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Standard archimedean integrals for $GL(2)$

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1. Basic identities involving $\Gamma(s)$, $B(\alpha, \beta)$, etc.

We take the Gamma function $\Gamma(s)$ to be defined by Euler's integral

$$\Gamma(s) = \int_0^\infty t^s e^{-t} \frac{dt}{t} \quad (\text{for } \operatorname{Re}(s) > 0)$$

Integration by parts proves the functional equation

$$\Gamma(s+1) = s \cdot \Gamma(s)$$

For $0 < s \in \mathbb{Z}$, this relation and induction show the connection to *factorials*,

$$\Gamma(n) = (n-1)! \quad (\text{for } n = 1, 2, \dots)$$

From the functional equation, we get a meromorphic continuation of $\Gamma(s)$ to the entire complex plane, except for poles at non-positive integers $-n$. The poles are *simple*, with residue $(-1)^n/n!$ at $-n$.

The identity

$$\int_0^\infty t^s e^{-ty} \frac{dt}{t} = \frac{\Gamma(s)}{y^s} \quad (\text{for } y > 0 \text{ and } \operatorname{Re}(s) > 0)$$

for $y > 0$ first follows for $\operatorname{Re}(s) > 0$ by replacing t by t/y in the integral. Then

$$\int_0^\infty t^s e^{-tz} \frac{dt}{t} = \frac{\Gamma(s)}{z^s} \quad (\text{for } \operatorname{Re}(z) > 0 \text{ and } \operatorname{Re}(s) > 0)$$

by complex analysis, since both sides are holomorphic in s and agree on the positive reals.

The latter identity allows non-obvious evaluation of a Fourier transform. Namely, let

$$f(x) = \begin{cases} x^\alpha \cdot e^{-x} & (\text{for } x > 0) \\ 0 & (\text{for } x < 0) \end{cases}$$

For $\operatorname{Re}(\alpha) > -1$ this function is locally integrable at 0, and in any case is of rapid decay at infinity. We can compute its Fourier transform:

$$\int_{\mathbb{R}} e^{-2\pi i \xi x} f(x) dx = \int_0^\infty e^{-2\pi i \xi x} x^{\alpha+1} e^{-x} \frac{dx}{x} = \int_0^\infty x^{\alpha+1} e^{-x(1+2\pi i \xi)} \frac{dx}{x} = \frac{\Gamma(\alpha+1)}{(1+2\pi i \xi)^{\alpha+1}}$$

Further, Fourier inversion gives the non-obvious

$$\int_{\mathbb{R}} e^{2\pi i \xi x} \frac{1}{(1 + 2\pi i \xi)^{\alpha+1}} d\xi = \frac{1}{\Gamma(\alpha + 1)} \cdot \begin{cases} x^\alpha \cdot e^{-x} & (\text{for } x > 0) \\ 0 & (\text{for } x < 0) \end{cases}$$

For $\alpha \in \mathbb{Z}$, the same conclusion can be reached by evaluation by residues.

Next, we recall the argument that expresses Euler's beta integral in terms of gamma functions, as

$$B(a, b) = \int_0^1 x^{a-1} (1-x)^{b-1} dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

Indeed, replacing x by $\frac{t}{t+1} = 1 - \frac{1}{t+1}$ in the integral gives

$$\int_0^1 x^{a-1} (1-x)^{b-1} dx = \int_0^\infty \left(\frac{t}{t+1}\right)^{a-1} \left(1 - \frac{t}{t+1}\right)^{b-1} \frac{dt}{(t+1)^2} = \int_0^\infty t^a \left(\frac{1}{t+1}\right)^{a+b} \frac{dt}{t}$$

Use the gamma identity in the form

$$\left(\frac{1}{t+1}\right)^s = \frac{1}{\Gamma(s)} \int_0^\infty e^{-u(t+1)} u^s \frac{du}{u}$$

to rewrite the beta integral further as

$$\frac{1}{\Gamma(a+b)} \int_0^\infty \int_0^\infty u^{a+b} t^a e^{-u(t+1)} \frac{du dt}{u t} = \frac{1}{\Gamma(a+b)} \int_0^\infty \int_0^\infty u^b t^a e^{-u} e^{-t} \frac{dt du}{t u} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

as claimed. ///

A similar sort of integral, with one more factor, is Euler's integral representation for **hypergeometric functions**, namely,

$$F(\alpha, \beta, \gamma; z) = \frac{1}{B(\beta, \gamma - \beta)} \int_0^1 x^{\beta-1} (1-x)^{\gamma-\beta-1} (1-xz)^{-\alpha} dx$$

where B is the beta function

$$B(a, b) = \int_0^1 x^{a-1} (1-x)^{b-1} dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

This F is the ${}_2F_1$ hypergeometric function, whose *series* definition is

$$F(\alpha, \beta, \gamma; z) = 1 + \frac{a b z}{c 1!} + \frac{a(a+1)b(b+1)z^2}{c(c+1)2!} + \dots = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

The notation $(a)_n$ is the *Pockhammer* symbol.

2. Holomorphic discrete series

Whittaker functions $W(g)$ for *holomorphic discrete series* for $GL_2(\mathbb{R})$ are completely elementary. For the holomorphic discrete series of weight $0 < \kappa \in \mathbb{Z}$, the Whittaker function corresponding to the lowest K -type is essentially an exponential function, namely,

$$W\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) = y^{\frac{\kappa}{2}} e^{-2\pi y} \quad (\text{for } y > 0)$$

The extension of this to the whole group is

$$W\left(\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}\right) \cdot \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} = \omega(z) e^{2\pi i x} \cdot y^{\frac{\kappa}{2}} e^{-2\pi y} \cdot e^{i\kappa\theta} \quad (\text{for } y > 0, z \in \mathbb{R}^\times)$$

where the central character ω must be compatible with κ , in the sense that

$$\omega\left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right) = e^{\pi i \kappa}$$

That is, the central character must have the same *parity*, as a character, as the weight κ has, as an integer.

Mellin transforms of holomorphic discrete series Whittaker functions are easy to evaluate, with the exponent $s - \frac{1}{2}$ rather than s , as usual, to have the functional equations $s \rightarrow 1 - s$ rather than $s \rightarrow -s$. Namely, regardless of the parity of κ ,

$$\begin{aligned} \int_{\mathbb{R}^\times} \text{sgn}(y)^\kappa \cdot |y|^{s-\frac{1}{2}} W\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) \frac{dy}{|y|} &= 2 \cdot \int_0^\infty |y|^{s-\frac{1}{2}} W\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) \frac{dy}{y} \\ &= 2 \cdot \int_0^\infty |y|^{s-\frac{1}{2}+\frac{\kappa}{2}} e^{-2\pi y} \frac{dy}{y} = 2 \cdot (2\pi)^{-(s-\frac{1}{2}+\frac{\kappa}{2})} \Gamma(s-\frac{1}{2}+\frac{\kappa}{2}) \end{aligned}$$

The archimedean integral arising in the *Rankin-Selberg convolution* L -function for two holomorphic cuspforms of the same weight κ is similarly easy to evaluate explicitly:

$$\int_0^\infty |y|^{s-1} W\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) \overline{W}\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) \frac{dy}{y} = \int_0^\infty |y|^{s-1+\kappa} e^{-4\pi y} \frac{dy}{y} = (4\pi)^{-(s-1+\kappa)} \cdot \Gamma(s-1+\kappa)$$

3. Spherical Whittaker functions for $GL_2(\mathbb{R})$

First, we want *integral representations* of Whittaker functions for *spherical principal series* with trivial central character for $G = GL_2(\mathbb{R})$. As usual, define subgroups

$$P = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \quad N = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$$

For $\mu \in \mathbb{C}$, let I_μ be the μ^{th} (naively normalized) spherical principal series of G with trivial central character, by definition consisting of the collection of all smooth functions f on G such that

$$f(p \cdot g) = \chi(p) \cdot f(g) \quad (\text{for } \chi(p) = |a/d|^\mu \text{ with } p = \begin{pmatrix} a & * \\ 0 & d \end{pmatrix} \in P, \text{ and } g \in G)$$

Computation of the Fourier expansion of Eisenstein series suggests presenting the Whittaker function as an image under an intertwining operator from I_μ to the *Whittaker space*

$$\text{Wh} = \{f \text{ on } GL_2(\mathbb{R}) : f/ng) = \psi(n) \cdot f(g) \text{ for all } n \in N, g \in G\} \quad (\text{where } \psi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = e^{2\pi i x})$$

Such an intertwining amounts to taking a Fourier transform, as follows. Let the spherical function be

$$\varphi\left(\begin{pmatrix} y & * \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}\right) = |y|^\mu \quad (\text{for } y > 0)$$

with *trivial* central character. This spherical function gets mapped to the Whittaker space by an intertwining

$$W_\mu^{\text{nf}}(g) = \int_{\mathbb{R}} e^{-2\pi ix} \varphi(w_o n_x g) dx$$

For diagonal elements $m_y = \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}$, we can explicitly determine the Iwasawa decomposition of $w_o n_x m_y$, namely (see the appendix)

$$w_o n_x m_y = \begin{pmatrix} y/r & * \\ 0 & r \end{pmatrix} \begin{pmatrix} y/r & x/r \\ -x/r & y/r \end{pmatrix} w_o \quad (\text{where } r = \sqrt{x^2 + y^2})$$

Therefore,

$$W_\mu^{\text{nf}} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} = \int_{\mathbb{R}} e^{-2\pi ix} \varphi \left(\begin{pmatrix} y/r & * \\ 0 & r \end{pmatrix} \begin{pmatrix} y/r & x/r \\ -x/r & y/r \end{pmatrix} w_o \right) dx$$

The spherical-ness is simply right K -invariance, and we are assuming trivial central character, so this simplifies to

$$W_\mu^{\text{nf}} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} = \int_{\mathbb{R}} e^{-2\pi ix} \left(\frac{|y|}{y^2 + x^2} \right)^\mu dx \quad (\text{for } \text{Re}(\mu) > \frac{1}{2})$$

Replace x by xy to obtain the useful integral representation

$$W_\mu^{\text{nf}} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} = |y|^{1-\mu} \int_{\mathbb{R}} e^{-2\pi ixy} \frac{1}{(1+x^2)^\mu} dx \quad (\text{for } \text{Re}(\mu) > \frac{1}{2})$$

It is awkward that $\text{Re}(\mu) > \frac{1}{2}$ is necessary for absolute convergence, since the *unitary* principal series have $\text{Re}(\mu) = \frac{1}{2}$, in the naive normalization. However, we can rearrange the integral to a form that shows the meromorphic continuation, and explains the normalization constant at the same time. Namely, using the identity involving $\Gamma(s)$,

$$|y|^{1-\mu} \int_{\mathbb{R}} e^{-2\pi ixy} \frac{1}{(1+x^2)^\mu} dx = \frac{|y|^{1-\mu}}{\Gamma(\mu)} \int_{\mathbb{R}} \int_0^\infty e^{-2\pi ixy} t^\mu e^{-t(1+x^2)} dx \frac{dt}{t}$$

Replace x by $x \cdot \frac{\sqrt{\pi}}{\sqrt{t}}$ to obtain

$$\frac{\sqrt{\pi} |y|^{1-\mu}}{\Gamma(\mu)} \int_0^\infty \int_{\mathbb{R}} e^{-2\pi ixy \cdot \frac{\sqrt{\pi}}{\sqrt{t}}} t^{\mu-\frac{1}{2}} e^{-t} e^{-\pi x^2} dx \frac{dt}{t}$$

Taking the Fourier transform of the Gaussian in x , this is

$$\frac{\sqrt{\pi} |y|^{1-\mu}}{\Gamma(\mu)} \int_0^\infty e^{-\pi y^2 \cdot \frac{\pi}{t}} t^{\mu-\frac{1}{2}} e^{-t} \frac{dt}{t}$$

Replacing t by $\pi|y| \cdot t$ gives

$$W_\mu^{\text{nf}} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} = \frac{\pi^\mu |y|^{1/2}}{\Gamma(\mu)} \int_0^\infty e^{-\pi|y|(\frac{1}{t}+t)} t^{\mu-\frac{1}{2}} \frac{dt}{t}$$

This last integral converges nicely for any complex μ .

A less naive normalization of the parametrization of the principal series I_μ^{nf} replaces μ by $\mu + \frac{1}{2}$, taking

$$I_\mu = I_{\mu+\frac{1}{2}}^{\text{nf}}$$

And a less naive normalization of the Whittaker function not only shifts μ , but also gets rid of the $\Gamma(\mu)$ and π^μ , by taking

$$W_\mu = \frac{\Gamma(\mu + \frac{1}{2})}{\pi^\mu} \cdot W_{\mu + \frac{1}{2}}^{\text{nf}}$$

That is,

$$W_\mu \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} = \frac{\Gamma(\mu + \frac{1}{2})}{\pi^\mu} \int_{\mathbb{R}} e^{-2\pi i x} \left(\frac{|y|}{x^2 + y^2} \right)^{\mu + \frac{1}{2}} dx = |y|^{1/2} \int_0^\infty e^{-\pi |y|(\frac{1}{t} + t)} t^\mu \frac{dt}{t}$$

Note that, in this normalization,

$$W_{-\mu} = W_\mu$$

One might further make a comparison to classical special functions.

4. Spherical Mellin transforms for $GL_2(\mathbb{R})$

The Hecke-Maaß integral representation of L -functions attached to *waveforms* already requires that we understand archimedean local integrals, namely, Mellin transforms of Whittaker functions, given by

$$\int_0^\infty y^s W_\mu^{\text{nf}} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \frac{dy}{y}$$

where W_μ^{nf} is the Whittaker function of the μ^{th} (naively normalized) spherical principal series I_μ^{nf} . As with holomorphic discrete series, we will later renormalize the exponent of y to $s - \frac{1}{2}$, as a global normalization to give L -functions functional equation $s \rightarrow 1 - s$ rather than $s \rightarrow -s$. The integral representation obtained above by consideration of an intertwining operator from I_μ^{nf} to the Whittaker space gives us what we need:

$$\int_0^\infty y^s W_\mu^{\text{nf}} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \frac{dy}{y} = \int_0^\infty y^s \int_{\mathbb{R}} e^{-2\pi i x} \left(\frac{y}{x^2 + y^2} \right)^\mu dx \frac{dy}{y}$$

The obvious thing to do is to interchange the order of integration, and apply the gamma identity, obtaining

$$\int_{\mathbb{R}} \int_0^\infty e^{-2\pi i x} y^{s+\mu} \left(\frac{1}{x^2 + y^2} \right)^\mu \frac{dy}{y} dx = \frac{1}{\Gamma(\mu)} \int_0^\infty \int_{\mathbb{R}} \int_0^\infty e^{-2\pi i x} y^{s+\mu} t^\mu e^{-t(x^2 + y^2)} \frac{dy}{y} dx \frac{dt}{t}$$

Replace y by y/\sqrt{t}

$$\frac{1}{\Gamma(\mu)} \int_0^\infty \int_{\mathbb{R}} \int_0^\infty e^{-2\pi i x} y^{s+\mu} t^{\frac{\mu-s}{2}} e^{-tx^2} e^{-y^2} \frac{dy}{y} dx \frac{dt}{t}$$

Replace y by \sqrt{y}

$$\frac{1}{2\Gamma(\mu)} \int_0^\infty \int_{\mathbb{R}} \int_0^\infty e^{-2\pi i x} y^{\frac{s+\mu}{2}} t^{\frac{\mu-s}{2}} e^{-tx^2} e^{-y} \frac{dy}{y} dx \frac{dt}{t} = \frac{\Gamma(\frac{s+\mu}{2})}{2\Gamma(\mu)} \int_0^\infty \int_{\mathbb{R}} e^{-2\pi i x} t^{\frac{\mu-s}{2}} e^{-tx^2} dx \frac{dt}{t}$$

Then replace x by $x \cdot \frac{\sqrt{\pi}}{\sqrt{t}}$ and take the Fourier transform of the Gaussian

$$\frac{\sqrt{\pi} \Gamma(\frac{s+\mu}{2})}{2\Gamma(\mu)} \int_0^\infty \int_{\mathbb{R}} e^{-2\pi i x \cdot \frac{\sqrt{\pi}}{\sqrt{t}}} t^{\frac{\mu-s-1}{2}} e^{-\pi x^2} dx \frac{dt}{t} = \frac{\sqrt{\pi} \Gamma(\frac{s+\mu}{2})}{2\Gamma(\mu)} \int_0^\infty e^{-\pi \cdot \frac{\pi}{t}} t^{\frac{\mu-s-1}{2}} \frac{dt}{t}$$

Replace t by $1/t$ and then by $t \cdot \pi^2$

$$\frac{\sqrt{\pi} \Gamma(\frac{s+\mu}{2}) \pi^{-(s+1-\mu)}}{2\Gamma(\mu)} \int_0^\infty e^{-t} t^{\frac{s+1-\mu}{2}} \frac{dt}{t} = \frac{\pi^{\mu-s-\frac{1}{2}} \Gamma(\frac{s+\mu}{2})}{2\Gamma(\mu)} \cdot \Gamma(\frac{s+1-\mu}{2})$$

That is, in a naive normalization,

$$\int_0^\infty |y|^s W_\mu^{\text{nf}} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \frac{dy}{y} = \frac{\pi^{\mu-s-\frac{1}{2}}}{2\Gamma(\mu)} \cdot \Gamma\left(\frac{s+\mu}{2}\right) \Gamma\left(\frac{s+1-\mu}{2}\right)$$

The less naive normalization replaces μ by $\mu + \frac{1}{2}$, s by $s - \frac{1}{2}$, and uses

$$W_\mu = \frac{\Gamma(\mu + \frac{1}{2})}{\pi^{\mu+\frac{1}{2}}} \cdot W_{\mu+\frac{1}{2}}^{\text{nf}}$$

This gives

$$\int_0^\infty |y|^{s-\frac{1}{2}} W_\mu \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \frac{dy}{y} = \frac{1}{2} \cdot \pi^{-s} \Gamma\left(\frac{s+\mu}{2}\right) \cdot \Gamma\left(\frac{s-\mu}{2}\right)$$

Note the nature of the product of gammas, specifically, the symmetry in $\pm\mu$.

5. Spherical Rankin-Selberg integrals for $GL_2(\mathbb{R})$

Let W_μ^{nf} and W_ν^{nf} be spherical Whittaker functions for the μ^{th} and ν^{th} spherical principal series for $GL_2(\mathbb{R})$ in the naive normalization. The local Rankin-Selberg integral is

$$\int_0^\infty y^{s-1} W_\mu^{\text{nf}} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} W_\nu^{\text{nf}} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \frac{dy}{y}$$

The exponent $s-1$ arises naturally from unwinding an Eisenstein series with a functional equation $s \rightarrow 1-s$. Also, one might complex conjugate the second Whittaker function, but this would only complex-conjugate the parameter ν (and replace y by $-y$, but the function is even). Replace the Whittaker functions by their integral representations, letting $\psi(x) = e^{2\pi i x}$, and apply the gamma identity:

$$\begin{aligned} & \int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^\infty y^{s-1} y^{1-\mu} \frac{\bar{\psi}(xy)}{(1+x^2)^\mu} y^{1-\nu} \frac{\bar{\psi}(uy)}{(1+u^2)^\nu} \frac{dy}{y} dx du \\ &= \frac{1}{\Gamma(\mu)\Gamma(\nu)} \int_0^\infty \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^\infty y^{s+1-\mu-\nu} \bar{\psi}(xy) t^\mu e^{-t(1+x^2)} \bar{\psi}(uy) \tau^\nu e^{-\tau(1+u^2)} \frac{dy}{y} dx du \frac{dt}{t} \frac{d\tau}{\tau} \end{aligned}$$

Replace x by $x \cdot \frac{\sqrt{\pi}}{\sqrt{t}}$ and u by $u \cdot \frac{\sqrt{\pi}}{\sqrt{\tau}}$ and take the Fourier transforms of the Gaussians, giving

$$\frac{\pi}{\Gamma(\mu)\Gamma(\nu)} \int_0^\infty \int_0^\infty \int_0^\infty y^{s+1-\mu-\nu} t^{\mu-\frac{1}{2}} e^{-t} e^{-\pi y^2(\frac{\pi}{t})} \tau^{\nu-\frac{1}{2}} e^{-\tau} e^{-\pi y^2(\frac{\pi}{\tau})} \frac{dy}{y} \frac{dt}{t} \frac{d\tau}{\tau}$$

Replace y by \sqrt{y} and then replace y by y/π^2 , to obtain

$$\frac{\pi^{\mu+\nu-s}}{2\Gamma(\mu)\Gamma(\nu)} \int_0^\infty \int_0^\infty \int_0^\infty y^{\frac{s+1-\mu-\nu}{2}} t^{\mu-\frac{1}{2}} e^{-t} e^{-y(\frac{1}{t}+\frac{1}{t})} \tau^{\nu-\frac{1}{2}} e^{-\tau} \frac{dy}{y} \frac{dt}{t} \frac{d\tau}{\tau}$$

Replace y by $y/(\tau^{-1} + t^{-1})$:

$$\begin{aligned} & \frac{\pi^{\mu+\nu-s}}{2\Gamma(\mu)\Gamma(\nu)} \Gamma\left(\frac{s+1-\mu-\nu}{2}\right) \int_0^\infty \int_0^\infty t^{\mu-\frac{1}{2}} e^{-t} \frac{1}{(\frac{1}{\tau} + \frac{1}{t})^{\frac{s+1-\mu-\nu}{2}}} \tau^{\nu-\frac{1}{2}} e^{-\tau} \frac{dt}{t} \frac{d\tau}{\tau} \\ &= \frac{\pi^{\mu+\nu-s}}{2\Gamma(\mu)\Gamma(\nu)} \Gamma\left(\frac{s+1-\mu-\nu}{2}\right) \int_0^\infty \int_0^\infty t^{\mu-\frac{1}{2}} e^{-t} \frac{(t\tau)^{\frac{s+1-\mu-\nu}{2}}}{(t+\tau)^{\frac{s+1-\mu-\nu}{2}}} \tau^{\nu-\frac{1}{2}} e^{-\tau} \frac{dt}{t} \frac{d\tau}{\tau} \\ &= \frac{\pi^{\mu+\nu-s}}{2\Gamma(\mu)\Gamma(\nu)} \Gamma\left(\frac{s+1-\mu-\nu}{2}\right) \int_0^\infty \int_0^\infty t^{\frac{s+\mu-\nu}{2}} e^{-t} \frac{1}{(t+\tau)^{\frac{s+1-\mu-\nu}{2}}} \tau^{\frac{s+\nu-\mu}{2}} e^{-\tau} \frac{dt}{t} \frac{d\tau}{\tau} \end{aligned}$$

Replace t by $t\tau$

$$\begin{aligned} &= \frac{\pi^{\mu+\nu-s}}{2\Gamma(\mu)\Gamma(\nu)} \Gamma\left(\frac{s+1-\mu-\nu}{2}\right) \int_0^\infty \int_0^\infty t^{\frac{s+\mu-\nu}{2}} e^{-t\tau} \frac{1}{(t\tau+\tau)^{\frac{s+1-\mu-\nu}{2}}} \tau^s e^{-\tau} \frac{dt}{t} \frac{d\tau}{\tau} \\ &= \frac{\pi^{\mu+\nu-s}}{2\Gamma(\mu)\Gamma(\nu)} \Gamma\left(\frac{s+1-\mu-\nu}{2}\right) \int_0^\infty \int_0^\infty t^{\frac{s+\mu-\nu}{2}} e^{-(t+1)\tau} \frac{1}{(t+1)^{\frac{s+1-\mu-\nu}{2}}} \tau^{\frac{s-1+\mu+\nu}{2}} \frac{dt}{t} \frac{d\tau}{\tau} \end{aligned}$$

Replace τ by $\tau/(t+1)$:

$$\begin{aligned} &= \frac{\pi^{\mu+\nu-s}}{2\Gamma(\mu)\Gamma(\nu)} \Gamma\left(\frac{s+1-\mu-\nu}{2}\right) \int_0^\infty \int_0^\infty t^{\frac{s+\mu-\nu}{2}} e^{-\tau} \frac{1}{(t+1)^s} \tau^{\frac{s-1+\mu+\nu}{2}} \frac{dt}{t} \frac{d\tau}{\tau} \\ &= \frac{\pi^{\mu+\nu-s}}{2\Gamma(\mu)\Gamma(\nu)} \Gamma\left(\frac{s+1-\mu-\nu}{2}\right) \Gamma\left(\frac{s-1+\mu+\nu}{2}\right) \int_0^\infty t^{\frac{s+\mu-\nu}{2}} \frac{1}{(t+1)^s} \frac{dt}{t} \end{aligned}$$

As usual,

$$\begin{aligned} &\int_0^\infty t^{\frac{s+\mu-\nu}{2}} \frac{1}{(t+1)^s} \frac{dt}{t} = \frac{1}{\Gamma(s)} \int_0^\infty \int_0^\infty t^{\frac{s+\mu-\nu}{2}} x^s e^{-x(1+t)} \frac{dt}{t} \frac{dx}{x} \\ &= \frac{1}{\Gamma(s)} \int_0^\infty \int_0^\infty t^{\frac{s+\mu-\nu}{2}} x^{\frac{s-\mu+\nu}{2}} e^{-x} e^{-t} \frac{dt}{t} \frac{dx}{x} = \frac{\Gamma\left(\frac{s+\mu-\nu}{2}\right) \Gamma\left(\frac{s-\mu+\nu}{2}\right)}{\Gamma(s)} \end{aligned}$$

Thus, the whole is

$$\frac{\pi^{\mu+\nu-s}}{2\Gamma(\mu)\Gamma(\nu)} \cdot \frac{\Gamma\left(\frac{s+\mu-\nu}{2}\right) \Gamma\left(\frac{s-\mu+\nu}{2}\right) \Gamma\left(\frac{s+1-\mu-\nu}{2}\right) \Gamma\left(\frac{s-1+\mu+\nu}{2}\right)}{\Gamma(s)}$$

That is, with the naive normalization,

$$\int_0^\infty y^{s-1} W_\mu^{\text{nf}} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} W_\nu^{\text{nf}} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \frac{dy}{y} = \frac{\pi^{\mu+\nu-s}}{2\Gamma(\mu)\Gamma(\nu)} \cdot \frac{\Gamma\left(\frac{s+\mu-\nu}{2}\right) \Gamma\left(\frac{s-\mu+\nu}{2}\right) \Gamma\left(\frac{s+1-\mu-\nu}{2}\right) \Gamma\left(\frac{s-1+\mu+\nu}{2}\right)}{\Gamma(s)}$$

For the less naive normalization, replace μ by $\mu + \frac{1}{2}$ and ν by $\nu + \frac{1}{2}$, and recall the renormalization

$$W_\mu = \frac{\Gamma\left(\mu + \frac{1}{2}\right)}{\pi^{\mu+\frac{1}{2}}} \cdot W_{\mu+\frac{1}{2}}^{\text{nf}}$$

Then the local Rankin-Selberg integral is

$$\int_0^\infty y^{s-1} W_\mu \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} W_\nu \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \frac{dy}{y} = \frac{\pi^{1-s}}{2} \cdot \frac{\Gamma\left(\frac{s+\mu-\nu}{2}\right) \Gamma\left(\frac{s-\mu+\nu}{2}\right) \Gamma\left(\frac{s-\mu-\nu}{2}\right) \Gamma\left(\frac{s+\mu+\nu}{2}\right)}{\Gamma(s)}$$

Note the pleasing symmetry in the parameters $\pm\mu$ and $\pm\nu$. The extra 2 in the denominator can be rationalized away in several different ways.

6. Moment integrals for discrete series of $GL_2(\mathbb{R})$

In the study of *moments* of L -functions, we encounter *local moment integrals* such as

$$\int_0^\infty \int_0^\infty y^{it} y'^{-it} W \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \overline{W} \begin{pmatrix} y' & 0 \\ 0 & 1 \end{pmatrix} \int_{\mathbb{R}} \overline{\psi}((y-y') \cdot x) \frac{dx}{(1+x^2)^{w/2}} \frac{dy}{y} \frac{dy'}{y'}$$

where the W 's are Whittaker functions for irreducibles of $GL_2(\mathbb{R})$ or $GL_2(\mathbb{C})$, with $\psi(x) = e^{2\pi ix}$ as usual, and where we have made the simplest useful choice $1/(1+x^2)^{w/2}$ of archimedean data. By Diaconu's computations, these integrals are not generally elementary, except for holomorphic discrete series of $GL_2(\mathbb{R})$,

where we can evaluate them as follows. For holomorphic discrete series with lowest weights both equal to κ , the local moment integral is

$$\begin{aligned} & \int_0^\infty \int_0^\infty y^{it} y'^{-it} y^{\kappa/2} e^{-2\pi y} y'^{\kappa/2} e^{-2\pi y'} \int_{\mathbb{R}} \overline{\psi}((y-y') \cdot x) \frac{dx}{(1+x^2)^{w/2}} \frac{dy}{y} \frac{dy'}{y'} \\ &= \int_0^\infty \int_0^\infty \int_{\mathbb{R}} y^{it+\frac{\kappa}{2}} y'^{-it+\frac{\kappa}{2}} e^{-(2\pi+2\pi ix)y} e^{-(2\pi-2\pi ix)y'} \frac{dx}{(1+x^2)^{w/2}} \frac{dy}{y} \frac{dy'}{y'} \end{aligned}$$

Using the gamma identity in both y and y' , this gives

$$(2\pi)^{-(1+\kappa)} \Gamma(it + \frac{\kappa}{2}) \Gamma(-it + \frac{\kappa}{2}) \int_{\mathbb{R}} \frac{1}{(1+ix)^{it+\frac{\kappa}{2}}} \cdot \frac{1}{(1-ix)^{-it+\frac{\kappa}{2}}} \cdot \frac{dx}{(1+x^2)^{w/2}}$$

Because of the simple choice of data, this is

$$(2\pi)^{-(1+\kappa)} \Gamma(it + \frac{\kappa}{2}) \Gamma(-it + \frac{\kappa}{2}) \int_{\mathbb{R}} \frac{1}{(1+ix)^{it+\frac{\kappa}{2}+\frac{w}{2}}} \cdot \frac{1}{(1-ix)^{-it+\frac{\kappa}{2}+\frac{w}{2}}} dx$$

This is the inner product on \mathbb{R} of two functions whose Fourier transforms we essentially know from the gamma identity. That is, we know

$$\int_{\mathbb{R}} e^{2\pi i \xi x} \frac{1}{(1+2\pi i \xi)^{\alpha+1}} d\xi = \frac{1}{\Gamma(\alpha+1)} \cdot \begin{cases} x^\alpha \cdot e^{-x} & (\text{for } x > 0) \\ 0 & (\text{for } x < 0) \end{cases}$$

Thus, replace x by $2\pi x$ in the given integral, to have

$$(2\pi)^{-\kappa} \Gamma(it + \frac{\kappa}{2}) \Gamma(-it + \frac{\kappa}{2}) \int_{\mathbb{R}} \frac{1}{(1+2\pi ix)^{it+\frac{\kappa}{2}+\frac{w}{2}}} \cdot \frac{1}{(1-2\pi ix)^{-it+\frac{\kappa}{2}+\frac{w}{2}}} dx$$

By the Plancherel identity,

$$\int_{\mathbb{R}} \frac{1}{(1+2\pi ix)^{\alpha+1}} \frac{1}{(1-2\pi ix)^{\beta+1}} dx = \frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)} \int_{\mathbb{R}} x^\alpha e^{-x} \cdot x^\beta e^{-x} dx = \frac{2^{-(\alpha+\beta+1)} \Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)}$$

In the case at hand,

$$\alpha+1 = it + \frac{\kappa}{2} + \frac{w}{2} \quad \text{and} \quad \beta+1 = -it + \frac{\kappa}{2} + \frac{w}{2}$$

Thus, the whole integral is

$$(2\pi)^{-\kappa} \Gamma(it + \frac{\kappa}{2}) \Gamma(-it + \frac{\kappa}{2}) \frac{2^{-(\kappa+w)} \Gamma(\kappa+w)}{\Gamma(it + \frac{\kappa}{2} + \frac{w}{2}) \Gamma(-it + \frac{\kappa}{2} + \frac{w}{2})}$$

Thus, for weight κ holomorphic discrete series on $GL_2(\mathbb{R})$, and with the simplest choice of data,

$$(\text{moment integral}) = (2\pi)^{-\kappa} \cdot 2^{-(\kappa+w)} \cdot \frac{\Gamma(it + \frac{\kappa}{2}) \Gamma(-it + \frac{\kappa}{2}) \Gamma(\kappa+w)}{\Gamma(it + \frac{\kappa}{2} + \frac{w}{2}) \Gamma(-it + \frac{\kappa}{2} + \frac{w}{2})}$$

From the asymptotic result

$$\frac{\Gamma(s)}{\Gamma(s+a)} = s^{-a} \cdot (1 + O(\frac{1}{|s|})) \quad (\text{for } a \text{ fixed})$$

for fixed w and for $s = \frac{1}{2} + it$ this moment integral has simple asymptotics, namely

$$(\text{moment integral}) \sim (\text{constant}) \cdot t^{-w}$$

To clarify the dependence upon w , use the *duplication* formula

$$\Gamma(s) \cdot \Gamma(s + \frac{1}{2}) = 2^{1-2s} \cdot \sqrt{\pi} \cdot \Gamma(2s)$$

to rewrite

$$\Gamma(\kappa + w) = \Gamma(2 \cdot \frac{\kappa + w}{2}) = 2^{\kappa+w-1} \cdot \sqrt{\pi} \cdot \Gamma(\frac{\kappa + w}{2}) \cdot \Gamma(\frac{\kappa + w + 1}{2}).$$

Thus, the exponential depending on w disappears, and

$$(\text{moment integral}) = (2\pi)^{-\kappa} \cdot \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma(it + \frac{\kappa}{2}) \Gamma(-it + \frac{\kappa}{2}) \Gamma(\frac{\kappa}{2} + \frac{w}{2}) \cdot \Gamma(\frac{\kappa}{2} + \frac{w}{2} + \frac{1}{2})}{\Gamma(it + \frac{\kappa}{2} + \frac{w}{2}) \Gamma(-it + \frac{\kappa}{2} + \frac{w}{2})}$$

7. Spherical Whittaker functions for $GL_2(\mathbb{C})$

First, we want *integral representations* of Whittaker functions for *spherical principal series* with trivial central character for $G = GL_2(\mathbb{C})$. As usual, let

$$P = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \quad N = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$$

For compatibility with global normalizations, take additive character and norm on \mathbb{C} to be

$$\psi \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) = e^{2\pi i(x+\bar{x})} \quad |\alpha|_{\mathbb{C}} = |\alpha \cdot \bar{\alpha}|$$

where the norm on the right-hand side is the usual norm on \mathbb{R} . For $\mu \in \mathbb{C}$, let I_{μ} be the μ^{th} (naively normalized) spherical principal series of G with trivial central character, by definition consisting of the collection of all smooth functions f on G such that

$$f(p \cdot g) = \chi(p) \cdot f(g) \quad (\text{for } \chi(p) = |a/d|_{\mathbb{C}}^{\mu} \text{ with } p = \begin{pmatrix} a & * \\ 0 & d \end{pmatrix} \in P, \text{ and } g \in G)$$

As over \mathbb{R} , computation of the Fourier expansion of Eisenstein series suggests presenting the Whittaker function as an image under an intertwining operator from I_{μ} to the *Whittaker space*

$$\text{Wh} = \{f \text{ on } GL_2(\mathbb{C}) : f(ng) = \psi(n) \cdot f(g) \text{ for all } n \in N, g \in G\}$$

Such an intertwining amounts to taking a Fourier transform, as follows. Let the spherical function be

$$\varphi \left(\begin{pmatrix} y & * \\ 0 & 1 \end{pmatrix} \cdot k \right) = |y|_{\mathbb{C}}^{\mu} = y^{2\mu} \quad (\text{for } y > 0, k \in U(2))$$

with *trivial* central character. This spherical function gets mapped to the Whittaker space by an intertwining

$$W_{\mu}^{\text{nf}}(g) = \int_{\mathbb{C}} \bar{\psi}(x) \varphi(w_o n_x g) dx$$

where the measure is *twice* the usual measure on \mathbb{C} , for compatibility with the normalization of the character. For diagonal elements $m_y = \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}$ with $y > 0$, we can explicitly determine the Iwasawa decomposition of $w_o n_x m_y$, namely (see the appendix)

$$w_o n_x m_y = \begin{pmatrix} y/r & * \\ 0 & r \end{pmatrix} \begin{pmatrix} y/r & \bar{x}/r \\ -x/r & y/r \end{pmatrix} w_o \quad (\text{where } r = \sqrt{x\bar{x} + y^2})$$

Therefore,

$$W_\mu^{\text{nf}} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} = \int_{\mathbb{C}} \bar{\psi}(x) \varphi \left(\begin{pmatrix} y/r & * \\ 0 & r \end{pmatrix} \begin{pmatrix} y/r & \bar{x}/r \\ -x/r & y/r \end{pmatrix} w_o \right) dx$$

The spherical-ness is right $U(2)$ -invariance, and we are assuming trivial central character, so this simplifies to

$$W_\mu^{\text{nf}} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} = \int_{\mathbb{C}} e^{-2\pi i(x+\bar{x})} \left| \frac{y}{y^2 + x\bar{x}} \right|^{2\mu} dx = \int_{\mathbb{C}} e^{-2\pi i(x+\bar{x})} \left| \frac{y}{y^2 + x\bar{x}} \right|^{2\mu} dx \quad (\text{for } \text{Re}(\mu) > \frac{1}{2})$$

where we reverted to the usual absolute value on \mathbb{R} , causing the exponent μ to effectively double. Replace x by xy to obtain a useful integral representation, valid for $y \in \mathbb{C}^\times$,

$$W_\mu^{\text{nf}} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} = |y|_{\mathbb{C}}^{1-\mu} \int_{\mathbb{C}} \bar{\psi}(x \cdot y) \frac{1}{(1 + x\bar{x})^{2\mu}} dx \quad (\text{for } \text{Re}(\mu) > \frac{1}{2})$$

As in the case of $GL_2(\mathbb{R})$, the integral can be rearranged to show the meromorphic continuation, and explain subsequent renormalization. Namely, using the identity involving $\Gamma(s)$, for $y > 0$,

$$y^{2-2\mu} \int_{\mathbb{C}} e^{-2\pi i(x+\bar{x})y} \frac{1}{(1 + x\bar{x})^{2\mu}} dx = \frac{y^{2-2\mu}}{\Gamma(2\mu)} \int_{\mathbb{C}} \int_0^\infty e^{-2\pi i(x+\bar{x})y} t^{2\mu} e^{-t(1+x\bar{x})} dx \frac{dt}{t}$$

Replace x by $x \cdot \frac{\sqrt{\pi}}{\sqrt{t}}$ to obtain

$$\frac{\pi |y|^{1-\mu}}{\Gamma(\mu)} \int_0^\infty \int_{\mathbb{C}} e^{-2\pi i(x+\bar{x})y \cdot \frac{\sqrt{\pi}}{\sqrt{t}}} t^{2\mu-1} e^{-t} e^{-\pi x\bar{x}} dx \frac{dt}{t}$$

Taking the Fourier transform of the Gaussian in $x \in \mathbb{C}$, this is

$$\frac{\pi |y|^{2-2\mu}}{\Gamma(2\mu)} \int_0^\infty e^{-\pi y^2 \cdot \frac{\pi}{t}} t^{2\mu-1} e^{-t} \frac{dt}{t}$$

Replacing t by $\pi|y| \cdot t$ gives

$$W_\mu^{\text{nf}} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} = \frac{\pi^{2\mu} |y|}{\Gamma(2\mu)} \int_0^\infty e^{-\pi|y|(\frac{1}{t}+t)} t^{2\mu-1} \frac{dt}{t}$$

This last integral converges nicely for any complex μ .

A less naive normalization of the parametrization of the principal series I_μ^{nf} replaces μ by $\mu + \frac{1}{2}$, taking

$$I_\mu = I_{\mu+\frac{1}{2}}^{\text{nf}}$$

and gets rid of the $\Gamma(2\mu)$ and $\pi^{2\mu}$, by taking

$$W_\mu = \frac{\Gamma(2\mu+1)}{\pi^{2\mu+1}} \cdot W_{\mu+\frac{1}{2}}^{\text{nf}}$$

That is,

$$W_\mu \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} = \frac{\Gamma(2\mu+1)}{\pi^{2\mu+1}} \int_{\mathbb{C}} e^{-2\pi i(x+\bar{x})} \left(\frac{|y|}{x\bar{x} + y^2} \right)^{2\mu+1} dx = |y| \int_0^\infty e^{-\pi|y|(\frac{1}{t}+t)} t^{2\mu} \frac{dt}{t}$$

Thus, in this normalization, visibly

$$W_{-\mu} = W_\mu$$

8. Spherical Mellin transforms for $GL_2(\mathbb{C})$

For $GL_2(\mathbb{C})$, the Mellin transform of the spherical Whittaker function gives an integral of the form

$$\int_0^\infty |y|_{\mathbb{C}}^s W_\mu^{\text{nf}} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \frac{dy}{y} = \int_0^\infty y^{2s} W_\mu^{\text{nf}} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \frac{dy}{y}$$

where W_μ^{nf} is the Whittaker function of the μ^{th} (naively normalized) spherical principal series I_μ^{nf} . As for $GL_2(\mathbb{R})$, later we will rewrite this with $|y|_{\mathbb{C}}^s$ replaced by $|y|_{\mathbb{C}}^{s-\frac{1}{2}}$, in order to give L -functions functional equation $s \rightarrow 1-s$ rather than $s \rightarrow -s$. To compute this integral in terms of gamma functions, use the integral representation obtained above from an intertwining operator from I_μ^{nf} to the Whittaker space:

$$\int_0^\infty y^{2s} W_\mu^{\text{nf}} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \frac{dy}{y} = \int_0^\infty y^{2s} \int_{\mathbb{C}} \bar{\psi}(x) \left(\frac{y}{x\bar{x} + y^2} \right)^{2\mu} dx \frac{dy}{y}$$

As over \mathbb{R} , the obvious thing to do is to interchange the order of integration, and apply the gamma identity, obtaining

$$\int_{\mathbb{C}} \int_0^\infty \bar{\psi}(x) y^{2s+2\mu} \left(\frac{1}{x\bar{x} + y^2} \right)^{2\mu} \frac{dy}{y} dx = \frac{1}{\Gamma(2\mu)} \int_0^\infty \int_{\mathbb{C}} \int_0^\infty \bar{\psi}(x) y^{2s+2\mu} t^{2\mu} e^{-t(x\bar{x} + y^2)} \frac{dy}{y} dx \frac{dt}{t}$$

Replace y by y/\sqrt{t}

$$\frac{1}{\Gamma(2\mu)} \int_0^\infty \int_{\mathbb{C}} \int_0^\infty \bar{\psi}(x) y^{2s+2\mu} t^{\mu-s} e^{-tx\bar{x}} e^{-y^2} \frac{dy}{y} dx \frac{dt}{t}$$

Replace y by \sqrt{y}

$$\frac{1}{2\Gamma(2\mu)} \int_0^\infty \int_{\mathbb{C}} \int_0^\infty \bar{\psi}(x) y^{s+\mu} t^{\mu-s} e^{-tx\bar{x}} e^{-y} \frac{dy}{y} dx \frac{dt}{t} = \frac{\Gamma(s+\mu)}{2\Gamma(2\mu)} \int_0^\infty \int_{\mathbb{C}} \bar{\psi}(x) t^{\mu-s} e^{-tx\bar{x}} dx \frac{dt}{t}$$

Then replace x by $x \cdot \frac{\sqrt{\pi}}{\sqrt{t}}$ and take the Fourier transform of the Gaussian

$$\frac{\pi \Gamma(s+\mu)}{2\Gamma(2\mu)} \int_0^\infty \int_{\mathbb{C}} \bar{\psi} \left(x \cdot \frac{\sqrt{\pi}}{\sqrt{t}} \right) t^{\mu-s-1} e^{-\pi x\bar{x}} dx \frac{dt}{t} = \frac{\pi \Gamma(s+\mu)}{2\Gamma(2\mu)} \int_0^\infty e^{-\pi \cdot \frac{\pi}{t}} t^{\mu-s-1} \frac{dt}{t}$$

Replace t by $1/t$ and then by t/π^2

$$\frac{\pi \Gamma(s+\mu) \pi^{-2(s+1-\mu)}}{2\Gamma(2\mu)} \int_0^\infty e^{-t} t^{s+1-\mu} \frac{dt}{t} = \frac{\pi^{2\mu-2s-1} \Gamma(s+\mu)}{2\Gamma(2\mu)} \cdot \Gamma(s+1-\mu)$$

That is, in a naive normalization,

$$\int_0^\infty |y|_{\mathbb{C}}^s W_\mu^{\text{nf}} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \frac{dy}{y} = \frac{\pi^{2\mu-2s-1}}{2\Gamma(2\mu)} \cdot \Gamma(s+\mu) \Gamma(s+1-\mu)$$

The less naive normalization replaces μ by $\mu + \frac{1}{2}$, s by $s - \frac{1}{2}$, and uses

$$W_\mu = \frac{\Gamma(2\mu+1)}{\pi^{2\mu+1}} \cdot W_{\mu+\frac{1}{2}}^{\text{nf}}$$

In this normalization, the Mellin transform of the spherical $GL_2(\mathbb{C})$ Whittaker function is

$$\int_0^\infty |y|_{\mathbb{C}}^{s-\frac{1}{2}} W_\mu \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \frac{dy}{y} = \frac{1}{2} \cdot \pi^{-2s} \Gamma(s+\mu) \cdot \Gamma(s-\mu)$$

As usual, note the nature of the product of gammas, especially, the symmetry in $\pm\mu$.

9. Appendix: explicit Iwasawa decompositions

To compute the images of various vectors under natural intertwinings from principal series to Whittaker spaces, as well as for other purposes, we need explicit Iwasawa decompositions of group elements. Let

$$w_o = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \quad m_y = \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}$$

Given x in \mathbb{R} or \mathbb{C} and $y > 0$, let $r = \sqrt{y^2 + x\bar{x}}$. We claim that we have an Iwasawa decomposition

$$w_o n_x m_y = \begin{pmatrix} y/r & -\bar{x}/r \\ 0 & r \end{pmatrix} \cdot \begin{pmatrix} y/r & \bar{x}/r \\ -x/r & y/r \end{pmatrix} \cdot w_o$$

Of course, once asserted, this can be *verified* by direct computation. However, instead, we should see how the decomposition could be found in the first place.

Over either \mathbb{R} or \mathbb{C} , the maximal compact K can be defined as

$$K = \{g : g^* g = 1_2\} = \{g : g \cdot g^* = 1_2\} \quad (\text{where } g^* \text{ is } g\text{-conjugate-transpose})$$

Of course, over \mathbb{R} the conjugation does nothing. Assuming that $g = pk$ with p upper triangular and k in K ,

$$g g^* = (pk)(pk)^* = p(kk^*)p = pp^*$$

Letting $p = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ with $a > 0$ and $d > 0$, we can solve for a, b, d . Here,

$$g = w_o n_x m_y w_o^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -x & y \end{pmatrix}$$

Then

$$g g^* = \begin{pmatrix} 1 & 0 \\ -x & y \end{pmatrix} \begin{pmatrix} 1 & -\bar{x} \\ 0 & y \end{pmatrix} = \begin{pmatrix} 1 & -\bar{x} \\ x & y^2 + x\bar{x} \end{pmatrix}$$

From an equality

$$\begin{pmatrix} a^2 + b\bar{b} & bd \\ \bar{b}d & d^2 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^* = \begin{pmatrix} 1 & -\bar{x} \\ x & y^2 + x\bar{x} \end{pmatrix}$$

immediately $d = r$, and $b = -\bar{x}/d = -\bar{x}/r$, and then

$$a^2 = 1 - b\bar{b} = 1 - x\bar{x}/r^2 = \frac{y^2 + x\bar{x} - x\bar{x}}{r^2} = \frac{y^2}{r^2}$$

from which $a = y/r$. Thus,

$$g = w_o n_x m_y w_o^{-1} = \begin{pmatrix} y/r & -\bar{x}/r \\ 0 & r \end{pmatrix} \cdot k$$

for some $k \in K$. Then

$$k = p^{-1} g = \begin{pmatrix} r/y & \bar{x}/ry \\ 0 & 1/r \end{pmatrix} = \begin{pmatrix} r/y - x\bar{x}/ry & \bar{x}/ry \\ -x/r & y/r \end{pmatrix} = \begin{pmatrix} y/r & \bar{x}/ry \\ -x/r & y/r \end{pmatrix}$$

Thus, we have the Iwasawa decomposition

$$w_o n_x m_y w_o^{-1} = \begin{pmatrix} y/r & -\bar{x}/r \\ 0 & r \end{pmatrix} \cdot \begin{pmatrix} y/r & \bar{x}/ry \\ -x/r & y/r \end{pmatrix}$$

This can be slightly rewritten as

$$w_o n_x m_y = \begin{pmatrix} y/r & -\bar{x}/r \\ 0 & r \end{pmatrix} \cdot \begin{pmatrix} y/r & \bar{x}/ry \\ -x/r & y/r \end{pmatrix} \cdot w_o$$

10. Appendix: $\Gamma(s) \cdot \Gamma(1-s) = \pi / \sin \pi s$

Take $0 < \text{Re}(s) < 1$ for convergence of both integrals, and compute

$$\Gamma(s) \cdot \Gamma(1-s) = \int_0^\infty \int_0^\infty u^s e^{-u} \cdot v^{1-s} e^{-v} \frac{du}{u} \frac{dv}{v} = \int_0^\infty \int_0^\infty u e^{-u(1+v)} v^{1-s} \frac{du}{u} \frac{dv}{v}$$

by replacing v by uv . Replacing u by $u/(1+v)$ (another instance of the basic *gamma identity*) and noting that $\Gamma(1) = 1$ gives

$$\int_0^\infty \frac{v^{-s}}{1+v} dv$$

Replace the path from 0 to ∞ by the *Hankel contour* H_ε described as follows. Far to the right on the real line, start with the branch of v^{-s} given by $(e^{2\pi i} v)^{-s} = e^{-2\pi i s} v^{-s}$, integrate from $+\infty$ to $\varepsilon > 0$ along the real axis, clockwise around a circle of radius ε at 0, then back out to $+\infty$, now with the standard branch of v^{-s} . For $\text{Re}(-s) > -1$ the integral around the little circle goes to 0 as $\varepsilon \rightarrow 0$. Thus,

$$\int_0^\infty \frac{v^{-s}}{1+v} dv = \lim_{\varepsilon \rightarrow 0} \frac{1}{1 - e^{-2\pi i s}} \int_{H_\varepsilon} \frac{v^{-s}}{1+v} dv$$

The integral of this integrand over a large circle goes to 0 as the radius goes to $+\infty$, for $\text{Re}(-s) < 0$. Thus, this integral is equal to the limit as $R \rightarrow +\infty$ and $\varepsilon \rightarrow 0$ of the integral

from R to ε
 from ε clockwise back to ε
 from ε to R
 from R counterclockwise to R

This integral is $2\pi i$ times the sum of the residues inside it, namely, that at $v = -1 = e^{\pi i}$. Thus,

$$\Gamma(s) \cdot \Gamma(1-s) = \int_0^\infty \frac{v^{-s}}{1+v} dv = \frac{2\pi i}{1 - e^{-2\pi i s}} \cdot (e^{\pi i})^{-s} = \frac{2\pi i}{e^{\pi i s} - e^{-\pi i s}} = \frac{\pi}{\sin \pi s}$$

11. Appendix: Duplication: $\Gamma(s) \cdot \Gamma(s + \frac{1}{2}) = 2^{1-2s} \cdot \sqrt{\pi} \cdot \Gamma(2s)$

From the Eulerian integral definition,

$$\Gamma(s) \cdot \Gamma(s + \frac{1}{2}) = \int_0^\infty e^{-t} t^s \frac{dt}{t} \cdot \int_0^\infty e^{-u} u^{s+\frac{1}{2}} \frac{du}{u}$$

Replace t by t/u

$$\int_0^\infty \int_0^\infty e^{-(\frac{t}{u}+u)} t^s u^{\frac{1}{2}} \frac{du}{u} \frac{dt}{t}$$