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A rationality principle

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The following rationality principle was used by Klingen *circa* 1960 in his result about special values of L-functions for finite-order characters on totally real number fields.

Theorem: (*Klingen*) The constant term c_o in the Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} c_n \, e^{2\pi i n z}$$

of a holomorphic elliptic modular form $f(z)$ lies in the field $\mathbf{Q}(c_1, c_2, \ldots)$ generated over $\mathbf{Q}$ by the higher Fourier coefficients. (*This formulation is slightly misleading.*)

Let V be a finite-dimensional vector space over a field K. Let Λ be a set of K-linear maps from V to K. Let k be a subfield of K, and let

$$V_o = \{v \in V : \lambda v \in k, \forall \lambda \in \Lambda\}$$

Suppose that V_o spans V over K. Let Λ^+ be a subset of Λ with the property that

$$\bigcap_{\lambda \in \Lambda^+} \ker \lambda = \{0\}$$

Note that the latter *separation* condition does *not* imply that all K-linear functionals on V are necessarily linear combinations of elements of Λ^+, unless (for example) we add the finite-dimensionality condition on V.

Theorem: An element $\mu \in \Lambda$, $\mu \notin \Lambda^+$, is a k-linear (not merely K-linear) combination of elements of Λ^+.

Proof: Let Λ_o^+ be a maximal linearly-independent subset of Λ^+. Since V is finite-dimensional,

$$\text{card}\, \Lambda_o^+ = \dim_K V$$

Thus, any $\mu \in \Lambda$ is a (finite) K-linear combination of elements of Λ_o^+

$$\mu + \sum_{\lambda \in \Lambda_o^+} c_\lambda \, \lambda$$

with $c_\lambda \in K$. Again by finite-dimensionality, we can choose a K-basis $\{v_\nu\}$ for V indexed by elements ν of Λ_o^+, such that

$$\lambda(v_\nu) = \begin{cases} 1 & (\lambda = \nu) \\ 0 & (\lambda \neq \nu) \end{cases}$$

Thus, V_o is a k-subspace of

$$\{ \sum_{\lambda \in \Lambda_o^+} b_\lambda \, v_\lambda : b_\lambda \in k\}$$

If V_o were strictly smaller it could not span V over K, so V_o must be exactly this space. Then all the v_λ's are in V_o. In particular, then,

$$c_\lambda = \mu(v_\lambda) \in k$$

Thus, μ is in fact a k-linear combination of the functionals in Λ_o^+, as claimed. ///

With a little attention to notions of rational structure on infinite-dimensional vector spaces, the latter result can be extended easily to certain infinite-dimensional cases.

Corollary: Let Let Λ be a collection of K-linear functionals on V. Let Λ^+ be a subset of Λ such that

$$\bigcap_{\lambda \in \Lambda^+} \ker \lambda = \{0\}$$

Suppose that $V = \text{colim} V_i$ be an ascending union (colimit) of finite-dimensional K-vectorspaces V_i such that

$$V_{i,o} = \{v \in V_i : \lambda v \in k \text{ for all } \lambda \in \Lambda\}$$

spans V_i over K for all i. Then $\mu \in \Lambda$ is a k-linear (not merely K-linear) combination of elements of Λ^+. ///