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Compactness of arithmetic quotients of anisotropic orthogonal groups

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These notes give the beginning of a treatment of reduction theory for classical groups following Tamagawa-Mostow and Godement's Bourbaki article. For the moment, the general non-compact quotient case is completely neglected.

- Affine heights
- Minkowski reduction
- Imbeddings of arithmetic quotients
- Mahler's criterion for compactness
- Compactness of anisotropic quotients of orthogonal groups

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1. Affine heights

Let $\mathbf{Q}$ be the rational number field and $\mathbf{A}$ its adeles. Let K_v be the standard (maximal!) compact subgroup of $GL(n,\mathbf{Q}_v)$: namely, for $\mathbf{Q}_v \approx \mathbf{R}$ it is the usual orthogonal group $O(n)$, and for $\mathbf{Q}_v$ non-archimedean it is $GL(n,\mathbf{Z}_v)$, where $\mathbf{Z}_v$ is the ring of local integers. (The fact that theses subgroups are *maximal* compact will not be needed.)

The adele group $GL(n,\mathbf{A})$ over $\mathbf{Q}$ is a restricted direct product of the local groups $GL(n,\mathbf{Q}_v)$ with respect to the subgroups K_v: that is, $GL(n,\mathbf{A})$ is the group consisting of elements

$$g = \{\{g_v\} : g_v \in GL(n,\mathbf{Q}_v),\ \text{almost all } g_v \in K_v\}$$

Let $V = \mathbf{Q}^n$, and $V_\mathbf{A} = V \otimes \mathbf{A}$. By definition, $GL(n,\mathbf{A})$ is the set of tuples

$$\{g_v : v \text{ is a place of } \mathbf{Q}\ \}$$

meeting the condition that $g_v \in GL(n,\mathbf{Q}_v)$ and for almost all places v the v^{th} component g_v actually lies in $GL(n,\mathbf{Z}_v)$, where $\mathbf{Z}_v$ is the local ring of integers. We let $GL(n,\mathbf{A})$ act **on the right** on $\mathbf{A}^n$, by matrix multiplication. (There are several reasons to choose this, rather than an action on the left.)

For the *real* prime v of $\mathbf{Q}$ define the **local height** function η_v on $x = (x_1,\ldots,x_n) \in V_{\mathbf{Q}_v} = \mathbf{Q}_v^n$ by

$$\eta_v(x) = \sqrt{x_1^2 + \ldots + x_n^2}$$

For a *non-archimedean* prime v of $\mathbf{Q}$ define the **local height** function η_v on $x = (x_1,\ldots,x_n) \in V_{\mathbf{Q}_v} = \mathbf{Q}_v^n$ by

$$\eta_v(x) = \sup_i |x_i|_v$$

where $|*|_v$ is the v^{th} from among the normalization of norms so that the product formula holds.

A vector $x \in V_\mathbf{A}$ is **primitive** if it is of the form $x_o g$ where $g \in GL(n,\mathbf{A})$ and $x_o \in V_\mathbf{Q}$. That is, it is an image of a *rational* point of the vectorspace by an element of the *adele* group. For $x = (x_1,\ldots,x_n) \in V_\mathbf{Q}$, at almost all non-archimedean primes v the x_i's are in $\mathbf{Z}_v$ and have greatest common divisor 1 (locally). Since elements of the adele group are in K_v almost everywhere, this property is not changed by multiplication by $g \in GL(n,\mathbf{A})$. That is, any primitive vector x has the property that at almost all v the components of x are locally integral and have (local) greatest common divisor 1.

For primitive $x \in V_{\mathbf{A}}$ we can define the **global height** function

$$\eta(x) = \prod_v \eta_v(x_v)$$

where the vector $x_v \in \mathbf{Q}_v^n$ is the v^{th}-prime component of x. Since x is primitive, at almost all finite primes the local height is 1, so this product has only finitely many factors which are not simply 1.

• For $t \in \mathbf{J}$ and primitive $x \in \mathbf{A}^n$, $\eta(tx) = |t|\eta(x)$, where $|t|$ is the idele norm.
• If a sequence of vectors in $\mathbf{A}^n$ goes to 0, then their heights go to zero also.
• If the heights of some (primitive) vectors x_i go to zero, then there are scalars $t_i \in \mathbf{Q}^\times$ so that $t_i x_i$ goes to 0 in $\mathbf{A}^n$.
• For $g \in GL(n, \mathbf{A})$ and $c > 0$, the set of non-zero vectors $x \in \mathbf{Q}^n$ so that $\eta(xg) < c$ is finite modulo $\mathbf{Q}^\times$. In particular, the infimum of $\{\eta(xg) : x \in \mathbf{Q}^n - 0\}$ is positive, and is assumed.
• For a compact subset E of $GL(n, \mathbf{A})$ there are constants $c, c' > 0$ so that for all primitive vectors x and for all $g \in E$

$$c\,\eta(x) \le \eta(xg) \le c'\,\eta(x)$$

Proof: The very first assertion follows from the product formula and from the fact that the local heights behave analogously.

For the second assertion: if a sequence of vectors x_i goes to 0, then for every large $N > 0$ and small $\varepsilon > 0$ there is i_o so that $i \ge i_o$ implies $\eta_v(x_v) < \varepsilon$ at archimedean primes, and $x_v \in N\mathbf{Z}_v^n$ for every finite v. Then $\eta(x) \le \varepsilon^\ell/N$ where ℓ is the number of archimedean primes. So the heights go to zero.

For the third assertion: suppose that $\eta(x_i)$ goes to 0, for some primitive vectors x_i. As noted when we defined primitive vectors, at almost all (depending upon x_i) finite primes v the vector x_i is in $\mathbf{Z}_v^n$ and the entries have local gcd 1. Since $\mathbf{Z}$ is a principal ideal domain, we can choose $s_i \in \mathbf{Q}$ to that at *every* finite prime v the components of $s_i x_i$ are locally integral, and have greatest common divisor 1. Then the local contribution to the height function from *all* finite primes is just 1. Therefore, the archimedean height of $s_i x_i$, which is merely Euclidean distance, goes to 0. Finally, we need some choice of trick to make the vectors go to 0 in $\mathbf{A}^n$. For example, for each index i let N_i be the greatest integer so that

$$\eta_\infty(s_i x_i) < \frac{1}{(N_i!)^2}$$

Let $t_i = s_i \cdot N_i!$. Then $t_i x_i$ goes to 0 in $\mathbf{A}^n$.

For the fourth assertion: fix $g \in GL(n, \mathbf{A})$. Since the compact groups K_v preserve heights, by the Iwasawa decomposition we may suppose that g is upper-triangular. Let g_{ij} be the ij^{th} entry. Choose representatives for $(\mathbf{Q}^n - 0)/\mathbf{Q}^\times$ consisting of vectors $x = (x_1, \ldots, x_n)$ so that if m is the first index with x_m non-zero then $x_m = 1$. Note that for any index j

$$\eta_v(x) \ge |x_j|_v$$

for all primes v so

$$\eta_v(xg) \ge |(xg)_j|_v$$

If $m < n$, taking $j = m+1$ gives

$$|x_{m+1} g_{m+1,m+1} + g_{m,m+1}|_v \prod_{v' \ne v} |g_{m+1,m+1}|_{v'} \le \eta(xg)$$

Here we use $|(xg)_m|_{v'} = |g_{mm}|_{v'}$ as a lower bound for the local height at primes v' different from v, and $|(xg)_{m+1}|_v$ as the lower bound at v. Then the inequality asserts that (for fixed g and c the quantity $x_{m+1} \in \mathbf{Q}_v$ lies in a compact subset of $\mathbf{Q}_v$. Since this holds for all v, it must be that x_{m+1} lies in a compact subset of $\mathbf{A}$. But since also x_{m+1} is in $\mathbf{Q}$, which is discrete (closed) in $\mathbf{A}$, there are only finitely-many x_{m+1} which can satisfy this inequality.

Continuing, for fixed x_{m+1}, use the $m+2$ entry of xg (if $m+1 \le n$) in analogous fashion to obtain

$$|x_{m+2}g_{m+2,m+2} + x_{m+1}g_{m+1,m+2} + g_{m,m+2}|_v \prod_{v' \neq v} |g_{m,m}| \le \eta(xg) \le c$$

As for x_{m+1}, this shows that x_{m+2} lies in a compact subset of $\mathbf{A}$, and since it lies in the discrete (closed) subset $\mathbf{Q}$ of $\mathbf{A}$ there are only finitely-many posibilities. Continuing further in this manner gives the asserted finiteness, proving the fourth assertion above.

For the last assertion: let E be a compact subgroup of $GL(n, \mathbf{A})$, and let $K = \prod_v K_v$ be the standard (maximal) compact subgroup of $GL(n, \mathbf{A})$. Then $K \cdot E \cdot K$ is still compact, being the continuous image of a compact set. So without loss of generality we may suppose that E is left and right K-stable. Then by local Cartan decompositions the compact set E of $GL(n, \mathbf{A})$ is equal to

$$K \, \Delta \, K$$

where Δ is a set of diagonal matrices in $GL(n, \mathbf{A})$. Necessarily Δ is compact. Let $g = \theta_1 \delta \theta_2$ with $\theta_i \in K$, and x a primitive vector. Then, using the K-invariance of the height function,

$$\eta(xg)/\eta(x) = \eta(x\theta_1\delta\theta_2)/\eta(x) = \eta(x\theta_1\delta)/\theta(x) = \eta(x\delta)/\eta(x\theta_1^{-1})$$

Thus, the set of ratios $\eta(xg)/\eta(x)$ for g in a compact set and x ranging over primitive vectors is exactly the set of values $\eta(x\delta)/\eta(x)$ where δ ranges over a compact set and x ranges. Let the diagonal entries of such δ be δ_i. Then

$$0 < \inf_{\delta \in \Delta} \inf_i |\delta_i| \le \eta(x\delta)/\eta(x) \le \sup_{\delta \in \Delta} \sup_i |\delta_i| < \infty$$

by compactness of Δ. This proves the last assertion above.

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2. Minkowski reduction

The previous preparations set things up to prove the basic reduction-theory result for non-compact quotients: we prove that there is a nice **approximate fundamental domain** for the action of $GL(n, \mathbf{Q})$ on $GL(n, \mathbf{A})$.

Theorem: *(Adelic form of Minkowski reduction)* Given $g \in GL(n, \mathbf{A})$, there are $\gamma \in GL(n, \mathbf{Q})$ and $\theta \in K$ so that

$$p = \gamma g \theta$$

is **upper-triangular** and so that the diagonal entries p_{ii} of p satisfy the **inequalities**

$$|p_{ii}/p_{i+1\, i+1}| \ge \frac{\sqrt{3}}{2} \quad \text{(idele norm)}$$

Further, for $i < j$, the entry p_{ij} of p can be arranged to lie in any specified set of representatives in $\mathbf{A}$ for the quotient

$$p_{ii}\mathbf{Q} \backslash \mathbf{A}$$

Remark: This result, combined with Strong Approximation for $SL(n)$, recovers the classical Minkowski reduction result for $SL(n, \mathbf{Z})$ acting on $SL(n, \mathbf{R})$. More importantly, it is the beginning of the sequence of results that gives the most general fundamental domain results (in terms of **Siegel sets**) later.

From the previous discussion, given $g \in GL(n, \mathbf{A})$ there is $x \in \mathbf{Q}^n - 0$ so that $\eta(xg) > 0$ is minimal among all values $\eta(x'g)$ with $x \in \mathbf{Q}^n - 0$. Take $\gamma \in GL(n, \mathbf{Q})$ so that $e_n\gamma = x$, where $\{e_i\}$ is the standard basis for $\mathbf{Q}^n$. By the Iwasawa decomposition, there is $\theta \in K$ so that $p = \gamma g \theta$ is upper-triangular. Then

$$\eta(\gamma g \theta) = |p_{nn}|$$

where p_{ij} is the ij^{th} entry of p. Let H be the subgroup of $GL(n, \mathbf{A})$ fixing e_n and stabilizing the subspace spanned by $e_1, \ldots, e_{n-1}$. Then $H \approx GL(n, \mathbf{A})$, and by induction we can suppose that $|p_i/p_{i+1,i+1}| \geq \frac{\sqrt{3}}{2}$ already for $i < n-1$. And by looking at just the lower-right two-by-two block inside these n-by-n matrices, it suffices to consider just the case that $n = 2$. This allows us to use a less cumbersome notation.

So, treating just the case $n = 2$, repeating a bit: given $g \in GL(2, \mathbf{A})$ there is $x \in \mathbf{Q}^2 - 0$so that $\eta(xg)$ is positive and minimal among all the values $\eta(x'g)$ with $x \in \mathbf{Q}^2 - 0$. Take $\gamma \in GL(2, \mathbf{Q})$ so that $(0\ 1)\gamma = x$. By the Iwasawa decomposition there is θ in the standard maximal compact subgroup K of $GL(2, \mathbf{A})$ so that $p = \gamma g \theta$ is upper-triangular, say

$$p = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

We wish to see that the minimality of$\eta(xg) = \eta((0\ 1)p)$ gives the inequality $|a/d| \geq \frac{\sqrt{3}}{2}$. Let $x' = (1, t) \in \mathbf{Q}^2$. The inequality

$$\eta((0\ 1)p) \leq \eta(x'p)$$

gives

$$|d| \leq \eta(a, b + dt)$$

for all $t \in \mathbf{Q}$. Let $r = a/d$ and $s = b/d$, for brevity. Dividing through by d gives, by elementary properties of the height function,

$$1 \leq \eta(r, s + t)$$

By changing the vector $(r, s + t)$ by an element of $\mathbf{Q}^\times$, we can suppose that the idele r is a local unit at all finite primes of $\mathbf{Q}$. Further, by right-multiplying by suitable elements of the form

$$\begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix}$$

in the standard maximal compact subgroups at finite primes, we may actually suppose that the idele r is 1 at all finite primes.

For given $s \in \mathbf{A}$, we can certainly choose $t \in \mathbf{Q}$ so that $s + t$ is integral at all finite primes and so that $|s + t|_\infty \leq \frac{1}{2}$. With this choice of $t \in \mathbf{Q}$, the height of $(r, s + t)$ is

$$\eta(r, s + t) = \eta_\infty(r, s + t) = \sqrt{|r|^2_\infty + |s + t|^2_\infty} \leq \sqrt{|r|^2_\infty + \frac{1}{4}}$$

Then from $1 \leq \eta(r, s + t)$ we square out to obtain

$$1 \leq |r|^2_\infty + \frac{1}{4}$$

which gives

$$\frac{\sqrt{3}}{2} \leq |r|_\infty$$

Since the idele r was arranged to be a local unit at all finite primes, this is

$$\frac{\sqrt{3}}{2} \leq |r|$$

Also, since $|r| = |a/d|$ we obtain the desired

$$\frac{\sqrt{3}}{2} \leq |\frac{a}{d}|$$

This proves the theorem. ///

Remark: The proof of Minkowski reduction certainly does make use of particulars about $\mathbf{Q}$, such as the fact that it is a Euclidean ring. This argument does not in any simple way generalize to more general situations. Rather, a relatively complicated argument based on this theorem and others below *reduces* the general case to this. In the end, the general *conclusion* is analogous but the proof is different.

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3. Imbeddings of arithmetic quotients

Let k be a number field. Let $Q = \langle , \rangle$ be a non-degenerate quadratic form on a k-vectorspace V, and $G = O(Q)$ the corresponding orthogonal group. We have the natural imbedding $G \to GL(V)$.

Proposition: The inclusion $G_k \to GL(V)_k$ induces an inclusion

$$G_k \backslash G_{\mathbf{A}} \to GL(V)_k \backslash GL(V)_{\mathbf{A}}$$

with closed image.

Before proving this, we prove a general topological lemma that is necessary here.

Lemma: Let X, Y be locally compact Hausdorff topological spaces. Further, suppose that there is a countable collection of opens U_i covering X so that each U_i has compact closure. Let G be a group acting continuously on X and acting continuously on Y. Suppose that the action of G on X is **transitive**. Let $f : X \to Y$ be a continuous injective map whose image is a closed subset of Y. Further, suppose that f **commutes** with the action of G in the sense that

$$g(f(x)) = f(g(x))$$

Then f is a homeomorphism of X to its image in Y.

Proof: This is an application of a version of the Baire Category argument. First, since $f(X)$ is closed in Y this image $f(X)$ is itself (with the subset topology) a locally compact Hausdorff space with countable basis. Therefore, without loss of generality, we may assume that f is surjective. Let C_i be the closure of U_i. Then the images $f(C_i)$ of the C_i are compact, hence closed (by Hausdorff-ness). We first claim that some $f(C_i)$ must have non-empty interior. If not, we do the usual Baire argument: fix a non-empty open set V_1 in Y with compact closure. Since $f(C_1)$ contains no non-empty open set, V_1 is not contained in $f(C_1)$, so there is a non-empty open set V_2 whose closure is compact and whose closure is contained in $V_1 - f(C_1)$. Since $f(C_2)$ can't contain V_2, there is a non-empty open set V_3 whose closure is compact and whose closure is contained in $V_2 - f(C_2)$. Continuing in this fashion, we have a descending chain of non-empty open sets

$$V_1 \supset \mathrm{clos}(V_2) \supset V_2 \supset \mathrm{clos}(V_2) \supset V_3 \supset \ldots$$

By construction, the intersection of the chain of compact sets $\mathrm{clos}(V_i)$ is disjoint from all the sets $f(C_i)$. Yet the intersection of a descending chain of compact sets is non-empty. Contradiction. Therefore, some $f(C_i)$ has non-empty interior. In particular, for y_o in the interior of $f(C_i)$, the map f is **open** at $x_o = f^{-1}(y_o)$.

Now we use the G-equivariance of the map f. Let U_o be an open set containing x_o so that $f(U_o)$ is open in Y. Then for any $g \in G$ the set gU_o is an open set containing gx_o. By the G-equivariance,

$$f(gU_o) = gf(U_o) = \text{ continuous image of open set } = \text{ open}$$

Therefore, since G is transitive on X, f is open at all points of X. ///

Proof: By the definition of the quotient topologies here, we must show that $GL(V)_k G_{\mathbf{A}}$ is closed in $GL(V)_A$.

Let X be the k-vectorspace of k-valued quadratic forms on V. We have a linear action ρ of $g \in GL(V)_k$ on $q \in X$ by

$$\rho(g)q\,(v, v) = q(g^{-1}v, g^{-1}v)$$

(with inverses simply for associativity). This extends to give a continuous group action of $GL(V)_{\mathbf{A}}$ on $X_{\mathbf{A}} = X \otimes \mathbf{A}$. Note that G_k is the subgroup of $GL(V)_k$ fixing the point $Q \in X$, essentially by definition.

Let Y be the set of images of Q under $GL(V)_k$. Then

$$GL(V)_k G_{\mathbf{A}} = \{g \in GL(V)_{\mathbf{A}} : g(Q) \in Y\}$$

That is, $GL(V)_k G_{\mathbf{A}}$ is the inverse image of Y. By the continuity of the group action, to prove that $GL(V)_k G_{\mathbf{A}}$ is closed in $GL(V)_{\mathbf{A}}$ it suffices to prove that the orbit

$$Y = GL(V)_k G_{\mathbf{A}}\,(Q)$$

is closed in $X_{\mathbf{A}}$. But, indeed, Y is a subset of the set $X \subset X_{\mathbf{A}}$, which is a (closed) discrete subset of $X_{\mathbf{A}}$. This proves the proposition, invoking the previous lemma. ///

If the global field we're working over is not $\mathbf{Q}$ itself, we need more preparation:

Proposition: Let k be a number field and K a finite extension of k. Let V be K^n viewed as a k-vectorspace. Let $H = GL(n, K)$ viewed as a k-group, and $G = GL_k(V)$. Then the natural inclusion

$$i : GL_K(K^n) = H \to G = GL_k(V)$$

gives rise to a homeomorphism of $H_k\backslash H_{\mathbf{A}}$ to its image in $G_k\backslash G_{\mathbf{A}}$, and this image is closed.

Proof: (This resembles, at least at an abstract level, the argument for the previous lemma. More will be said here in the next version of these notes.)

Theorem: *Mahler's criterion for compactness:* Let G be an orthogonal group attached to an n-dimensional non-degenerate k-valued quadratic form. For a subset X of $G_{\mathbf{A}} \subset GL(n, \mathbf{A})$ to be compact left modulo G_k, it is necessary and sufficient that the following property holds: given $x_i \in X$ and $v_i \in k^n$ such that $x_i v_i \to 0$ in $\mathbf{A}^n$, for sufficiently large index i we have simply $v_i = 0$.

Proof: The propositions above reduce the problem to proving an analogous assertion for $G = GL(n, k)$ with $k = \mathbf{Q}$. Indeed, that was the present purpose of those propositions. In particular, in the case of $GL(n)$ we additionally suppose that there are positive constants c' and c'' so that

$$X \subset \{g \in GL(n, \mathbf{A}) : c' \leq |\det g| \leq c''\}$$

The serious direction of implication is to show that, if the condition is satisfied, then X is compact modulo G_k. Let η be the affine height function on k^n. Then $\eta(xv) \geq c_1$ for some c_1 for any non-zero $v \in k^n$. By the Iwasawa decomposition, can write $x = p\theta$ with $\theta \in GL(n, \mathbf{o}_k)$ and p upper-triangular, where $\mathbf{o}_k$ is the ring of integers in k. Further, since we consider x modulo G_k, and using the fact that actually $k = \mathbf{Q}$, the Minkowski reduction allows us to suppose that the diagonal entries p_i of p satisfy $|p_i/p_{i+1}| \geq c$ for some $c > 0$. Therefore, letting e_i be the usual basis vectors in k^n, $c_1 \leq |p_i| = \eta(xe_1)$. And our extra hypothesis gives us

$$c' \leq |p_1 \ldots p_n| \leq c''$$

Thus, (by Fujisaki's lemma, for example) the diagonal entries of elements p coming from elements of X lie inside some compact subset of $\mathbf{J}/k^\times$.

Certainly the superdiagonal entries, left-modulo k-rational upper-triangular matrices, can be put into a compact set.

Therefore, X is compact left modulo $GL(n, k)$, for $k = \mathbf{Q}$. But, as remarked at the outset, the propositions above about imbeddings of arithmetic quotients reduce the general case and the orthogonal group case to this. ///

Theorem: Let G be the orthgonal group of a non-degenerate quadratic form $Q = \langle , \rangle$ on a vectorspace $V \approx k^n$ over a number field k. Then $G_k\backslash G_{\mathbf{A}}$ is compact if and only if Q is k-anisotropic.

Proof: On one hand, suppose Q is k-anisotropic. If $g_n v_n \to 0$ in $\mathbf{A}^n$ with $g_n \in G_{\mathbf{A}}$ and $v_n \in \mathbf{A}^n$, then $Q(v_n g_n)$ also goes to $Q(0) = 0$, by the continuity of Q. But $Q(g_n v_n) = Q(v_n)$, because $G_{\mathbf{A}}$ preserves values of Q. Since Q has no non-zero k-rational isotropic vectors and k^n is discrete in $\mathbf{A}^n$, this means that eventually $v_n = 0$. By Mahler's criterion this implies that the quotient is compact.

On the other hand, suppose that Q is isotropic. Then there is a non-zero isotropic vector $v \in k^n$. Let H be the subgroup of $G_{\mathbf{A}}$ fixing v. For all indices i let $v_i = v$. So certainly v_i does not go to 0. Now we'll need to exploit the fact that the topology on $\mathbf{J}$ is *not* simply the subspace topology from $\mathbf{A}$, but is inherited from the imbedding $\alpha \to (\alpha, \alpha^{-1})$ of $\mathbf{J} \to \mathbf{A} \times \mathbf{A}$: we can find a sequence t_i of ideles which go to 0 in the $\mathbf{A}$-topology (but certainly not in the $\mathbf{J}$-topology). Then $t_i v_i \to 0$. And certainly still $Q(t_i v_i) = 0$, so by Witt's theorem there is $g_i \in G_{\mathbf{A}}$ so that $g_i v_i = t_i v_i$. Thus, $g_i v_i \to 0$, but certainly v_i does not do so. Thus, Mahler's criterion says that the quotient is not compact. ///