+1$ and leaving all other elements fixed. The corresponding elements τ_j of W act by conjugation on diagonal matrices by interchanging j^{th} and $(j+1)^{th}$ entries, and leaving the others fixed. Noting that $\rho + i\mathfrak{a}^*$ is stable under W, for $w \in W$ and reflection τ, we have

$$|c_{w\tau}(s)| \;=\; |c_w(\tau \cdot s)| \cdot |c_\tau(s)| \;=\; 1 \cdot 1 \qquad\qquad \text{(for } s \in \rho + i\mathfrak{a}^*\text{)}$$

by induction on the length ℓ of an expression $w = \sigma_1 \ldots \sigma_\ell$ as a product of reflections σ_j from among the generators τ_j of W.

To show that $|c_\tau(s)| = 1$ for a reflection τ, we first show that, with $s = (s_1, \ldots, s_n) \in \rho + i\mathfrak{a}^*$ and j^{th} reflection τ_j, we have $c_{\tau_j}(s) = c(s_j - s_{j+1})$, where $c(s)$ with $s \in \mathbb{C}$ appears in the constant term $y^s + c(s)y^{1-s}$ of the usual GL_2 Eisenstein series. Indeed, $c_{\tau_j}(s)$ is the coefficient of the summand of the constant term $c_B E_s$ coming from the Bruhat cell $B\tau_j B$: computing adelically, with $\mathbb{A}$ the adeles of a number field k,

$$\tau_j^{th} \text{ summand of } c_B E_s \;=\; \int_{N_k \backslash N_\mathbb{A}} \sum_{\gamma \in B_k \backslash B_k \tau_j B_k} \varphi_s^{\mathrm{sph}}(\gamma \cdot n \cdot a)\; dn \qquad\qquad \text{(with } a \in A^+\text{)}$$

Conjugation by τ_j interchanges the $a \to (a_j/a_{j+1})^{\pm 1}$ root spaces and permutes the other root spaces. Thus, $B_k \backslash B_k \tau_j B_k$ has irredundant representatives

$$n'_x \;=\; \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & x & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \qquad\qquad \text{(with } x \in k \text{ at the } (j, j+1) \text{ position)}$$

Let $N' = \{n'_x\}$ and let N'' be its complement in N, namely,

$$N'' \;=\; \{\begin{pmatrix} 1 & * & * & * & * & * \\ & \ddots & * & * & * & * \\ & & 1 & 0 & * & * \\ & & & 1 & * & * \\ & & & & \ddots & * \\ & & & & & 1 \end{pmatrix}\} \qquad \text{(with 0 at } (j,j+1)^{th} \text{ position)}$$

Unwinding and simplifying,

$$\tau_j^{th} \text{ summand } \;=\; \int_{N''_k\backslash N''_{\mathbb{A}}} \int_{N'_k\backslash N'_{\mathbb{A}}} \sum_{B_k\backslash B_k\tau_j B_k} \varphi_s^{\rm sph}(\gamma n' n'' a)\, dn'\, dn'' \;=\; \int_{N''_k\backslash N''_{\mathbb{A}}} \int_{N'_{\mathbb{A}}} \varphi_s^{\rm sph}(\tau_j n' n'' a)\, dn'\, dn''$$

$$=\; \Big(\int_{N''_k\backslash N''_{\mathbb{A}}} dn''\Big) \cdot \int_{N'_{\mathbb{A}}} \varphi_s^{\rm sph}(\tau_j n' a)\, dn' \;=\; (\tau_j a \tau_j)^s \cdot (a_j/a_{j+1}) \int_{N'_{\mathbb{A}}} \varphi_s^{\rm sph}(\tau_j n')\, dn'$$

The character of $a \in A^+$ has exponent

$$(s_1, \ldots, s_{j-1}, s_{j+1}+1, s_j - 1, s_{j+2}, \ldots, s_n)$$

Since τ_j does not affect coordinates other than s_j, s_{j+1}, we suppress those other coordinates. Noting that $\rho_j - \rho_{j+1} = 1$, as expected we find

$$(\ldots, s_{j+1}+1, s_j-1, \ldots) \;=\; (\ldots, s_{j+1}+\rho_j-\rho_{j+1}, s_j+\rho_{j+1}-\rho_j, \ldots)$$

$$=\; (\ldots, \rho_j, \rho_{j+1}, \ldots) + \tau_j(\ldots, s_j-\rho_j, s_{j+1}-\rho_{j+1}, \ldots) \;=\; \rho + \tau_j(s-\rho) \;=\; \tau_j \cdot s$$

The integral over $N'_{\mathbb{A}}$ is identical to that for GL_2 with character

$$\begin{pmatrix} a & * \\ 0 & d \end{pmatrix} \longrightarrow |a|^{s_j}\, |d|^{s_{j+1}} \;=\; |ad|^{\frac{s_l+s_{j+1}}{2}}\, |a/d|^{\frac{s_j-s_{j+1}}{2}}$$

The corresponding GL_2 Eisenstein series is $g \to |\det g|^{\frac{s_j+s_{j+1}}{2}} \cdot E_{\frac{s_j-s_{j+1}}{2}}$ where the latter has trivial central character, and constant term

$$y^{\frac{s_j-s_{j+1}}{2}} \;+\; c(\frac{s_j-s_{j+1}}{2})\, y^{1-\frac{s_j-s_{j+1}}{2}}$$

For GL_2, the cocycle relation $c(s)c(1-s) = 1$ and the conjugation $\overline{c(s)} = c(\overline{s})$ *do* suffice to prove that $|c(s)| = 1$ for $\mathrm{Re}\,(s) = \frac{1}{2}$. Thus, for $\mathrm{Re}\,(s_j) = \rho_j$ and $\mathrm{Re}\,(s_{j+1}) = \rho_{j+1}$, we have $\mathrm{Re}\,(\frac{s_j-s_{j+1}}{2}) = \frac{1}{2}$, so

$$|c_{\tau_j}(s)| \;=\; |c(\frac{s_j-s_{j+1}}{2})| \;=\; 1 \qquad \text{(for } s \in \rho + i\mathfrak{a}^*)$$

This sets up the induction for the general assertion that $|c_w(s)| = 1$ for $s \in \rho + i\mathfrak{a}^*$ and $w \in W$, as above.

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2. *Spectral decomposition of pseudo-Eisenstein series*

The minimal-parabolic pseudo-Eisenstein series arise naturally in elaboration of the Gelfand condition of vanishing of constant term along the minimal parabolic B, by an adjunction relation. This adjunction, combined with spectral decomposition along A^+ and the functional equations of E_s, yield the spectral decomposition of the pseudo-Eisenstein series.

[2.1] Adjunction and pseudo-Eisenstein series For f a reasonable function on $\Gamma\backslash G/K$, the minimal-parabolic constant term is

$$c_B f(g) \;=\; \int_{N\cap\Gamma\backslash N} f(ng)\, dn$$

with N the unipotent radical of the minimal parabolic B. The function $g \to c_B f(g)$ is left $N(B\cap\Gamma)$-invariant.

The Gelfand condition on *cuspforms* that all constant terms vanish requires $c_B f = 0$ in particular. It is best to describe $c_B f$ *as a distribution*, and its vanishing in that sense.

That is, for $\varphi \in C_c^\infty(N(B\cap\Gamma)\backslash G)^K \approx C_c^\infty(A^+)$, letting $\langle,\rangle_X$ be the pairing of distributions and test functions on a space X,

$$\langle c_B f, \varphi\rangle_{N(B\cap\Gamma)\backslash G} \;=\; \int_{N(B\cap\Gamma)\backslash G} c_B f\cdot\varphi \;=\; \int_{N(B\cap\Gamma)\backslash G} \Big(\int_{N\cap\Gamma\backslash N} f(ng)\, dn\Big)\cdot\varphi(g)\, dg$$

$$=\; \int_{N(B\cap\Gamma)\backslash G} \Big(\int_{N\cap\Gamma\backslash N} f(ng)\,\varphi(ng)\, dn\Big)\, dg \;=\; \int_{B\cap\Gamma\backslash G} f(g)\,\varphi(g)\, dg$$

$$=\; \int_{\Gamma\backslash G} \sum_{\gamma\in B\cap\Gamma\backslash\Gamma} f(\gamma g)\,\varphi(\gamma g)\, dg \;=\; \int_{\Gamma\backslash G} f(g)\Big(\sum_{\gamma\in B\cap\Gamma\backslash\Gamma} \varphi(\gamma g)\Big)\, dg$$

This exhibits the *pseudo-Eisenstein series*

$$\Psi_\varphi(g) \;=\; \sum_{\gamma\in B\cap\Gamma\backslash\Gamma} \varphi(\gamma g) \qquad\qquad (\text{for } \varphi\in C_c^\infty(N(B\cap\Gamma)\backslash G/K))$$

entering the *adjunction*

$$\langle c_B f, \varphi\rangle_{N(B\cap\Gamma)\backslash G} \;=\; \langle f, \Psi_\varphi\rangle_{\Gamma\backslash G}$$

That is, $\varphi \to \Psi_\varphi$ is adjoint to $f \to c_B f$. Then $c_B f = 0$ is equivalent to $\langle f, \Psi_\varphi\rangle_{\Gamma\backslash G} = 0$ for all φ.

[2.2] Fourier inversion To decompose the pseudo-Eisenstein series Ψ_φ as an integral of minimal-parabolic Eisenstein series, begin with Fourier transform on the Lie algebra $\mathfrak{a} \approx \mathbb{R}^{n-1}$ of A^+. Let $\langle,\rangle : \mathfrak{a}^*\times\mathfrak{a} \to \mathbb{R}$ be the $\mathbb{R}$-bilinear pairing of $\mathfrak{a}$ with its $\mathbb{R}$-linear dual $\mathfrak{a}^*$. For $f \in C_c^\infty(\mathfrak{a})$, the Fourier transform is

$$\widehat{f}(\xi) \;=\; \int_{\mathfrak{a}} e^{-i\langle x,\xi\rangle}\, f(x)\, dx$$

Fourier inversion is

$$f(x) \;=\; \frac{1}{(2\pi)^{\dim\mathfrak{a}}} \int_{\mathfrak{a}^*} e^{i\langle x,\xi\rangle}\, \widehat{f}(\xi)\, d\xi$$

[2.3] Mellin inversion Let $\exp : \mathfrak{a} \to A^+$ be the Lie algebra exponential, and $\log : A^+ \to \mathfrak{a}$ the inverse. Given $\varphi \in C_c^\infty(A^+)$, let $f = \varphi\circ\exp$ be the corresponding function in $C_c^\infty(\mathfrak{a})$. The *Mellin transform* $\mathcal{M}\varphi$ of φ is the Fourier transform of f:

$$\mathcal{M}\varphi(i\xi) \;=\; \widehat{f}(\xi)$$

Mellin inversion is Fourier inversion in these coordinates:

$$\varphi(\exp x) \;=\; f(x) \;=\; \frac{1}{(2\pi)^{\dim\mathfrak{a}}} \int_{\mathfrak{a}^*} e^{i\langle\xi,x\rangle}\, \widehat{f}(\xi)\, d\xi \;=\; \frac{1}{(2\pi)^{\dim\mathfrak{a}}} \int_{\mathfrak{a}^*} e^{i\langle\xi,x\rangle}\, \mathcal{M}\varphi(i\xi)\, d\xi$$

Extend the pairing $\langle,\rangle$ on $\mathfrak{a}^*\times\mathfrak{a}$ to a $\mathbb{C}$-bilinear pairing on the complexification. Use the convention

$$(\exp x)^{i\xi} \;=\; e^{i\langle\xi,x\rangle} \;=\; e^{\langle i\xi,x\rangle}$$

With $a = \exp x \in A^+$, Mellin inversion is

$$\varphi(a) \;=\; \frac{1}{(2\pi)^{\dim \mathfrak{a}}} \int_{\mathfrak{a}^*} a^{i\xi}\, \mathcal{M}\varphi(i\xi)\, d\xi \;=\; \frac{1}{(2\pi i)^{\dim \mathfrak{a}}} \int_{i\mathfrak{a}^*} a^{s}\, \mathcal{M}\varphi(s)\, ds \qquad (\text{with } a \in A^+ \text{ and } s = i\xi)$$

With this notation, the Mellin transform itself is

$$\mathcal{M}\varphi(s) \;=\; \int_{A^+} a^{-s}\, \varphi(a)\, da \qquad (\text{with } s \in i\mathfrak{a}^*)$$

Since φ is a test function, its Fourier-Mellin transform is *entire* on $\mathfrak{a}^* \otimes_{\mathbb{R}} \mathbb{C}$. (It is in the Paley-Wiener space.) Thus, for any $\sigma \in \mathfrak{a}^*$, Mellin inversion can be written

$$\varphi(a) \;=\; \frac{1}{(2\pi i)^{\dim \mathfrak{a}}} \int_{\sigma + i\mathfrak{a}^*} a^{s}\, \mathcal{M}\varphi(s)\, ds$$

[2.4] Spectral decomposition of pseudo-Eisenstein series: first step Identifying $N(B\cap\Gamma)\backslash G/K \approx A^+$, let $g \to a(g)$ be the function that picks out the A^+ component in an Iwasawa decomposition $G = NA^+K$. For $\sigma \in \mathfrak{a}^*$ suitable for convergence, the following rearrangement is legitimate:

$$\Psi_\varphi(g) \;=\; \sum_{\gamma \in B\cap\Gamma\backslash\Gamma} \varphi(a(\gamma\circ g)) \;=\; \sum_{\gamma \in B\cap\Gamma\backslash\Gamma} \frac{1}{(2\pi i)^{\dim \mathfrak{a}}} \int_{\sigma + i\mathfrak{a}^*} a(\gamma g)^s\, \mathcal{M}\varphi(s)\, ds$$

$$=\; \frac{1}{(2\pi i)^{\dim \mathfrak{a}}} \int_{\sigma + i\mathfrak{a}^*} \Big(\sum_{\gamma \in B\cap\Gamma\backslash\Gamma} a(\gamma g)^s \Big)\, \mathcal{M}\varphi(s)\, ds \;=\; \frac{1}{(2\pi i)^{\dim \mathfrak{a}}} \int_{\sigma + i\mathfrak{a}^*} E_s(g)\, \mathcal{M}\varphi(s)\, ds$$

Note that the parameter $s \in \mathfrak{a}^* \otimes_{\mathbb{R}} \mathbb{C}$ in the Eisenstein series E_s is *not* written in coordinates as earlier.

This does express the pseudo-Eisenstein series as a superposition of Eisenstein series, as desired. However, the coefficients $\mathcal{M}\varphi$ are not expressed in terms of Ψ_φ itself. This is rectified as follows.

[2.5] Adjunctions involving Eisenstein series Note that $dn\, da\, dk/a^{2\rho}$ is a Haar measure on $G = NA^+K$, so $da\, dk/a^{2\rho}$ is a right G-invariant measure on $N\backslash G$, and $da/a^{2\rho}$ is the associated measure on $N\backslash G/K$. In the region of convergence, for $f \in C_c^\infty(\Gamma\backslash G)$, using a complex-bilinear pairing rather than hermitian,

$$\langle f, E_s\rangle_{\Gamma\backslash G} \;=\; \int_{\Gamma\backslash G} f(g)\, E_s(g) \;=\; \int_{B\cap\Gamma\backslash G} f(g)\, a(g)^s\, dg$$

$$=\; \int_{N(B\cap\Gamma)\backslash G} \int_{N\cap\Gamma\backslash N} f(ng)\, a(ng)^s\, dg \;=\; \int_{N(B\cap\Gamma)\backslash G} c_B f(g)\; a(g)^s\; dg \;=\; \int_{A^+} c_B f(a)\; a^s\; \frac{da}{a^{2\rho}}$$

$$=\; \int_{A^+} c_B f(a)\; a^{-(2\rho - s)}\; da \;=\; \mathcal{M} c_B f(2\rho - s)$$

That is, with $f = \Psi_\varphi$,

$$\langle \Psi_\varphi, E_s\rangle_{\Gamma\backslash G} \;=\; \mathcal{M} c_B \Psi_\varphi(2\rho - s) \qquad (\text{with } \mathbb{C}\text{-bilinear pairing})$$

On the other hand, a similar unwinding of the pseudo-Eisenstein series, and recollection of the constant term $c_B E_s$, gives

$$\langle \Psi_\varphi, E_s\rangle_{\Gamma\backslash G} \;=\; \int_{B\cap\Gamma\backslash G} \varphi(g)\, E_s(g)\, dg \;=\; \int_{N(B\cap\Gamma)\backslash G} \int_{N\cap\Gamma\backslash N} \varphi(ng)\, E_s(ng)\, dg$$

$$= \int_{N(B\cap\Gamma)\backslash G} \varphi(g)\; c_B E_s(g)\, dg \;=\; \int_{A^+} \varphi(a)\; c_B E_s(a)\, \frac{da}{a^{2\rho}} \;=\; \int_{A^+} \varphi(a)\; \sum_w c_w(s)\, a^{w\cdot s}\, \frac{da}{a^{2\rho}}$$

$$= \sum_w c_w(s) \int_{A^+} \varphi(a)\; a^{-(2\rho - w\cdot s)}\, da \;=\; \sum_w c_w(s)\; \mathcal{M}\varphi(2\rho - w\cdot s)$$

Combining these,

$$\mathcal{M} c_B \Psi_\varphi(2\rho - s) \;=\; \langle \Psi_\varphi, E_s\rangle_{\Gamma\backslash G} \;=\; \sum_w c_w(s)\; \mathcal{M}\varphi(2\rho - w\cdot s)$$

Replacing s by $2\rho - s$, noting that $2\rho - w\cdot(2\rho - s) = w\cdot s$,

$$\mathcal{M} c_B \Psi_\varphi(s) \;=\; \sum_w c_w(2\rho - s)\; \mathcal{M}\varphi(w\cdot s)$$

[2.6] Complex conjugation on the unitary hyperplane $\rho + i\mathfrak{a}^*$ The Eisenstein series E_s behaves reasonable under complex conjugation: $\overline{E_s} = E_{\overline{s}}$. This is visible in the region of convergence, and persists under analytic continuation, since $\overline{E_{\overline{s}}} = E_s$ is an equality of meromorphic functions. This relation is inherited by the constant term along B:

$$\sum_w \overline{c_w(s)}\cdot \overline{a^{w\cdot s}} \;=\; c_B\overline{E_s} \;=\; c_B E_{\overline{s}} \;=\; \sum_w c_w(\overline{s})\cdot a^{w\cdot\overline{s}}$$

Since $\overline{a^{w\cdot s}} = a^{w\cdot\overline{s}}$, this gives $\overline{c_w(s)} = c_w(\overline{s})$. For s on the unitary hyperplane $\rho + i\mathfrak{a}^*$, conveniently $\overline{s} = 2\rho - s$. For such s,

$$\overline{c_w(s)} \;=\; c_w(2\rho - s) \qquad\qquad (\text{for } s\in\rho + i\mathfrak{a}^*)$$

Also, as proven earlier, $|c_w(s)| = 1$ for $s\in\rho + i\mathfrak{a}^*$ for all $w\in W$. Thus,

$$c_w(2\rho - s) \;=\; \overline{c_w(s)} \;=\; \frac{1}{c_w(s)} \qquad\qquad (\text{for } s\in\rho + i\mathfrak{a}^*)$$

[2.7] Spectral expansion of pseudo-Eisenstein series: second part To convert the earlier expression

$$\Psi_\varphi(g) \;=\; \frac{1}{(2\pi i)^{\dim\mathfrak{a}}} \int_{\sigma + i\mathfrak{a}^*} E_s(g)\, \mathcal{M}\varphi(s)\, ds$$

into a W-symmetric expression, to obtain an expression in terms of $c_B\Psi_\varphi$, we must use the functional equations of E_s. However, $\sigma + i\mathfrak{a}^*$ is W-stable only for $\sigma = \rho$. Thus, the integral over $\sigma + i\mathfrak{a}^*$ must be viewed as an iterated contour integral, and *moved* to $\rho + i\mathfrak{a}^*$. *For simplicity, we assume Ψ_φ is orthogonal to any residues.* Then

$$\Psi_\varphi \;=\; \frac{1}{|W|} \sum_w \frac{1}{(2\pi i)^{\dim\mathfrak{a}}} \int_{\rho + i\mathfrak{a}^*} E_{w\cdot s}\, \mathcal{M}\varphi(w\cdot s)\, ds \;=\; \frac{1}{|W|}\, \frac{1}{(2\pi i)^{\dim\mathfrak{a}}} \int_{\rho + i\mathfrak{a}^*} E_s \Big(\sum_w \frac{1}{c_w(s)}\, \mathcal{M}\varphi(w\cdot s) \Big)\, ds$$

As just observed, on $\rho + i\mathfrak{a}^*$ we have $1/c_w(s) = c_w(2\rho - s)$. Thus,

$$\sum_w \frac{1}{c_w(s)}\, \mathcal{M}\varphi(w\cdot s) \;=\; \sum_w c_w(2\rho - s)\, \mathcal{M}\varphi(w\cdot s) \;=\; \mathcal{M} c_B \Psi_\varphi(s)$$

This gives the desired spectral expansion of Ψ_φ: with complex-bilinear pairing $\langle,\rangle$,

$$\Psi_\varphi \;=\; \frac{1}{|W|}\frac{1}{(2\pi i)^{\dim \mathfrak{a}}}\int_{\rho+i\mathfrak{a}^*} E_s\cdot \mathcal{M}\Psi_\varphi(s)\,ds \;=\; \frac{1}{|W|}\frac{1}{(2\pi i)^{\dim \mathfrak{a}}}\int_{\rho+i\mathfrak{a}^*} E_s\cdot \langle \Psi_\varphi, E_{2\rho-s}\rangle_{\Gamma\backslash G}\,ds$$

[2.8] Isometry for most-continuous spectrum Let $f\in C_c^\infty(\Gamma\backslash G)$, $\varphi\in C_c^\infty(N\backslash G)$, and assume Ψ_φ is orthogonal to residues of E_s *above* ρ. Using the expression for Ψ_φ in terms of Eisenstein series,

$$\langle \Psi_\varphi, f\rangle \;=\; \Big\langle \frac{1}{|W|}\frac{1}{(2\pi i)^{\dim \mathfrak{a}}}\int_{\rho+i\mathfrak{a}^*} \langle \Psi_\varphi, E_{2\rho-s}\rangle\cdot E_s\,ds,\;\; f\Big\rangle \;=\; \frac{1}{|W|}\frac{1}{(2\pi i)^{\dim \mathfrak{a}}}\int_{\rho+i\mathfrak{a}^*} \langle \Psi_\varphi, E_{2\rho-s}\rangle\cdot\langle E_s, f\rangle\,ds$$

This shows that $f\to(s\to\langle f,E_s\rangle)$ is an inner-product-preserving map from the Hilbert-space span of the pseudo-Eisenstein series to its image in $L^2(\rho+i\mathfrak{a})$.

The map $\Psi_\varphi\to\langle\Psi_\varphi,E_{2\rho-s}\rangle$, with $s=\rho+it$ and $t\in\mathfrak{a}^*$, produces functions $u(t)=\langle\Psi_\varphi,E_{\rho-it}\rangle$ satisfying

$$u(wt) \;=\; \langle \Psi_\varphi, E_{2\rho-w\cdot s}\rangle \;=\; \langle \Psi_\varphi, E_{w\cdot(2\rho-s)}\rangle \;=\; \langle \Psi_\varphi, \frac{E_{2\rho-s}}{c_w(2\rho-s)}\rangle \;=\; c_w(s)\cdot u(t) \qquad\qquad (\text{for all } w\in W)$$

since $c_w(2\rho-s)=\overline{c_w(s)}=1/c_w(s)$ on $\rho+i\mathfrak{a}^*$.

[2.9] Toward Plancherel We claim that *any* $u\in L^2(\rho+i\mathfrak{a}^*)$ satisfying $u(wt)=c_w(s)\cdot u(t)$ for all $w\in W$ is in the image. First, for compactly-supported u meeting this condition, we claim

$$\Phi_u \;=\; \frac{1}{|W|}\frac{1}{(2\pi i)^{\dim \mathfrak{a}}}\int_{\rho+i\mathfrak{a}^*} u(t)\cdot E_{\rho+it}\,dt \;\neq\; 0$$

It suffices to show $c_B\Phi_u$ is not 0. With $s=\rho+it$, the relation implies $u(t)E_{2\rho-s}$ is *invariant* by W. Let

$$C \;=\; \{t\in\mathfrak{a}^* \;:\; \langle t,\alpha\rangle>0 \text{ for all simple } \alpha>0\}$$

be the positive Weyl chamber in $\mathfrak{a}^*$, where $\langle,\rangle$ is the Killing form transported to $\mathfrak{a}^*$ by duality. Then

$$\Phi_u \;=\; \frac{1}{|W|}\frac{1}{(2\pi i)^{\dim \mathfrak{a}}}\int_{\rho+i\mathfrak{a}^*} u(t)\cdot E_s\,dt \;=\; \frac{1}{(2\pi i)^{\dim \mathfrak{a}}}\int_{\rho+iC} u(t)\cdot E_s\,dt$$

Since $u(tw)=u(t)\cdot c_w(\rho+it)$, the constant term of Φ_u is

$$c_B\Phi_u = \frac{1}{(2\pi i)^{\dim \mathfrak{a}}}\int_{\rho+iC} u(t)\cdot\sum_w c_w a^{w\cdot s}\,dt = \frac{1}{(2\pi i)^{\dim \mathfrak{a}}}\int_{\rho+iC}\sum_w u(wt)\cdot a^{w\cdot s}\,dt = \frac{1}{(2\pi i)^{\dim \mathfrak{a}}}\int_{\rho+i\mathfrak{a}^*} u(t)\cdot a^s\,dt$$

This Fourier transform does not vanish for non-vanishing u.

It seems necessary to invoke the *complete* spectral decomposition of $L^2(\Gamma\backslash G/K)$, that *cuspforms* and *cuspidal data* Eisenstein series attached to *non-minimal* parabolics, and their L^2 residues, as well as the minimal-parabolic pseudo-Eisenstein series, span $L^2(\Gamma\backslash G/K)$. *And* we must know the orthogonality of integrals of minimal-parabolic Eisenstein series to all the other spectral components.

Granting this, necessarily Φ_u is in the topological closure of minimal-parabolic pseudo-Eisenstein series Ψ_φ with test-function data φ. Thus, given u, there is φ such that $\langle\Psi_\varphi,\Phi_u\rangle\neq 0$. Then

$$0 \;\neq\; \langle \Psi_\varphi,\Phi_u\rangle \;=\; \frac{1}{|W|}\frac{1}{(2\pi i)^{\dim \mathfrak{a}}}\int_{\rho+i\mathfrak{a}^*} u(t)\cdot\langle\Psi_\varphi,E_{2\rho-s}\rangle\,dt$$

Thus, the functions $s \to \langle \Psi_\varphi, E_{2\rho-s}\rangle$ are dense in the space of $L^2(\rho + i\mathfrak{a}^*)$ functions u satisfying $u(wt) = c_w(s) \cdot u(t)$ for all $w \in W$.

Recall that we have suppressed (multi-)residues of E_s encountered in moving the contour of integration. In fact, the only residue in that region is *constant*. Thus, there is an *isometry*

$$\{L^2(\rho + i\mathfrak{a}^*) \text{ integrals of minimal-parabolic } E_s\} \oplus \mathbb{C} \;\approx\; L^2 - \text{closure of} \{\text{minimal-parabolic } \Psi_\varphi\}$$

However, we have not quite proven this fragment of a Plancherel theorem. We did prove that the map from the space of pseudo-Eisenstein series to integrals of Eisenstein series is an isometry to its image.

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3. *Convergence of Eisenstein series*

We derive Godement's criterion for absolute convergence of Eisenstein series, as in [Borel 1966], in the context of minimal-parabolic Eisenstein series.

[3.0.1] Claim: *(In coordinates)* The minimal-parabolic Eisenstein series E_s on PGL_n converges absolutely for $\frac{\sigma_j - \sigma_{j+1}}{2} > 1$ for $j = 1, \ldots, n-1$, where $s = (s_1, \ldots, s_n) \in \mathbb{C}$ and $\sigma = (\mathrm{Re}(s_1), \ldots, \mathrm{Re}(s_n))$.

Let $\langle , \rangle$ be the Killing form on the Lie algebra $\mathfrak{a}$ of $G = PGL_n$. It is a scalar multiple of $\langle x, y\rangle = \mathrm{tr}(xy)$. Let 2ρ be the sum of positive roots.

[3.0.2] Claim: *(Intrinsic/conceptual version)* The minimal-parabolic Eisenstein series E_s on PGL_n converges absolutely for $\langle \alpha, \sigma - 2\rho\rangle > 0$ for all positive simple roots α.

[3.0.3] Remark: That is, the Eisenstein series E_s converges absolutely for $\sigma \in \mathfrak{a}^*$ in the translate by 2ρ of

$$\text{positive Weyl chamber} \;=\; \{\beta \in \mathfrak{a}^* : \langle \beta, \alpha\rangle > 0, \text{ for all positive roots } \alpha\}$$

Proof: Fix a number field k. Let h be the standard *height* function on a k-vectorspace with specified basis (not necessarily ordered). Let $e_1, \ldots, e_n$ be the standard basis of k^n. Any exterior power $\wedge^\ell k^n$ has (unordered) basis of wedges of the e_j, so has an associated height function. Let

$$\eta_j(g) \;=\; \frac{h\big((e_j \wedge \ldots \wedge e_n) \cdot \wedge^{n-j+1} g\big)}{h(e_j \wedge \ldots \wedge e_n)} \qquad (\text{for } g \in GL_n(\mathbb{A}))$$

where $\wedge^\ell g$ is the natural action of g on $\wedge^\ell k^n$. Note that η_j is the standard spherical vector in principal series attached to the character $\chi_{(0,\ldots,0,1,1,\ldots,1)}$ with $j-1$ zeros. Thus, the spherical vector φ_s^{sph} from which is made the s^{th} minimal-parabolic Eisenstein series E_s is expressible as

$$\varphi_s^{\mathrm{sph}} \;=\; \eta_1^{s_1}\, \eta_2^{s_2 - s_1}\, \eta_3^{s_3 - s_1 - s_2} \ldots \eta_n^{s_n - s_1 - s_2 - \ldots - s_{n-1}} \qquad (\text{where } s = (s_1, \ldots, s_n))$$

From *reduction theory*, given compact $C \subset G_{\mathbb{A}} = PGL_n(\mathbb{A})$,

$$h(v) \;\le_C\; h(v \cdot g) \;\le_C\; h(v) \qquad (\text{for all } 0 \neq v \in k^n \text{ and } g \in C)$$

and similarly for heights on $\wedge^\ell k^n$. Therefore, convergence of the series defining the Eisenstein series $E_s(g_o)$ is equivalent to convergence of

$$\int_C \sum_{\gamma \in B_k \backslash G_k} \varphi_s^{\mathrm{sph}}(\gamma g)\, dg$$

Shrinking C sufficiently so that $\gamma \cdot C \cap C \neq \phi$ implies $\gamma = 1$,

$$\int_C \sum_{\gamma \in B_k \backslash G_k} \varphi_s^{\rm sph}(\gamma g)\, dg \;=\; \int_{B_k \backslash G_k \cdot C} \varphi_s^{\rm sph}(g)\, dg$$

Also from reduction theory,

$$\inf_{0 \neq v} h(v) \;>\; 0 \qquad\qquad (\text{inf over } 0 \neq v \in \wedge^{n-j+1} k^n)$$

Therefore, $G_k \cdot C$ is contained in the set

$$Y \;=\; \{g \in G_{\mathbb A} \;:\; 1 \ll_C \eta_j(g) \;\text{ for } j = 1, \ldots, n\}$$

Thus, convergence of the Eisenstein series is *implied* by convergence of

$$\int_{B_k \backslash Y} \big|\varphi_s^{\rm sph}(g)\big|\; dg$$

The set Y is stable by right multiplication by the maximal compact subgroup $K_v \subset G_v$ at all places v, so this integral is

$$\int_{B_k \backslash (Y \cap P_{\mathbb A})} \big|\varphi_s^{\rm sph}(p)\big|\; dp \qquad\qquad (\text{left Haar measure on } B)$$

Let ρ be the half-sum of positive roots, so the left Haar measure on $B_{\mathbb A}$ is $d(na) = dn\, da/a^{2\rho}$, where dn is Haar measure on the unipotent radical and da is Haar measure on diagonal matrices A. Since $\varphi_s^{\rm sph}$ is left $N_{\mathbb A}$-invariant and $N_k \backslash N_{\mathbb A}$ is compact, convergence of the latter integral is equivalent to convergence of

$$\int_{A_k \backslash (Y \cap A_{\mathbb A})} \big|\varphi_s^{\rm sph}(a)\big|\; \frac{da}{a^{2\rho}} \;=\; \int_{A_k \backslash (Y \cap A_{\mathbb A})} a^{\sigma - 2\rho}\;\, da \qquad\qquad (\text{where } \sigma = (\text{Re}\,(s_1), \ldots, \text{Re}\,(s_n)))$$

The quotient $k^\times \mathbb{J}^1$ of norm-one ideles $\mathbb{J}^1$ is *compact*, by Fujisaki's Lemma. The discrepancy between $PSL_n(\mathbb R)$ and $PGL_n(\mathbb R)$ is absorbed by $A \cap \prod_v K_v$. Thus, convergence of the following archimedean integral suffices.

We parametrize $A \subset SL_n(\mathbb R)$ by $n-1$ maps from $GL_1(\mathbb R)$, namely,

$$h_j \;:\; t \;\longrightarrow\; \begin{pmatrix} 1 &&&&&& \\ & \ddots &&&&& \\ && 1 &&&& \\ &&& t &&& \\ &&&& t^{-1} && \\ &&&&& 1 && \\ &&&&&& \ddots & \\ &&&&&&& 1 \end{pmatrix} \qquad\qquad (\text{at } j^{th} \text{ and } (j+1)^{th} \text{ positions})$$

From

$$\eta_i(h_j(t)) \;=\; \begin{cases} |t|^{-1} & (\text{for } i = j+1) \\ \;1 & (\text{otherwise}) \end{cases}$$

we have

$$Y \cap A_{\mathbb R} \cap SL_n(\mathbb R) \;\;=\;\; \{\; \prod_j h_j(t_j) \;:\; t_j \in \mathbb{J} \;\text{ and }\; |t_j^{-1}| \gg 1 \;\}$$

Noting that $h_j(t)^{2\rho} = |t|^2$ for all j, convergence of the Eisenstein series is implied by convergence of

$$\int_0^{\ll 1} t^{\sigma_j - \sigma_{j+1} - 2}\; \frac{dt}{t} \qquad\qquad (\text{for } j = 1, \ldots, n-1)$$

These integrals are absolutely convergent for $\sigma_i - \sigma_{i+1} - 2 > 0$ for all i.

More intrinsically, identifying $\mathfrak{a}^*$ and $\mathfrak{a}$ via $\langle,\rangle$, the positive simple roots $\alpha_1, \ldots, \alpha_{n-1}$ are

$$\alpha_j \;=\; \begin{pmatrix} 0 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & & & \\ & & & 1 & & & \\ & & & & -1 & & \\ & & & & & 0 & \\ & & & & & & \ddots \\ & & & & & & & 0 \end{pmatrix} \qquad \text{(at } j^{th} \text{ and } (j+1)^{th} \text{ positions)}$$

The absolute convergence condition immediately becomes $\langle \sigma - 2\rho, \alpha\rangle > 0$ for all simple roots α. ///

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4. *Meromorphic continuation of Eisenstein series*

This discussion is in part an adaptation of the appendix of [Langlands 1967/76] treating Eisenstein series for the minimal parabolic in GL_n over a number field, explicitly accommodating non-trivial ideal class groups. The tone of that appendix suggests that the treatment of $GL_n(\mathbb{Z})$ was known, although there is no obvious extant source.

[4.1] Overview In brief, the idea is to view the minimal-parabolic Eisenstein series as an iterated object, an Eisenstein series for a not-quite-minimal parabolic P, formed from data including a suitably normalized GL_2 Eisenstein series $\widetilde{E}$ on the Levi component of P. Phragmén-Lindelöf gives whose that the analytic continuation of $\widetilde{E}$ is bounded in vertical strips, yielding convergence of the P Eisenstein series to a larger region of the form

$$\Omega_i \;=\; \{s \in \mathbb{C}^n \;:\; \mathrm{Re}(s_j) - \mathrm{Re}(s_{j+1}) > 2 \text{ for } j \neq i\}$$

That is, in Ω_i there is no constraint on $\mathrm{Re}(s_i) - \mathrm{Re}(s_{i+1})$.

This applies to all the 2-by-2 blocks along the diagonal, giving a meromorphic continuation of E_s to $\bigcup_i \Omega_i$. Then Bochner's lemma (see appendix) analytically continues the whole Eisenstein series to the convex hull of $\bigcup_i \Omega_i$, namely, $\mathbb{C}^n$.

[4.2] Iterated Eisenstein series For each fixed index $1 \le i < n$, there is the next-to-minimal standard parabolic P with standard Levi components and unipotent radicals given by $P = N^P \cdot M^P$ with

$$N^P \;=\; \begin{pmatrix} 1 & * & * & * & * & * & * & * \\ & \ddots & * & * & * & * & * & * \\ & & 1 & * & * & * & * & * \\ & & & 1 & 0 & * & * & * \\ & & & 0 & 1 & * & * & * \\ & & & & & 1 & * & * \\ & & & & & & \ddots & * \\ & & & & & & & 1 \end{pmatrix} \qquad M^P \;=\; \begin{pmatrix} * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & \ddots & 0 & 0 & 0 & 0 & 0 & 0 \\ & & * & 0 & 0 & 0 & 0 & 0 \\ & & & * & * & 0 & 0 & 0 \\ & & & * & * & 0 & 0 & 0 \\ & & & & & * & 0 & 0 \\ & & & & & & \ddots & 0 \\ & & & & & & & * \end{pmatrix}$$

with the anomalous block at the (i,i), $(i,i+1)$, $(i+1,i)$, and $(i+1,i+1)$ positions. The minimal-parabolic Eisenstein series can be written as an iterated sum

$$E_s(g) \;=\; \sum_{\gamma \in B_k\backslash G_k} \varphi_s^{\mathrm{sph}}(\gamma g) \;=\; \sum_{\gamma \in P_k\backslash G_k} \Big(\sum_{\delta \in B_k\backslash P_k} \varphi_s^{\mathrm{sph}}(\delta\gamma g)\Big)$$

The coset space $B_k\backslash P_k$ has representatives

$$B_k\backslash P_k \;\approx\; M_k\backslash M_k^P \;\approx\; \{\delta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & \ddots & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 & 0 & 0 \\ & & & a & b & 0 & 0 & 0 \\ & & & c & d & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & \ddots & 0 \\ & & & & & & & 1 \end{pmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in P_k^{1,1}\backslash GL_2(k)\} \;\approx\; P_k^{1,1}\backslash GL_2(k)$$

where $P^{1,1}$ is the standard upper-triangular parabolic in GL_2. Further,

$$\varphi_s^{\rm sph}\begin{pmatrix} a_1 & * & * & * & * & * & * & * \\ & \ddots & * & * & * & * & * & * \\ & & a_{i-1} & * & * & * & * & * \\ & & & a_i & * & * & * & * \\ & & & 0 & a_{i+1} & * & * & * \\ & & & & & a_{i+2} & * & * \\ & & & & & & \ddots & * \\ & & & & & & & a_n \end{pmatrix} \;=\; |a_1|^{s_1}\ldots|a_n|^{s_n}$$

$$=\; |a_1|^{s_1}\ldots|a_{i-1}|^{s_{i-1}}|a_i/a_{i+1}|^{\frac{s_i-s_{i+1}}{2}}\,|a_ia_{i+1}|^{\frac{s_i+s_{i+1}}{2}}\,|a_{i+2}|^{s_{i+2}}\ldots|a_n|^{s_n}$$

Thus, the inner sum is

$$\sum_{\delta\in B_k\backslash P_k}\varphi_s^{\rm sph}\Big(\delta\cdot\begin{pmatrix} a_1 & * & * & * & * & * & * & * \\ & \ddots & * & * & * & * & * & * \\ & & a_{i-1} & * & * & * & * & * \\ & & & a & b & * & * & * \\ & & & c & d & * & * & * \\ & & & & & a_{i+2} & * & * \\ & & & & & & \ddots & * \\ & & & & & & & a_n \end{pmatrix}\Big)$$

$$=\; |a_1|^{s_1}\ldots|a_{i-1}|^{s_{i-1}}\cdot E^{1,1}_{\frac{s_i-s_{i+1}}{2}}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\cdot\Big|\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\Big|^{\frac{s_i+s_{i+1}}{2}}\cdot|a_{i+2}|^{s_{i+2}}\ldots|a_n|^{s_n} \qquad (\text{for } h\in GL_2)$$

where $E^{1,1}$ is the usual GL_2 Eisenstein series with trivial central character. Therefore, let $g = nmk$ be an Iwasawa decomposition with $n\in N^P$, $m\in M^P$, and $k\in\prod_v K_v$ with m in the form just displayed, and put

$$\Phi_s^{[i]}(g) \;=\; |a_1|^{s_1}\ldots|a_{i=1}|^{s_{i-1}}\cdot E^{1,1}_{\frac{s_i-s_{i+1}}{2}}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\cdot\Big|\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\Big|^{\frac{s_i+s_{i+1}}{2}}\cdot|a_{i+2}|^{s_{i+2}}\ldots|a_n|^{s_n}$$

Then

$$E_s(g) \;=\; \sum_{\gamma\in P_k\backslash G_k}\Phi_s^{[i]}(\gamma\, g) \qquad (\text{for } g\in PGL_n)$$

This expresses the PGL_n minimal-parabolic Eisenstein series P-Eisenstein series attached to the $P^{1,1}$ Eisenstein series on the GL_2 part of its Levi component.

[4.3] Convergence estimate The normalization of the GL_2 Eisenstein series to eliminate poles, to be bounded on vertical strips for g in compacts in $GL_2(\mathbb{A})$, and to be *invariant* under $s\to 1-s$, is

$$\widetilde{E}_s(g) \;=\; s(1-s)\cdot\xi(2s)\cdot E_s^{1,1}(g) \qquad (\text{for } s\in\mathbb{C},\text{ with } \xi(s)=\pi^{-s/2}\Gamma(s/2)\zeta(s))$$

Thus, let

$$\widetilde{\Phi}_s^{[i]} \;=\; (\frac{s_i - s_{i+1}}{2})(1 - \frac{s_i - s_{i+1}}{2}) \cdot \xi(s_i - s_{i+1}) \cdot \Phi_s^{[i]}$$

An argument similar to that for convergence of the minimal-parabolic Eisenstein series E_s will prove the absolute convergence of

$$(\frac{s_i - s_{i+1}}{2})(1 - \frac{s_i - s_{i+1}}{2}) \cdot \xi(s_i - s_{i+1}) \cdot E_s(g) \;=\; \sum_{\gamma \in P_k \backslash G_k} \widetilde{\Phi}_s^{[i]}(\gamma\, g)$$

for $\frac{\mathrm{Re}(s_j) - \mathrm{Re}(s_{j+1})}{2} > 1$ for $j \neq i$, with no condition on $s_i - s_{i+1}$.

Indeed, for g in a fixed compact and $s_i - s_{i+1}$ in a fixed vertical strip, $\Phi_s^{[i]}(g)$ is dominated by the function obtained by replacing $\widetilde{E}^{1,1}$ by a constant, namely, with $\sigma_j = \mathrm{Re}(s_j)$,

$$\theta(g) \;=\; |a_1|^{\sigma_1} \ldots |a_{i-1}|^{\sigma_{i-1}} \cdot \left| \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right|^{\frac{\sigma_i + \sigma_{i+1}}{2}} \cdot |a_{i+2}|^{\sigma_{i+2}} \ldots |a_n|^{\sigma_n}$$

We prove the absolute convergence of the degenerate Eisenstein series $E(g) = \sum_{\gamma \in P_k \backslash G_k} \theta(\gamma g)$.

As in the earlier convergence argument, convergence is equivalent to convergence of an integrated form, namely

$$\int_C \sum_{\gamma \in P_k \backslash G_k} \theta(\gamma g)\; dg$$

Shrinking C sufficiently so that $\gamma \cdot C \cap C \neq \phi$ implies $\gamma = 1$,

$$\int_C \sum_{\gamma \in P_k \backslash G_k} \theta(\gamma g)\; dg \;=\; \int_{B_k \backslash G_k \cdot C} \theta(g)\; dg$$

As in the earlier convergence argument, letting η_j be the norm of the determinant of the lower right $n - j$ minor, $G_k \cdot C$ is contained in

$$Y \;=\; \{g \in G_{\mathbb{A}} \;:\; 1 \ll_C \; \eta_j(g) \;\text{ for } j = 1, \ldots, n\}$$

To compare with M^P, drop the $(i+1)^{th}$ condition: $G_k \cdot C$ is contained in

$$Y' \;=\; \{g \in G_{\mathbb{A}} \;:\; 1 \ll_C \; \eta_j(g) \;\text{ for } j \neq i+1\}$$

Thus, convergence of the Eisenstein series is implied by convergence of

$$\int_{P_k \backslash Y'} \theta(g)\;\; dg$$

As Y' is stable by right multiplication by the maximal compact subgroup $K_v \subset G_v$ at all places v, this integral is

$$\int_{P_k \backslash (Y' \cap P_{\mathbb{A}})} \theta(p)\;\; dp \qquad\qquad \text{(left Haar measure on } P\text{)}$$

Let $\alpha = \alpha_i$ be the i^{th} simple positive root, and ρ the half-sum of positive roots. The left Haar measure on $P_{\mathbb{A}}$ is $d(nm) = dn\, dm/m^{2\rho - \alpha}$, where dn is Haar measure on N^P and dm is Haar measure on the Levi component M^P. Since θ is left $N_{\mathbb{A}}$-invariant and $N_k \backslash N_{\mathbb{A}}$ is compact, convergence of the latter integral is equivalent to convergence of

$$\int_{M_k \backslash (Y' \cap M_{\mathbb{A}})} \theta(m)\;\; \frac{dm}{m^{2\rho - \alpha}}$$

As in the earlier convergence argument, Fujisaki's Lemma and right action of $M \cap \prod_v K_v$ reduce the convergence question to that of the following archimedean integral.

Parametrize $M \cap SL_n(\mathbb{R})$ by maps from $GL_1(\mathbb{R})$ as earlier, with the i^{th} replaced by the obvious map from $SL_2(\mathbb{R})$, namely,

$$h_i' \;:\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \begin{pmatrix} 1 &&&&&& \\ & \ddots &&&&& \\ && 1 &&&& \\ &&& a & b && \\ &&& c & d && \\ &&&&& 1 && \\ &&&&&& \ddots & \\ &&&&&&& 1 \end{pmatrix} \qquad \text{(at } i^{th} \text{ and } (i+1)^{th} \text{ positions)}$$

Then

$$Y' \cap M_{\mathbb{R}} \cap SL_n(\mathbb{R}) \;=\; \{ \prod_{j \neq i} h_j(t_j) \;:\; t_j \in \mathbb{J} \text{ and } |t_j^{-1}| \gg 1 \;\} \;\times\; \{h_i'(T) : T \in SL_2(\mathbb{R}),\; |\det T| \gg 1\}$$

Noting that $h_j(t)^{2\rho} = |t|^2$, convergence is implied by convergence of

$$\begin{cases} \displaystyle\int_0^{\ll 1} t^{\sigma_j - \sigma_{j+1} - 2}\, \frac{dt}{t} & (\text{for } j \neq i) \\[2ex] \displaystyle\int_{SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})} 1\, dt & (\text{right invariant measure } dt) \end{cases}$$

The GL_1 integrals are absolutely convergent for $\sigma_j - \sigma_{j+1} - 2 > 0$ for $j \neq i$. Since $SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})$ has finite volume, the SL_2 integral is convergent. This is the desired convergence conclusion: there is no constraint on $\sigma_i - \sigma_{i+1}$. Thus, the iterated expression for the Eisenstein series analytically continues as indicated.

[4.4] Functional equations for reflections In addition to a partial analytic continuation, the previous argument gives the functional equation for the reflection $\tau = \tau_\alpha \in W$ attached to the i^{th} simple root $\alpha = \alpha_i$. A first version is

$$\xi(2 - (s_i - s_{i+1})) \cdot E_{\tau \cdot s} \;=\; \xi(s_i - s_{i+1}) \cdot E_s \qquad (\text{as earlier, } \tau \cdot s = \rho + \tau(s - \rho))$$

The reflection τ interchanges $\pm\alpha$ and permutes the other positive roots. Thus,

$$\tau(\rho) \;=\; \tfrac{1}{2}\tau\Big(\alpha + \sum_{\beta > 0,\; \beta \neq \alpha} \beta\Big) \;=\; \tfrac{1}{2}\Big(-\alpha + \sum_{\beta > 0,\; \beta \neq \alpha} \beta\Big) \;=\; \rho - \alpha$$

This also shows that $\langle \rho, \alpha \rangle = \langle \alpha, \alpha \rangle / 2$ for simple α. Using $s_i - s_{i+1} = \langle s, \alpha_i \rangle$, and $\tau x = x - 2\frac{\langle x, \alpha \rangle\, \alpha}{\langle \alpha, \alpha \rangle}$,

$$\langle \tau \cdot s, \alpha \rangle \;=\; \langle \rho + \tau(s - \rho), \alpha \rangle \;=\; \langle \rho + \tau s - \tau\rho, \alpha \rangle \;=\; \langle \rho + \tau s - \rho + \alpha, \alpha \rangle \;=\; \langle \tau s + \alpha, \alpha \rangle \;=\; \langle \alpha, \alpha \rangle - \langle s, \alpha \rangle$$

Thus, the α^{th} functional equation is

$$E_{\tau \cdot s} \;=\; = \; \frac{\xi\langle s, \alpha \rangle}{\xi\langle \tau \cdot s, \alpha \rangle} \cdot E_s \;=\; \frac{\xi\langle s, \alpha \rangle}{\xi(\langle \alpha, \alpha \rangle - \langle s, \alpha \rangle)} \cdot E_s \qquad (\text{for reflection } \tau = \tau_\alpha)$$

Since also $E_{\tau \cdot s} = E_s / c_\tau(s)$, using the functional equation $\xi(1 - z) = \xi(z)$,

$$c_\tau(s) \;=\; \frac{\xi\langle \tau \cdot s, \alpha \rangle}{\xi\langle s, \alpha \rangle} \;=\; \frac{\xi(\langle \alpha, \alpha \rangle - \langle s, \alpha \rangle)}{\xi\langle s, \alpha \rangle} \;=\; \frac{\xi(1 - \langle \alpha, \alpha \rangle + \langle s, \alpha \rangle)}{\xi\langle s, \alpha \rangle} \;=\; \frac{\xi(\langle s, \alpha \rangle - 1)}{\xi\langle s, \alpha \rangle} \qquad (\text{reflection } \tau = \tau_\alpha)$$

The cocycle relation $c_{w\tau}(s) = c_w(\tau\cdot s)\cdot c_\tau(s)$ for the coefficients $c_w(s)$ of the constant term give the general case inductively. For example, with reflections σ,τ attached to simple roots α,β, respectively,

$$c_{\sigma\tau}(s) \;=\; c_\sigma(\tau\cdot s)\cdot c_\tau(s) \;=\; \frac{\xi\langle\sigma\tau\cdot s,\alpha\rangle}{\xi\langle\tau\cdot s,\alpha\rangle}\cdot\frac{\xi\langle\tau\cdot s,\beta\rangle}{\xi\langle s,\beta\rangle} \;=\; \frac{\xi(\langle\tau\cdot s,\alpha\rangle-1)}{\xi\langle\tau\cdot s,\alpha\rangle}\cdot\frac{\xi(\langle s,\beta\rangle-1)}{\xi\langle s,\beta\rangle}$$

Qualitatively, the number of factors in both numerator and denominator of $c_w(s)$ is the *length* of w.

[4.5] Application of Bochner's Lemma

The $n-1$ partial analytic continuations can be organized to allow application of Bochner's Lemma.

Above, for $\alpha=\alpha_i$ the i^{th} simple root, we showed that (suppressing some parentheses)

$$E_s^{[\alpha]} \;=\; \frac{\langle s,\alpha\rangle}{2}\cdot\Big(1-\frac{\langle s,\alpha\rangle}{2}\Big)\cdot\xi\langle s,\alpha\rangle\cdot E_s$$

admits an analytic continuation in which $s_i - s_{i+1} = \langle s,\alpha\rangle$ is not constrained, and this normalized version of E_s is *invariant* under $s\to\tau_\alpha\cdot s = \rho+\tau(s-\rho)$. This *might* suggest normalization factors for *all* positive roots, to obtain a W-invariant expression:

$$E_s \;\cdot\; \prod_{\beta>0}\frac{\langle s,\beta\rangle}{2}\cdot\Big(1-\frac{\langle s,\beta\rangle}{\langle\beta,\beta\rangle}\Big)\cdot\xi\langle s,\beta\rangle \qquad\qquad \text{(???... will fail)}$$

The intention is that, for each simple root α, the product $E_s^{[\alpha]}$ is invariant under the reflection τ_α, and the remaining factors should be permuted among themselves, since the other positive roots are permuted among themselves by τ_α. However, this is not quite so: in the normalization in which $\tau\cdot s = \rho+\tau(s-\rho)$, the collection of pairing values $\{\langle s,\beta\rangle : \beta\neq\alpha\}$ is not stabilized.

Instead, $\tau\cdot s-\rho = \tau(s-\rho)$. That is, the *affine* action $s\to\tau\cdot s$ becomes conveniently *linear* on $s-\rho$. Therefore, for simple α, rewrite

$$\langle s,\alpha\rangle \;=\; \langle s-\rho+\rho,\alpha\rangle \;=\; \langle s-\rho,\alpha\rangle+\langle\rho,\alpha\rangle \;=\; \langle s-\rho,\alpha\rangle+1$$

and consider

$$E_s \;\cdot\; \prod_{\beta>0}\Big(\tfrac{1}{2}+\frac{\langle s-\rho,\beta\rangle}{2}\Big)\cdot\Big(\tfrac{1}{2}-\frac{\langle s-\rho,\beta\rangle}{2}\Big)\cdot\xi\big(\langle s-\rho,\beta\rangle+1\big)$$

All indicated values of the completed zeta function ξ are in the convergent range when $\langle s-\rho,\beta\rangle>0$ for all positive β.

A technical issue arises: while for simple α the pole at $\langle s,\alpha\rangle=1$ of $\xi(\langle s,\alpha\rangle-1)$ is cancelled by the vanishing of E_s there, there is no obvious cancellation of these poles for the other factors. To most easily justify application of Bochner's lemma, add additional polynomial factors to be sure to cancel these poles: let

$$E_s^{\#} \;=\; E_s \;\cdot\; \prod_{\beta>0}\Big(\tfrac{1}{2}+\frac{\langle s-\rho,\beta\rangle}{2}\Big)\cdot\Big(\frac{\langle s-\rho,\beta\rangle}{2}\Big)^2\cdot\Big(\tfrac{1}{2}-\frac{\langle s-\rho,\beta\rangle}{2}\Big)\cdot\xi\big(\langle s-\rho,\beta\rangle+1\big)$$

The exponential decay of the gamma factor in ξ is more than sufficient to preserve boundedness in vertical strips for real part s in compacts.

[4.5.1] Claim: $E_s^{\#}$ has an analytic continuation to a holomorphic function on $\mathbb{C}^n$, and is invariant under $s\to w\cdot s$ for all $w\in W$.

Proof: By the GL_2 discussion and the above adaptations, $E_s^{\#}$ has an analytic continuation to the tube domain Ω over $\Omega_o\subset\mathbb{R}^n$ given by

$$\Omega_o \;=\; \{\sigma\in\mathbb{R}^n \;:\; \langle\sigma-\rho,\alpha\rangle>1 \text{ for all but possibly a single simple root } \alpha\}$$

In Ω, for $\mathrm{Re}(s)$ in compacts, $E_s^\#$ is *bounded*, so certainly has sufficiently modest growth for application of Bochner's Lemma. Thus, $E_s^\#$ has an analytic continuation to the convex hull of Ω, which is $\mathbb{C}^n$. ///

[4.5.2] Corollary: The meromorphic continuation of E_s is holomorphic off the zero-sets of $\xi(\langle s-\rho,\beta\rangle+1)$ for positive roots β, and off $\langle s-\rho,\beta\rangle = 0, \pm\frac{1}{2}$. ///

[4.5.3] Corollary: The meromorphic continuation of E_s satisfies $E_{w\cdot s} = E_s/c_w(s)$ for $w \in W$.

Proof: The partial analytic continuations of E_s, and the analytic continuation of $E_s^\#$ to a W-invariant function, prove the functional equations of E_s for reflections attached to simple roots, and also prove the cocycle relation on constant terms, by induction on length of $w \in W$. ///

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5. *Example:* PGL_3

For $G = PGL_3$ there are two simple positive roots,

$$\langle x,\alpha\rangle \;=\; x_1 - x_2 \qquad\qquad \langle x,\beta\rangle \;=\; x_2 - x_3 \qquad\qquad (\text{for } x \in \mathfrak{a} \text{ with diagonal entries } x_i)$$

The other positive root is $\alpha+\beta$, so $\rho = \frac{1}{2}(\alpha+\beta+(\alpha_\beta)) = \alpha+\beta$. Let σ,τ be the reflections corresponding to α,β, respectively. The whole Weyl group is

$$W \;=\; \{1,\,\sigma,\,\tau,\,\sigma\tau,\,\tau\sigma,\,\sigma\tau\sigma\}$$

and we note that $\sigma\tau\sigma = \tau\sigma\tau$. From the GL_2 computation,

$$c_\sigma(s) \;=\; \frac{\xi\langle s-\rho,\alpha\rangle}{\xi(\langle s-\rho,\alpha\rangle+1)} \qquad\qquad c_\tau(s) \;=\; \frac{\xi\langle s-\rho,\beta\rangle}{\xi(\langle s-\rho,\beta\rangle+1)}$$

By the cocycle relation $c_{wr}(s) = c_w(r\cdot s)\cdot c_r(s)$ for reflection r and $w \in W$,

$$c_{\sigma\tau}(s) \;=\; c_\sigma(\tau\cdot s)\cdot c_\tau(s) \;=\; \frac{\xi\langle \tau(s-\rho),\alpha\rangle}{\xi(\langle \tau(s-\rho),\alpha\rangle+1)}\cdot\frac{\xi\langle s-\rho,\beta\rangle}{\xi(\langle s-\rho,\beta\rangle+1)}$$

Since $\langle \tau x,\alpha\rangle = \langle x,\tau\alpha\rangle = \langle x,\alpha+\beta\rangle$,

$$c_{\sigma\tau}(s) \;=\; \frac{\xi\langle s-\rho,\alpha+\beta\rangle}{\xi(\langle s-\rho,\alpha+\beta\rangle+1)}\cdot\frac{\xi\langle s-\rho,\beta\rangle}{\xi(\langle s-\rho,\beta\rangle+1)}$$

Similarly,

$$c_{\tau\sigma}(s) \;=\; \frac{\xi\langle s-\rho,\alpha+\beta\rangle}{\xi(\langle s-\rho,\alpha+\beta\rangle+1)}\cdot\frac{\xi\langle s-\rho,\alpha\rangle}{\xi(\langle s-\rho,\alpha\rangle+1)}$$

Finally,

$$c_{\tau\sigma\tau}(s) \;=\; c_{\sigma\tau\sigma}(s) \;=\; c_{\sigma\tau}(\sigma\cdot s)\cdot c_\sigma(s) \;=\; \frac{\xi\langle \sigma(s-\rho),\alpha+\beta\rangle}{\xi(\langle \sigma(s-\rho),\alpha+\beta\rangle+1)}\cdot\frac{\xi\langle \sigma(s-\rho),\beta\rangle}{\xi(\langle \sigma(s-\rho),\beta\rangle+1)}\cdot\frac{\xi\langle s-\rho,\alpha\rangle}{\xi(\langle s-\rho,\alpha\rangle+1)}$$

Using $\sigma\beta = \alpha+\beta$ and $\sigma(\alpha+\beta) = \beta$, this is

$$c_{\tau\sigma\tau}(s) \;=\; c_{\sigma\tau\sigma}(s) \;=\; \frac{\xi\langle s-\rho,\beta\rangle}{\xi(\langle s-\rho,\beta\rangle+1)}\cdot\frac{\xi\langle s-\rho,\alpha+\beta\rangle}{\xi(\langle s-\rho,\alpha+\beta\rangle+1)}\cdot\frac{\xi\langle s-\rho,\alpha\rangle}{\xi(\langle s-\rho,\alpha\rangle+1)}$$

6. *Appendix: some reduction theory*

This is an adaptation of a small part of [Godement 1963].

[6.1] Height functions Let k be a global field with adeles $\mathbb{A}$. For completion $k_v \approx \mathbb{R}$, let h_v be the usual real Hilbert-space norm on $k_v^n \approx \mathbb{R}^n$. For $k_v \approx \mathbb{C}$, let h_v be the *square* of the usual complex Hilbert-space norm on $k_v^n \approx \mathbb{C}^n$. For k_v non-archimedean, let $h_v(x)$ be the sup of the v-adic norms of the coordinates of $x \in k_v^n$. The family of absolute values on all the k_v is normalized to make the product formula hold. These h_v are *local height functions*. The (global) *height function* h is

$$h(x) \;=\; \prod_v h_v(x_v) \qquad\qquad (\text{for } x = \{x_v\})$$

Sufficient conditions for finiteness of this product are given below.

The isometry groups $K_v \subset GL_n(k_v)$ of the height functions h_v are as follows. For $k_v \approx \mathbb{R}$, the isotropy group is the standard orthogonal group $K_v = O(n, \mathbb{R})$. For $k_v \approx \mathbb{C}$, the isotropy group is the standard unitary group $K_v = U(n)$. For k_v non-archimedean, the isotropy group is $K_v = GL_n(\mathfrak{o}_v)$, the group of matrices over the local integers $\mathfrak{o}_v$ in k_v, with determinant in the local units $\mathfrak{o}_v^\times$. Let

$$K \;=\; \prod_v K_v \;\subset GL_n(\mathbb{A})$$

Let P be the standard *parabolic* subgroup of upper-triangular matrices. Recall the *Iwasawa decompositions* $GL_n(k_v) \;=\; P_v \cdot K_v$.

Now we identify a class of vectors with *finite height*. First, given $x \in k^n - \{0\}$, for all but finitely-many v all the components of the vector x are v-integral, *and* generate the local integers $\mathfrak{o}_v$. In particular, for all but finitely-many v the v^{th} local height $h_v(x)$ of $x \in k^n$ is 1, and the infinite product for $h(x)$ is a *finite* product.

For each prime v the group K_v is *transitive* on the collection of vectors in k_v^n with given norm. (The arguments for this differ somewhat between archimedean and non-archimedean places.)

Consider vectors to be *row* vectors, and let $GL_n(\mathbb{A})$ act on the *right* by matrix multiplication. Say that a non-zero vector $x \in \mathbb{A}^n$ is *primitive* if $x \in k^n \cdot GL_n(\mathbb{A})$.

[6.1.1] Theorem:
- For idele t of k and primitive x, $h(tx) = |t| \cdot h(x)$. In particular, $k^\times$ preserves heights.
- For fixed $g \in GL_n(\mathbb{A})$ and for fixed $c > 0$

$$\{x \in k^n : h(x \cdot g) < c\} \;/\; k^\times \;=\; \text{finite}$$

- For a compact subset C of $GL_n(\mathbb{A})$ there are positive constants c, c' (depending only upon C) so that for all primitive vectors x and for all $g \in C$

$$c \cdot h(x) \le h(x \cdot g) \le c' \cdot h(x)$$

Proof: The first assertion is immediate, and the *product formula* shows that $k^\times$ leaves heights invariant.

For the second assertion, fix $g \in GL_n(\mathbb{A})$. Since K preserves heights, via Iwasawa we may suppose that g is in the group $P_{\mathbb{A}}$ of upper triangular matrices in $GL_n(\mathbb{A})$. Choose representatives $x = (x_1, \ldots, x_n)$ for non-zero vectors in k^n modulo $k^\times$ such that, letting μ be the first index with $x_\mu \neq 0$, then $x_\mu = 1$. That is, x is of the form

$$x \;=\; (0, \ldots, 0, 1, x_{\mu+1}, \ldots, x_n)$$

To illustrate the idea of the argument in a light notation, first consider $n = 2$, let $g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ and $x = (1, y)$. Thus,

$$x \cdot g = (1, y) \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = (a, b + yd)$$

From the definition of the local heights, at each v

$$\max(|a|_v, |b + yd|_v) \;\leq\; h_v(xg)$$

so

$$|b + yd|_v \prod_{w \neq v} |a|_w \;\leq\; \prod_{\text{all } w} h_w(xg) = h(xg)$$

Since g is fixed, a is fixed, and at almost all places $|a|_w = 1$. Thus, for $h(xg) < c$ there is a uniform constant c' so that for all places v

$$|b + yd|_v \;\leq\; c'$$

Since for almost all v the residue class field cardinality q_v is strictly greater than c', for almost all v

$$|b + yd|_v \;\leq\; 1$$

Therefore, $b + yd$ lies in a compact subset C of $\mathbb{A}$. Since b, d are fixed, and since k is discrete (and closed) in $\mathbb{A}$, the collection of images $\{b + dy : y \in k\}$ is discrete in $\mathbb{A}$. Thus, the collection of y so that $b + dy$ lies in C is finite, as desired.

For general n and $x \in k^n$ such that $h(xg) < c$, let $\mu - 1$ be the least index such that $x_\mu \neq 0$. Adjust by $k^\times$ so that $x_\mu = 1$.From $h(xg) < c$

$$|g_{\mu-1,\mu} + x_\mu g_{\mu,\mu}|_v \prod_{w \neq v} |g_{\mu-1,\mu-1}|_w \;\leq\; h(gx) < c \qquad \text{(for each } v\text{)}$$

For almost all v we have $|g_{\mu-1,\mu-1}|_v = 1$, so there is a uniform constant c' such that

$$|g_{\mu-1,\mu} + x_\mu g_{\mu,\mu}|_v \;<\; c' \qquad \text{(for all } v\text{)}$$

For almost all places v the residue field cardinality q_v is strictly greater than c', so for almost all v

$$|g_{\mu-1,\mu} + x_\mu g_{\mu,\mu}|_v \leq 1$$

Therefore, $g_{\mu-1,\mu} + x_\mu g_{\mu,\mu}$ lies in a compact subset C of $\mathbb{A}$. Since k is discrete, the collection of x_μ is finite.

Continue similarly to show that there are only finitely many choices for the other entries of x. Inductively, suppose that $x_i = 0$ for $i < \mu - 1$, and that $x_\mu, \ldots, x_{\nu-1}$ are fixed, and show that x_ν has only finitely many possibilities. Looking at the ν^{th} component $(xg)_\nu$ of xg,

$$|g_{\mu-1,\nu} + x_\mu g_{\mu,\nu} + \ldots + x_{\nu-1} g_{\nu-1,\nu} + x_\nu g_{\nu,\nu}|_v \prod_{w \neq v} |g_{\mu-1,\mu-1}|_w \;\leq\; h(xg) \;\leq\; c$$

For almost all v we have $|g_{\mu-1,\mu-1}|_w = 1$, so there is a uniform constant c' such that

$$|(xg)_\nu|_v = |g_{\mu-1,\nu} + x_\mu g_{\mu,\nu} + \ldots + x_{\nu-1} g_{\nu-1,\nu} + x_\nu g_{\nu,\nu}|_v \;<\; c' \qquad \text{(for all places } v\text{)}$$

For almost all places v the residue field cardinality q_v is strictly greater than c', so

$$|g_{\mu-1,\nu} + x_\mu g_{\mu,\nu} + \ldots + x_{\nu-1} g_{\nu-1,\nu} + x_\nu g_{\nu,\nu}|_v \leq 1 \qquad \text{(for almost all } v\text{)}$$

Therefore,

$$g_{\mu-1,\nu} + x_\mu g_{\mu,\nu} + \ldots + x_{\nu-1} g_{\nu-1,\nu} + x_\nu g_{\nu,\nu}$$

lies in the intersection of a compact subset C of $\mathbb{A}$ with a (closed) discrete set, so lies in a finite set. Thus, the number of possibilities for x_ν is finite. By induction we obtain the finiteness.

For the *third* and last assertion, recall the Cartan decompositions

$$GL_n(k_v) \;= K_{\;v} \cdot A_v \cdot K_v$$

where A_v is the subgroup of $GL_n(k_v)$ of diagonal matrices (v archimedean or not). Since the map

$$\theta_1 \times a \times \theta_2 \longrightarrow \theta_1 a \theta_2$$

is not an injection, one cannot immediately infer that for a given compact set C in $GL_n(k_v)$ the set

$$\{a \in A_v : \text{ for some } c \in C,\ c \in K_v a\, K_v\}$$

is compact. Since K_v is compact, $C' = K_v \cdot C \cdot K_v$ is compact, and now $\theta_1 a \theta_2 \in C'$ with $\theta_i \in K_v$ implies $a \in C' \cap A_v$, which *is* compact.

Thus, any compact subset of $GL_n(\mathbb{A})$ is contained in a set

$$\{\theta_1 \delta \theta_2 : \theta_1, \theta_2 \in K, \delta \in C_D\}$$

where C_D is a suitable *compact* set of diagonal matrices. Since K preserves heights and since the set of primitive vectors is stable under K, the set of values

$$\{\frac{h(xg)}{h(x)} : x \text{ primitive},\ g \in C\}$$

is contained in a set

$$\{\frac{h(x\delta)}{h(x)} : x \text{ primitive},\ \delta \in C_D\}$$

for some compact set C_D of diagonal matrices. Letting the diagonal entries of δ be δ_i, we have

$$0 < \inf_{\delta \in C_D} \inf_i |\delta_i| \leq \frac{h(x\delta)}{h(x)} \leq \sup_{\delta \in C_D} \sup_i |\delta_i| < +\infty$$

This gives the desired bound. ///

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7. *Appendix: Bochner's Lemma*

Bochner's Lemma is a one-of-a-kind device for meromorphic continuation in two or more complex variables.

Let Ω_o be a non-empty, connected, open set in $\mathbb{R}^n$. The *tube domain* Ω over Ω_o is $\Omega = \Omega_o + i\mathbb{R}^n$, that is, the collection of $z \in \mathbb{C}^n$ with real part in Ω_o.

Let f be a holomorphic $\mathbb{C}$-valued function on Ω, of not-too-awful vertical growth, in the sense that, for x in fixed compact $C \subset \Omega_o$, there is $1 \leq N \in \mathbb{Z}$ such that

$$|f(x+iy)| \;\ll_C\; e^{|y|^N} \qquad \text{(with } |(y_1, \ldots, y_n)|^2 = y_1^2 + \ldots + y_n^2\text{)}$$

[7.0.1] Claim: f extends to a holomorphic function on the convex hull of Ω.

Proof: First, let x, ξ be two points in Ω_o, such that the line segment connecting them lies entirely within Ω_o. We will specify a rectangle inside Ω with x, ξ the midpoints of opposite sides. Let $\gamma = \gamma_{x,\xi,R}$ parametrize the rectangle with sides individually parametrized by

$$\begin{cases} \text{side through } x\text{:} & x + it(x - \xi) & (\text{with } -R \le t \le R) \\ \text{top:} & (1-t)(x + iR(x-\xi)) + t(\xi + iR(x-\xi)) & (\text{with } 0 \le t \le 1) \\ \text{side through } \xi\text{:} & \xi - it(x-\xi) & (\text{with } -R \le t \le R) \\ \text{bottom:} & (1-t)(\xi - iR(\xi)) + t(x - iR(x-\xi)) & (\text{with } 0 \le t \le 1) \end{cases}$$

The expressions for the top and bottom simplify to

$$\begin{cases} \text{top:} & (1-t)x + t\xi + iR(x-\xi) & (\text{with } 0 \le t \le 1) \\ \text{bottom:} & (1-t)\xi + tx - iR(x-\xi) & (\text{with } 0 \le t \le 1) \end{cases}$$

This rectangle lies inside $Z = x + \mathbb{C} \cdot (x - \xi) \approx \mathbb{C}$, and is contractible in Ω. Let $j(\zeta) = x + \zeta \cdot (x - \xi)$. In Z, Cauchy's formula in one variable is

$$f \circ j(\zeta_o) \;=\; \frac{1}{2\pi i} \int_\gamma \frac{f \circ j(\zeta)\; d\zeta}{\zeta - \zeta_o}$$

To legitimately push the top and bottom of the rectangle to infinity, use the growth assumption on f, and the modified integral expression

$$f \circ j(\zeta_o) \;=\; e^{-\zeta_o^{2N}} \frac{1}{2\pi i} \int_\gamma \frac{e^{\zeta^{2N}} \cdot\; f \circ j(\zeta)\; d\zeta}{\zeta - \zeta_o}$$

Thus, taking the limit $R \to +\infty$,

$$e^{\zeta_o^{2N}} \cdot f(\zeta_o \cdot (x - \xi)) \;=\; \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{(x+it(x-\xi))^{2N}} f(x + it(x-\xi))\; dt}{it - \zeta_o} + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{(\xi - it(x-\xi))^{2N}} f(\xi - it(x-\xi))\; dt}{-1 - it - \zeta_o}$$

The right-hand side makes sense for any $x, \xi \in \Omega_o$, whether or not the line segment connecting them lies in Ω_o. Further, the right-hand side is holomorphic in $x, \xi \in \Omega$. Thus, the left-hand side is holomorphic, and gives the extension to the convex hull of Ω. ///

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8. *Appendix: continuous spectrum for $PSL_2(\mathbb{Z})$*

This appendix gives a more elementary, familiar example.

There are two usual points: constant terms of Eisenstein series determine their functional equations, and these functional equations enter the spectral decomposition of pseudo-Eisenstein series. Here $G = SL_2(\mathbb{R})$, $\Gamma = SL_2(\mathbb{Z})$, and $K = SO(2)$, P is the standard parabolic of upper-triangular matrices, and its unipotent radical N is upper-triangular unipotent matrices. Let $\langle f_1, f_2 \rangle = \int_{\Gamma \backslash G} f_1 \cdot f_2$, so the pairing is $\mathbb{C}$-bilinear, not hermitian.

We also prove the analytic continuation, and give some estimates necessary for the GL_n case.

[8.1] Constant term The constant term $c_P f$ of a function f on $\Gamma\backslash G$ is

$$c_P f(g) \;=\; \int_{N\cap\Gamma\backslash N} f(ng)\, dn$$

[8.2] Pseudo-Eisenstein series For $\varphi \in C_c^\infty(N\backslash G/K) \approx C_c^\infty(0,\infty)$ the **pseudo-Eisenstein series** is

$$\Psi_\varphi(g) \;=\; \sum_{P\cap\Gamma\backslash\Gamma} \varphi(\gamma g) \;\;\in\;\; C_c^\infty(\Gamma\backslash G/K)$$

[8.3] Fourier-Laplace-Mellin transforms Fourier inversion for Schwartz functions on the real line is

$$f(x) \;=\; \int_{-\infty}^{\infty} \Big(\int_{-\infty}^{\infty} f(x)\, e^{-2\pi i\xi x}\, dx\Big)\, e^{2\pi i\xi x}\, d\xi$$

Replacing ξ by $\xi/(2\pi)$ gives another version:

$$f(x) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} \Big(\int_{-\infty}^{\infty} f(t)\, e^{-it\xi}\, dt\Big)\, e^{i\xi x}\, d\xi$$

A change of coordinates gives the multiplicative form, *Mellin* inversion, as follows. Take $F \in C_c^\infty(0,+\infty)$, and put

$$f(x) \;=\; F(e^x)$$

Let $y = e^x$ and $r = e^t$

$$F(y) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} \Big(\int_{0}^{\infty} F(r)\, r^{-i\xi}\, \frac{dr}{r}\Big)\, y^{i\xi}\, d\xi$$

Define the transform $\mathcal{M}F$ by

$$\mathcal{M}F(i\xi) \;=\; \int_{-\infty}^{\infty} F(r)\, r^{-i\xi}\, \frac{dr}{r}$$

or, for complex s,

$$\mathcal{M}F(s) \;=\; \int_{-\infty}^{\infty} F(r)\, r^{-s}\, \frac{dr}{r}$$

Then we have the inversion formula

$$F(y) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{M}F(i\xi)\, y^{i\xi}\, d\xi$$

With $s = i\xi$ and $d\xi = -i\, ds$,

$$F(y) \;=\; \frac{1}{2\pi i}\int_{0-i\infty}^{0+i\infty} \mathcal{M}F(s)\, y^{s}\, ds$$

For $f \in C_c^\infty(\mathbb{R})$ the Fourier transform $\hat{f}(\xi)$ is in the Paley-Wiener space: it is *entire* in ξ and of rapid decay on horizontal lines, so the same is true of the transform $\mathcal{M}F$ of $F \in C_c^\infty(0,+\infty)$. For such F, for *any* real σ, there is an inversion formula

$$F(y) \;=\; \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{M}F(s)\, y^{s}\, ds$$

[8.4] Functional equations of Eisenstein series Taking

$$\varphi(g) \;=\; y^s \qquad\qquad (\text{with } g = z\begin{pmatrix} y & * \\ 0 & 1\end{pmatrix}\cdot k,\ z \text{ central},\ k \in O(2))$$

the usual spherical Eisenstein series E_s is

$$E_s(g) \;=\; \sum_{\gamma \in B\cap\Gamma\backslash\Gamma} \varphi(\gamma \cdot g)$$

Granting *meromorphic continuation* of E_s, the *functional equation* of E_s is determined by the constant term, as follows. Recall that the constant term of E_s is of the form

$$c_P E_s \;=\; y^s + c_s\, y^{1-s} \qquad\qquad \text{(with meromorphic } c_s\text{)}$$

Both E_s and E_{1-s} are eigenfunctions with the same eigenvalue $s(s-1)$ for the (image of the) Casimir operator

$$\Delta \;=\; y^2 \cdot \Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big)$$

From the *theory of the constant term*, a moderate-growth eigenfunction for Casimir, with (standard) constant term subtracted, is of rapid decay in (standard) Siegel sets. Consider a subtraction suggested by potential cancellation of parts of the constant term, namely

$$c_P\Big(E_{1-s} \;-\; \frac{E_s}{c_s}\Big) \;=\; \Big(y^{1-s} + c_{1-s}y^s\Big) \;-\; \Big(\frac{y^s + c_s y^{1-s}}{c_s}\Big) \;=\; (c_{1-s} - \frac{1}{c_s})\, y^s$$

For $\mathrm{Re}\, s < 0$ and off the real line, the Casimir eigenvalue $s(s-1)$ is not real, yet $\mathrm{Re}\, s < 0$ assures that y^s is square-integrable on any standard Siegel set. That is, the difference $E_{1-s} - \frac{1}{c_s} \cdot E_s$ is in $L^2(\Gamma\backslash\mathfrak{H})$. The Casimir operator is *self-adjoint* in at least the weak sense that any eigenvalue must be real. Therefore, the difference $E_{1-s} - \frac{1}{c_s} \cdot E_s$ is identically zero, which gives the functional equation and relation

$$E_{1-s} \;=\; \frac{E_s}{c_s} \qquad\qquad c_s \cdot c_{1-s} \;=\; 1$$

[8.5] Decomposition of pseudo-Eisenstein series For $\varphi \in C_c^\infty(N\backslash G/K) \approx C_c^\infty(0,+\infty)$ the inversion formula is

$$\varphi \;=\; \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} M\varphi(s)\, y^s\, ds$$

Winding up, the corresponding pseudo-Eisenstein series Ψ_φ is therefore

$$\Psi_\varphi \;=\; \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{M}\varphi(s) \cdot E_s\, ds \qquad\qquad \text{(with } \sigma = \mathrm{Re}\, s > 1 \text{ for convergence of } E_s\text{)}$$

Granting the meromorphic continuation of the Eisenstein series, move the vertical line of integration to the left, to $\sigma = 1/2$:

$$\Psi_\varphi \;=\; \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \mathcal{M}\varphi(s)\, E_s \;+\; \sum_{s_o} \mathrm{Res}_{s=s_o}\big(E_s \cdot \mathcal{M}\varphi(s)\big)$$

We prefer to have $\mathcal{M}c_P\Psi_\varphi$ enter the formula, not $\mathcal{M}\varphi$, to express things in terms of the automorphic forms Ψ_φ, not in terms of the auxiliary functions φ from which they're made. To this end, note the standard unwinding

$$\int_{\Gamma\backslash G} E_s \cdot f \;=\; \int_{P_{\mathbb{Z}}\backslash G} y^s\, c_P f\, \frac{dx\, dy}{y^2} \;=\; \int_0^\infty y^{-(1-s)}\, c_P f\, \frac{dy}{y} \;=\; \mathcal{M}c_P f(1-s)$$

On the other hand, unwinding the pseudo-Eisenstein series Ψ_φ gives

$$\int_{\Gamma\backslash G} E_s\, \Psi_\varphi \;=\; \int_{P_{\mathbb{Z}}\backslash G} c_P E_s \cdot \varphi\, \frac{dx\, dy}{y^2} \;=\; \int_0^\infty \big(y^s + c_s y^{1-s}\big) \cdot \varphi\, \frac{1}{y} \cdot \frac{dy}{y} \;=\; \mathcal{M}\varphi(1-s) + c_s\, \mathcal{M}\varphi(s)$$

Combining these two unwindings explains the constant term of pseudo-Eisenstein series without direct computation:

$$\mathcal{M}c_P\Psi_\varphi(1-s) \;=\; \int_{\Gamma\backslash G} E_s\cdot\Psi_\varphi \;=\; \mathcal{M}\varphi(1-s)+c_s\,\mathcal{M}\varphi(s)$$

Then the expression of Ψ_φ in terms of Eisenstein series is

$$\Psi_\varphi-(\text{residual part}) \;=\; \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\mathcal{M}\varphi(s)\,E_s\,ds \;=\; \frac{1}{2}\cdot\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\mathcal{M}\varphi(s)\,E_s+\mathcal{M}\varphi(1-s)\,E_{1-s}\,ds$$

$$=\;\frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\mathcal{M}\varphi(s)\,E_s+\mathcal{M}\varphi(1-s)\,c_{1-s}E_s\,ds \qquad\text{(by functional equation of } E_s)$$

$$=\;\frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\big(\mathcal{M}\varphi(s)+c_{1-s}\mathcal{M}\varphi(1-s)\big)\cdot E_s\,ds$$

$$=\;\frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\mathcal{M}c_P\Psi_\varphi(s)\cdot E_s\,ds \qquad\text{(recognizing constant term of } \Psi_\varphi)$$

That is, an pseudo-Eisenstein series is expressible as an integral of Eisenstein series E_s on the line $\mathrm{Re}(s)=1/2$, plus residues:

$$\Psi_\varphi-(\text{residual part}) \;=\; \frac{1}{2\pi i}\int_{\frac{1}{2}+i0}^{\frac{1}{2}+i\infty}\mathcal{M}c_P\Psi_\varphi(s)\cdot E_s\,ds \;=\; \frac{1}{2\pi i}\int_{\frac{1}{2}+i0}^{\frac{1}{2}+i\infty}\langle\Psi_\varphi,E_{1-s}\rangle\cdot E_s\,ds$$

[8.6] Plancherel for continuous spectrum

Let

$$\langle f_1,f_2\rangle \;=\; \int_{\Gamma\backslash G} f_1\,f_2\,dg \qquad (\mathbb{C}\text{-bilinear})$$

Let $f\in C_c^\infty(\Gamma\backslash G)$, $\varphi\in C_c^\infty(N\backslash G)$, and assume Ψ_φ is orthogonal to residues of Eisenstein series, that is, to constants. Using the expression for Ψ_φ in terms of Eisenstein series,

$$\langle\Psi_\varphi,f\rangle \;=\; \Big\langle\frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\langle\Psi_\varphi,E_{1-s}\rangle\cdot E_s\,ds,\;\; f\Big\rangle \;=\; \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\langle\Psi_\varphi,E_{1-s}\rangle\cdot\langle E_s,f\rangle\,ds$$

This proves that $f\to(s\to\langle f,E_s\rangle)$ is an inner-product-preserving map from the Hilbert-space span of the pseudo-Eisenstein series to $L^2(\frac{1}{2}+i\mathbb{R})$.

The map $\Psi_\varphi\to\langle\Psi_\varphi,E_{1-s}\rangle$ produces functions $u(t)=\langle\Psi_\varphi,E_{1-s}\rangle$ satisfying the relation

$$u(-t) \;=\; \langle\Psi_\varphi,E_s\rangle \;=\; \langle\Psi_\varphi,c_sE_{1-s}\rangle \;=\; c_s\langle\Psi_\varphi,E_{1-s}\rangle \;=\; c_s\cdot u(t)$$

We claim that *any* $u\in L^2(\frac{1}{2}+i\mathbb{R})$ satisfying $u(-t)=c_s\,u(t)$ is in the image. First, claim that, for compactly-supported u satisfying $u(-t)=c_s\,u(t)$

$$\Phi_u \;=\; \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}u(t)\cdot E_{\frac{1}{2}+it}\,dt \;\neq\; 0$$

It suffices to show $c_P\Phi_u$ is not 0. With $s=\frac{1}{2}+it$, the relation implies $u(-t)E_{1-s}=u(t)c_s\cdot E_{1-s}/c_s=u(t)E_s$. Then

$$\Phi_u \;=\; \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}u(t)\cdot E_s\,dt \;=\; \frac{1}{2\pi i}\int_{\frac{1}{2}+0}^{\frac{1}{2}+i\infty}u(t)\cdot E_s\,dt$$

The constant term of Φ_u is

$$c_P\Phi_u = \frac{1}{2\pi i}\int_{\frac{1}{2}+0i}^{\frac{1}{2}+i\infty} u(t)\cdot(y^s+c_s y^{1-s})\;dt \;=\; \frac{1}{2\pi i}\int_{\frac{1}{2}+0i}^{\frac{1}{2}+i\infty} u(t)y^{\frac{1}{2}+it}+u(-t)y^{\frac{1}{2}-it}\;dt \;=\; \frac{\sqrt{y}}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} u(t)\,e^{it\log y}\;dt$$

This Fourier transform does not vanish for non-vanishing u.

Since the E_s integrate to 0 against cuspforms, an integral Φ_u of them does, also. Thus, Φ_u is in the topological closure of pseudo-Eisenstein series Ψ_φ with test-function data φ. Thus, given u, there is φ such that $\langle\Psi_\varphi,\Phi_u\rangle \neq 0$. Then

$$0 \;\neq\; \langle\Psi_\varphi,\Phi_u\rangle \;=\; \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} u(t)\cdot\langle\Psi_\varphi,E_{1-s}\rangle\;dt$$

Thus, the functions $s\to\langle\Psi_\varphi,E_s\rangle$ are dense in the space of $L^2(\frac{1}{2}+i\mathbb{R})$ functions u satisfying $u(-t)=c_s\,u(t)$. Thus, there is an *isometry*

$$\{\text{cuspforms}\}^\perp\cap L^2(\Gamma\backslash G)^K \quad\approx\quad \{u\in L^2(\Gamma\backslash G/K) : u(-t)=c_s\cdot u(t)\}$$

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9. *Appendix: Meromorphic continuation for* $PSL_2(\mathbb{Z})$

[9.1] Analytic continuation and functional equation

For $(c\;d)=v\in\mathbb{R}^2$, consider the Gaussian

$$\varphi(v) \;=\; e^{-\pi|v|^2} \;=\; e^{-\pi(c^2+d^2)}$$

where $v\to|v|$ is the usual length function on $\mathbb{R}^2$. For $g\in GL_2(\mathbb{R})$, define

$$\Theta(g) \;=\; \sum_{v\in\mathbb{Z}^2}\varphi(v\cdot g) \;=\; \sum_{(c,d)\in\mathbb{Z}^2} e^{-\pi|(c,d)g|^2}$$

where $v\in\mathbb{R}^2$ is a row vector. Consider the integral (a Mellin transform)

$$\int_0^\infty t^{2s}\,(\Theta(tg)-1)\,\frac{dt}{t}$$

where the t in the argument of Θ simply acts by scalar multiplication on $g\in GL_2(\mathbb{R})$. On one hand, integrating term-by-term gives

$$\int_0^\infty t^{2s}\,(\Theta(tg)-1)\,\frac{dt}{t} \;=\; \sum_{v\neq(0,0)}\int_0^\infty t^{2s}\,e^{-\pi|tvg|^2}\,\frac{dt}{t}$$

Since

$$\pi|tvg|^2 \;=\; (t\cdot\sqrt{\pi}|vg|)^2$$

we can change variables by replacing t by $t/(\sqrt{\pi}|vg|)$ to obtain

$$\sum_{v\neq(0,0)}(\sqrt{\pi}|vg|)^{-2s}\int_0^\infty t^{2s}\,e^{-t^2}\,\frac{dt}{t} \;=\; \frac{1}{2}\pi^{-s}\sum_{v\neq(0,0)}|vg|^{-2s}\int_0^\infty t^s\,e^t\,\frac{dt}{t}$$

$$= \frac{1}{2}\pi^{-s}\,\Gamma(s) \sum_{v\neq(0,0)} |vg|^{-2s}$$

Now we want $g \in SL(2,\mathbb{R})$ of a simple sort chosen to map $i \to x+iy$. One reasonable choice is

$$g \;=\; \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix}$$

Using this choice of G and writing out $v = (c,d)$ gives

$$vg \;=\; (c,d)g \;=\; (\,c \quad d\,) \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} \;=\; (c\sqrt{y},\; (cx+d)/\sqrt{y})$$

and thus

$$\sum_v |vg|^{-2s} \;=\; \sum_v |(c\sqrt{y}, (cx+d)/\sqrt{y})|^{-2s} \;=\; \sum_v (c^2y + (cx+d)^2/y)^{-s}$$

$$= \sum_v \frac{y^s}{(c^2y^2 + (cx+d)^2)^s} \;=\; \sum_v \frac{y^s}{|ciy + cx + d|^{2s}} \;=\; \sum_v \frac{y^s}{|cz+d|^{2s}}$$

Letting $1 \le \delta = \gcd(c,d)$, this is

$$\sum_v \frac{y^s}{|cz+d|^{2s}} \;=\; \sum_\delta \frac{1}{\delta^{2s}} \sum_{\text{coprime}\, c,d} \frac{y^s}{|cz+d|^{2s}} \;=\; 2\,\zeta(2s)\cdot E_s(z)$$

The expression

$$2\,\zeta(2s)\,E_s(z) \;=\; \sum_{(c,d)\neq(0,0)} \frac{y^s}{|cz+d|^{2s}} \qquad \text{(summing } (c,d) \text{ over } \textit{all} \text{ non-zero vectors in } \mathbb{Z}^2\text{)}$$

is convenient, being a sum over a lattice with 0 removed.

Thus, we see that the integral representation yields the Eisenstein series with a leading power of π, a gamma function, and a factor of $\zeta(2s)$:

$$\int_0^\infty t^{2s}\,(\Theta(tg) - 1)\,\frac{dt}{t} \;=\; 2\,\pi^{-s}\,\Gamma(s)\,\zeta(2s)\,E_s(g)$$

On the other hand, to prove the meromorphic continuation, use the integral representation as in Riemann's corresponding argument for $\zeta(s)$, first breaking the integral into two parts, one from 0 to 1, and the other from 1 to $+\infty$. Keep $g \in SL(2,\mathbb{R})$ in a compact subset of $SL(2,\mathbb{R})$. Then

$$\int_1^\infty t^{2s}\,(\Theta(tg) - 1)\,\frac{dt}{t} \;=\; \text{entire in } s$$

since elementary estimates show that the integral is uniformly and absolutely convergent. Apply Poisson summation to the kernel: first note that the Gaussian $\varphi(v) = e^{-\pi|v|^2}$ is its own Fourier transform, and that

$$\text{Fourier transform of } (v \to \varphi(tvg)) \;=\; (v \to t^{-2}\det(g)^{-1}\cdot\varphi(t^{-1}v\,{}^\top g^{-1}))$$

where ${}^\top g$ is g-transpose. Then Poisson summation asserts

$$\Theta(tg) \;=\; t^{-2}\,\det(g)^{-1}\cdot\Theta(t^{-1}\,{}^\top g^{-1})$$

The modification for the kernel gives

$$\Theta(tg) - 1 \;=\; t^{-2}\,\det(g)^{-1}\cdot[\Theta(t^{-1}\,{}^\top g^{-1}) - 1] + t^{-2}\,\det(g)^{-1} - 1$$

Then tranform the integral from 0 to 1: at first only for $\mathrm{Re}(s) > 1$,

$$\int_0^1 t^{2s}\,(\Theta(tg)-1)\,\frac{dt}{t} \;=\; \int_0^1 t^{2s}\,\left(t^{-2}\,\det(g)^{-1}\cdot[\Theta(t^{-1}\,{}^\top g^{-1})-1] + t^{-2}\,\det(g)^{-1} - 1\right)\,\frac{dt}{t}$$

Replacing t by $1/t$ turns this into

$$\int_1^\infty t^{-2s}\,\left(t^{2}\,\det(g)^{-1}\cdot[\Theta(t\,{}^\top g^{-1})-1] + t^{2}\,\det(g)^{-1} - 1\right)\,\frac{dt}{t}$$

Explicitly evaluating the last two elementary integrals of powers of t from 1 to ∞, using $\mathrm{Re}(s) > 1$, this is

$$\det(g)^{-1}\int_1^\infty t^{2-2s}\,(\Theta(t\,{}^\top g^{-1})-1)\,\frac{dt}{t} + \frac{\det(g)^{-1}}{2s-2} - \frac{1}{2s}$$

That g has determinant 1 to simplifies this to

$$\int_1^\infty t^{2-2s}\,(\Theta(t\,{}^\top g^{-1})-1)\,\frac{dt}{t} + \frac{1}{2s-2} - \frac{1}{2s}$$

Further, for g in $SL(2)$,

$${}^\top g^{-1} \;=\; wgw^{-1}$$

where w is the *long Weyl element*

$$w \;=\; \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Since $\mathbb{Z}^2 - (0,0)$ is stable under w, and since the length function $v \to |v|^2$ is invariant under w,

$$\Theta(g) \;=\; \Theta(wg) \;=\; \Theta(gw^{-1})$$

so

$$\Theta({}^\top g^{-1}) \;=\; \Theta(g)$$

Thus, the original integral from 0 to 1 becomes

$$\int_1^\infty t^{2-2s}\,(\Theta(tg)-1)\,\frac{dt}{t} + \frac{1}{2s-2} - \frac{1}{2s}$$

and the *whole* equality, with g of the special form above, is

$$\frac{1}{2}\pi^{-s}\Gamma(s)\,\zeta(2s)\,E_s(z) \;=\; \int_1^\infty t^{2s}\,(\Theta(tg)-1)\,\frac{dt}{t} + \int_1^\infty t^{2-2s}\,(\Theta(tg)-1)\,\frac{dt}{t} + \frac{1}{2s-2} - \frac{1}{2s}$$

or (multiplying through by 2)

$$\pi^{-s}\Gamma(s)\,\zeta(2s)\,E_s(z) \;=\; 2\int_1^\infty t^{2s}\,(\Theta(tg)-1)\,\frac{dt}{t} + 2\int_1^\infty t^{2-2s}\,(\Theta(tg)-1)\,\frac{dt}{t} - \frac{1}{1-s} - \frac{1}{s}$$

The integral from 1 to ∞ is nicely convergent for all $s \in \mathbb{C}$, uniformly in g in compacts. The elementary rational expressions of s have meromorphic continuations. Thus, the right-hand side gives a meromorphic continuation of the Eisenstein series, and is visibly invariant under $s \to 1-s$.

It is also visible that the only poles are at $s = 1, 0$, that the residue at $s = 1$ is the constant function 1, and at $s = 0$ the residue is the constant function 0. At $s = 1$ the factor $\pi^{-s}\Gamma(s)$ is holomorphic and has value $1/\pi$, so

$$\mathrm{Res}_{s=1}\,\zeta(2s)\,E_s \;=\; \pi$$

At $s = 0$ the factor $\pi^{-s}\Gamma(s)$ has a simple pole with residue 1, so $\zeta(2s)\, E_s$ itself is holomorphic at $s = 0$, and is the constant function 1.

Now we recover the assertions for E_s itself. The convergence of the infinite product

$$\zeta(2s) \;=\; \sum_n \frac{1}{n^{2s}} \;=\; \prod_{p \text{ prime}} \frac{1}{1-p^{-2s}}$$

for $\mathrm{Re}\,(s) > 1/2$ assures that $\zeta(2s)$ is not zero for $\mathrm{Re}\,(s) > 1/2$. And $\zeta(2) = \pi^2/6$. These standard facts and the previous discussion give the full result. ///

[9.2] Vertical growth in s

As should be expected, estimates on vertical growth are applications of Phragmén-Lindelöf to the entire function

$$\widetilde{E}_s(z) \;=\; s(1-s)\cdot \pi^{-s}\,\Gamma(s)\,\zeta(2s)\cdot E_s(z)$$

for z in a fixed compact subset C of $\mathfrak{H}$. For $\mathrm{Re}\,(s) = 1+\delta$ with $\delta > 0$, for $z \in C$ the Eisenstein series $E_s(z)$ is *bounded*. Similarly, $\zeta(2s)$ is bounded there, as is the power of π. The gamma function is bounded on $\mathrm{Re}\,(s) = 1+\delta$, in fact, of rapid decay, so $\widetilde{E}_s(z)$ is bounded there.

Via the functional equation, $\widetilde{E}_s(z)$ is bounded on $\mathrm{Re}\,(s) = -\delta$, uniformly for $z \in C$. By Phragmén-Lindelöf, $\widetilde{E}_s(z)$ is uniformly bounded for $z \in C$ and $-\delta \le \mathrm{Re}\,(s) \le 1+\delta$.

Bibliography

The following are standard representatives of prior art and larger context. The above discussion is not literally logically dependent on these, but is an explication of a worked example.

[Borel 1965/6] A. Borel, *Introduction to automorphic forms*, in *Algebraic Groups and Discontinuous Subgroups, Boulder, 1965*, Proc. Symp. Pure Math. IX, AMS, 1966, 199-210.

[Casselman 2005] W. Casselman, *A conjecture about the analytical behavior of Eisenstein series*, Pure and Applied Math. Q **1** (2005) no. 4, part 3, 867-888.

[Godement 1963] R. Godement, *Domaines fondamentaux des groupes arithmetiques*, Sem. Bourb. **257** (1962-3).

[Godement 1966] R. Godement, *Decomposition of* $L^2(\Gamma\backslash G)$ *for* $\Gamma = SL(2,\mathbb{Z})$, in Proc. Symp. Pure Math. **9** (1966), AMS, 211-24.

[Harish-Chandra 1959] Harish-Chandra, *Automorphic forms on a semi-simple Lie group*, Proc. Nat. Acad. Sci. **45** (1959), 570-573.

[Harish-Chandra 1968] Harish-Chandra, *Automorphic forms on semi-simple Lie groups*, notes by G.J.M. Mars, SLN **62**, Springer-Verlag, 1968.

[Hejhal 1976], D. Hejhal, *The Selberg trace formula for* $PSL_2(\mathbb{R})$*, I*, SLN **548**, Springer-Verlag, 1976

[Hejhal 1983], D. Hejhal, *The Selberg trace formula for* $PSL_2(\mathbb{R})$*, II*, SLN **1001**, Springer-Verlag, 1983.

[Iwaniec 2002] H. Iwaniec, *Spectral methods of automorphic forms*, Graduate Studies in Mathematics **53**, AMS, 2002.

[Jacquet 1983] H. Jacquet, *On the residual spectrum of* $GL(n)$, in Lie Group Representations, II, Lecture Notes in Math. **1041**, Springer-Verlag, 1983.

[Langlands 1967/1976] R.P. Langlands, *On the functional equations satisfied by Eisenstein series*, Lecture Notes in Mathematics, vol. 544, Springer-Verlag, Berlin and New York, 1976.

[Moeglin-Waldspurger 1989] C. Moeglin, J.-L. Waldspurger, *Le spectre résiduel de* GL_n, with appendix *Poles des fonctions L de pairs pour* GL_n, Ann. Sci. École Norm. Sup. **22** (1989), 605-674.

[Moeglin-Waldspurger 1995] C. Moeglin, J.-L. Waldspurger, *Spectral Decompositions and Eisenstein series*, Cambridge Univ. Press, Cambridge, 1995.

[Rankin 1939] R. Rankin, *Contributions to the theory of Ramanujan's function* $\tau(n)$ *and similar arithmetic functions, I*, Proc. Cam. Phil. Soc. **35** (1939), 351-372.

[Selberg 1956] A. Selberg, *Harmonic analysis and discontinuous groups in weakly symmetric spaces, with applications to Dirichlet series*, J. Indian Math. Soc. **20** (1956), 47-87

[Shahidi 2010] F. Shahidi, *Eisenstein series and automorphic L-functions*, AMS Colloquium Publ. **58**, AMS, 2010.

[Wong 1990] S.-T. Wong, *The meromorphic continuation and functional equations of cuspidal Eisenstein series for maximal cuspidal groups*, Memoirs of AMS vol. 83, no. 423, 1990.