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Standard estimates for $SL_2(\mathbb{Z}[i]) \backslash SL_2(\mathbb{C}) / SU(2)$

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This is a simple example of a general phenomenon. Examples in $SL_2(\mathbb{R})$ are carried out in [Iwaniec 2002]. For $\Gamma = SL_2(\mathbb{Z}[i])$, $G = SL_2(\mathbb{C})$, and $K = SU(2)$, we prove the *standard estimate*

$$\sum_{|s_F| \leq T} |F(g)|^2 + \frac{1}{2\pi} \int_{-T}^T |E_{\frac{1}{2}+it}(g)|^2 dt \ll_C T^3 \quad (\text{uniformly for } g \text{ in a compact } C \subset G)$$

for cuspforms F with eigenvalues

$$\lambda_F = s_F(s_F - 1)$$

for the Laplacian \mathcal{D} , and Eisenstein series E_s . We normalize the dependence of E_s on the parameter s so that the functional equation relates E_s and E_{1-s} .

As usual, we consider *integral* operators attached to compactly supported measures η on the group G , and exploit the intrinsic sense of such operators on any reasonable representation space for G , for example, Hilbert, Banach, Fréchet, and LF (strict colimits of Fréchet), or, generally, quasi-complete, locally convex spaces. For a representation π, V of G , and a compactly-supported measure η , the action is

$$\eta \cdot v = \int_G \pi(g)(v) d\eta(g) \quad (\text{for } v \in V)$$

The general theory of Gelfand-Pettis integrals assures the reasonable behavior of such integrals.

The non-trivial but memorable fact used in the proof, illustrated in the case of $G = SL_2(\mathbb{C})$, is that a waveform f , an eigenfunction for the G -invariant Laplacian \mathcal{D} in $L^2(\Gamma \backslash G / K)$, generates an irreducible representation of G under right translation, specifically, an *unramified principal series* I_s .^[1] The same is true of Eisenstein series E_s more immediately. We index the character defining the unramified principal series I_s so that the standard intertwining operators go from I_s to I_{1-s} .

Thus, a waveform f (or Eisenstein series E_s) is the unique spherical vector in the copy of the unramified principal series representation it generates, up to a constant. Thus, for any left-and-right K -invariant compactly-supported measure η the integral operator action

$$(\eta \cdot f)(x) = \int_G \pi(y)f(xy) d\eta(y)$$

produces another right K -invariant vector in the representation space of f . Necessarily $\eta \cdot f$ is a scalar multiple of f . Let $\chi_f(\eta)$ denote the eigenvalue:

$$\eta \cdot f = \chi_f(\eta) \cdot f \quad (\text{with } \chi_f(\eta) \in \mathbb{C})$$

This is an intrinsic representation-theoretic relation, so the scalar $\chi_f(\eta)$ can be computed in any model of the representation. We choose an unramified principal series

$$I_s = \left\{ \text{smooth } K\text{-finite } \varphi : \varphi \left(\begin{pmatrix} a & * \\ 0 & d \end{pmatrix} \cdot g \right) = \left| \frac{a}{d} \right|^{2s} \cdot \varphi(g) \right\} \quad (\text{with } s \in \mathbb{C})$$

[1] The general theory of spherical functions shows that, generally, eigenfunctions for all left G -invariant differential operators on G/K generate principal series. Often, the center of the enveloping algebra *surjects* to that collection of differential operators: for classical groups this holds. However, [Helgason 1984] gives examples of non-surjection among exceptional groups.

On I_s , the Laplacian or Casimir has eigenvalue $\lambda_f = s(s-1)$.

[1.1] Choice of integral operator Let $\|g\|$ be the square of the *operator norm* on G for the standard representation of G on \mathbb{C}^2 by matrix multiplication. In a Cartan decomposition,

$$\|k_1 \cdot \begin{pmatrix} e^{r/2} & 0 \\ 0 & e^{-r/2} \end{pmatrix} \cdot k_2\| = e^r \quad (\text{with } k_1, k_2 \in K, r \geq 0)$$

This norm gives a left G -invariant metric $d(\cdot, \cdot)$ on G/K by

$$d(gK, hK) = \|g^{-1}h\| = \|h^{-1}g\|$$

The triangle inequality follows from the submultiplicativity of the norm.

Take η to be the characteristic function of the left and right K -invariant set of group elements of norm at most e^δ , with small $\delta > 0$. That is,

$$\eta(g) = \begin{cases} 1 & (\text{for } \|g\| \leq e^\delta) \\ 0 & (\text{for } \|g\| > e^\delta) \end{cases}$$

or

$$\eta\left(k_1 \cdot \begin{pmatrix} e^{r/2} & 0 \\ 0 & e^{-r/2} \end{pmatrix} \cdot k_2\right) = \begin{cases} 1 & (\text{for } r \leq \delta) \\ 0 & (\text{for } r > \delta) \end{cases} \quad (\text{with } r \geq 0)$$

[1.2] Upper bound on a kernel The map $f \rightarrow (\eta \cdot f)(x)$ on automorphic forms f can be expressed as integration of f against a sort of automorphic form q_x by winding up the integral, as follows.

$$\begin{aligned} (\eta \cdot f)(x) &= \int_G f(xy) \eta(y) dy = \int_G f(y) \eta(x^{-1}y) dy = \int_{\Gamma \backslash G} \left(\sum_{\gamma \in \Gamma} f(\gamma y) \eta(x^{-1}\gamma y) \right) dy \\ &= \int_{\Gamma \backslash G} f(y) \cdot \left(\sum_{\gamma \in \Gamma} \eta(x^{-1}\gamma y) \right) dy \end{aligned}$$

Thus, for $x, y \in G$ put

$$q_x(y) = \sum_{\gamma \in \Gamma} \eta(x^{-1}\gamma y)$$

The norm-squared of q_x , as a function of y alone, is

$$|q_x|_{L^2(\Gamma \backslash G)}^2 = \int_{\Gamma \backslash G} \sum_{\gamma \in \Gamma} \sum_{\gamma' \in \Gamma} \eta(x^{-1}\gamma\gamma y) \bar{\eta}(x^{-1}\gamma'y) dy = \int_G \sum_{\gamma \in \Gamma} \eta(x^{-1}\gamma y) \bar{\eta}(x^{-1}y) dy$$

after unwinding. For both $\eta(x^{-1}\gamma y)$ and $\eta(x^{-1}y)$ to be non-zero, the distance from x to both y and γy must be at most δ . By the triangle inequality, the distance from y to γy must be at most 2δ . For x in a fixed compact C , this requires that y be in ball of radius δ , and that $\gamma y = y$. Since K is compact and Γ is discrete, the isotropy groups of all points in G/K are finite. Thus,

$$|q_x|_{L^2(\Gamma \backslash G)}^2 \ll \int_{d(x,y) \leq \delta} 1 dy \asymp \delta^3$$

[1.3] Lower bound on eigenvalues A non-trivial *lower* bound for $\chi_f(\eta)$ can be given for $\delta \ll 1/t_f$, as follows. With spherical function φ^o in the s^{th} principal series, the corresponding eigenvalue is

$$\chi_s(\eta) = \int_G \eta(g) \varphi^o(g) dg = \int_{r \leq \delta} \varphi^o\left(k \cdot \begin{pmatrix} e^{r/2} & 0 \\ 0 & e^{-r/2} \end{pmatrix}\right) dg$$

In fact, a qualitative argument clearly indicates the outcome, although we will also carry out a more explicit computation. For the qualitative argument, we need qualitative metrical properties of the Iwasawa decomposition. Let P^+ be the upper-triangular matrices in G with positive real entries, and $K = SU(2)$. Let $g \rightarrow p_g k_g$ be the decomposition. We claim that $\|g\| \leq \delta$ implies $\|p_g\| \ll \delta$ for small $\delta > 0$. This is immediate, since the Jacobian of the map $P^+ \rightarrow G/K$ near $e \in P^+$ is *invertible*.

But, also, the Iwasawa decomposition is easily computed here, and the integral expressing the eigenvalue can be estimated explicitly: elements of K can be parametrized as

$$k = \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ -\beta & \alpha \end{pmatrix} \quad (\text{where } |\alpha|^2 + |\beta|^2 = 1)$$

and let $a = e^{r/2}$. Then

$$k \cdot \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} * & * \\ -a\beta & \alpha/a \end{pmatrix}$$

Right multiplication by a suitable element k_2 of $SU(2)$ rotates the bottom row to put the matrix into P^+ :

$$k \cdot \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot k_2 = \begin{pmatrix} * & * \\ 0 & \sqrt{(-a|\beta|)^2 + (|\alpha|/a)^2} \end{pmatrix}$$

Thus,

$$\chi_s(\eta) = \int_{r \leq \delta} \left((-a|\beta|)^2 + (|\alpha|/a)^2 \right)^{-s} dg$$

Rather than compute the integral exactly, make δ small enough to give a lower bound on the integrand, such as would arise from

$$\left| \left((-a|\beta|)^2 + (|\alpha|/a)^2 \right)^{-s} - 1 \right| < \frac{1}{2} \quad (\text{for all elements of } K)$$

Since $|\alpha|^2 + |\beta|^2 = 1$, for small r ,

$$(-e^{r/2}|\beta|)^2 + (|\alpha|/e^{r/2})^2 = e^r|\beta|^2 + |\alpha|^2/e^r \asymp (1+r)|\beta|^2 + (1-r)|\alpha|^2 \ll 1+r$$

Thus, for small $0 \leq r \leq \delta$,

$$\left| (e^r|\beta|^2 + |\alpha|^2/e^r)^{-s} - 1 \right| \ll |s| \cdot r$$

Thus, $0 \leq r \leq \delta \ll \frac{1}{|s|}$ suffices to make this less than $\frac{1}{2}$. That is, with η the characteristic function of the δ -ball, we have the lower bound

$$|\chi_s(\eta)| = \int_G \eta(g) \varphi^o(g) dg \gg \int_{r \leq \delta} 1 = \text{vol}(\delta\text{-ball}) \asymp \delta^3 \quad (\eta \text{ char fcn of } \delta\text{-ball, for } |s| \ll 1/\delta,)$$

Taking δ as large as possible compatible with $\delta \ll 1/|s|$ gives the bound

$$\chi_s(\eta) \gg \delta^3 \quad (\text{for } |s| \ll 1/\delta, \eta \text{ the characteristic function of } \delta\text{-ball})$$

From the L^2 automorphic spectral expansion of q_x , apply Plancherel

$$\sum_F |\langle q_x, F \rangle|^2 + \frac{|\langle q_x, 1 \rangle|^2}{\langle 1, 1 \rangle} + \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\langle q_x, E_s \rangle|^2 dt = |q_x|_{L^2(\Gamma \backslash G/K)}^2 \ll \delta^3$$

Truncating this to Bessel's inequality and dropping the single residual term,

$$\sum_{|s_F| \leq T} |\langle q_x, F \rangle|^2 + \frac{1}{2\pi} \int_{-T}^{+T} |\langle q_x, E_s \rangle|^2 dt \ll \delta^3$$

Recall that for a the spherical vector $f \in I_s$

$$\langle q_x, f \rangle = \chi_s(\eta) \cdot f$$

and use the inequality $\chi_s(\eta) \gg \delta^3$ from above for this restricted parameter range, obtaining

$$\sum_{|s_F| \leq T} (\delta^3 \cdot |F(x)|)^2 + \int_{-T}^{+T} (\delta^3 \cdot |E_s(x)|)^2 dt \ll \delta^3$$

Multiply through by $T^6 \asymp 1/\delta^6$ to obtain the **standard estimate**

$$\sum_{|s_F| \leq T} |F(x)|^2 + \int_{-T}^{+T} |E_s(x)|^2 dt \ll T^3$$

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