(*February 19, 2005*)

Meromorphic continuation of higher-rank Eisenstein series

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Expanded version of a talk given in Tel Aviv, Israel, 22 March 2001.

To *trivialize* the proof of meromorphic continuation of Eisenstein series, we want to use only standard facts, together with Bernstein's formulation of a *continuation principle*. In particular, one should prove meromorphic continuation *as a vector-valued function*, from which one infers that the continuation is of moderate growth, as are any residues.

To avoid inessential complications, we only consider automorphic forms on groups $G = GL(n, \mathbf{R})$, left invariant under the full group $\Gamma = GL(n, \mathbf{Z})$, right invariant under the corresponding orthogonal group $K = O(n, \mathbf{R})$, and with trivial central character.

A slightly more serious assumption is that the parabolic from which we 'induce' will be maximal (proper). This assumption is mostly a matter of convenience and simplicity.

An undesirable but essential assumption (for the moment) is that the 'data' on the Levi component is *cuspidal*. For now, it does not seem that this can be dropped, although one may hope.

- Weak-to-strong issues

- A continuation principle

- Cuspidal-data Eisenstein series on $GL(n)$

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1. Weak-to-strong issues

Definition: Let V be a topological vectorspace. A V-valued function $s \to f(s)$ (with s in a connected complex manifold) is *weakly holomorphic* if for every continuous linear functional λ on V the $\mathbf{C}$-valued function $s \to \lambda(f(s))$ is holomorphic.

Proposition: For a locally convex topological vectorspace V, a weakly holomorphic V-valued function f is *continuous.*

Proof: This is a standard result, following from the Banach-Alaoglu theorem and a variant of the Banach-Steinhaus theorem. ///

An immediate goal is to determine useful contexts in which *weak holomorphy* implies *holomorphy.* To emphasize the distinction, we may use the phrase *strong holomorphy* rather than simply *holomorphy.*

Definition: A family of operators $T_s : V \to W$ from one topological vectorspace to another is *weakly holomorphic* in a parameter s (in a connected complex manifold, for example a connected open subset of $\mathbf{C}$) if for every vector $v \in V$ and for every continuous functional $\mu \in W^*$ the $\mathbf{C}$-valued function $\mu(T_s v)$ is holomorphic in s.

In a different direction, but likewise essential:

Proposition: Let $S_s : X \to Y$ and $T_s : Y \to Z$ be two weakly holomorphic families of continuous linear operators on topological vectorspaces X, Y, Z. Then the composition $T_s \circ S_s : X \to Z$ is weakly holomorphic. Similarly, for a weakly holomorphic Y-valued function $s \to f(s)$, the composite $T_s \circ f$ is a weakly holomorphic Z-valued function.

Proof: This is an immediate corollary of Hartogs' theorem that separate analyticity of a function of several complex variables implies joint analyticity (without any other hypotheses). Consider the family of operators

$$T_t \circ S_s$$

By the definition of weak holomorphy, for $x \in X$ and a linear functional $\mu \in Z^*$ the **C**-valued function

$$(s,t) \to \mu(T_t(S_s(x))$$

is separately analytic. By Hartogs' theorem, it is jointly analytic. It follows that the diagonal function

$$s \to (s,s) \to \mu(T_s(S_s(x))$$

is analytic. The second assertion has a nearly identical proof. ///

It is striking that no hypotheses on the topological vectorspaces are necessary for this composition closure property of weakly holomorphic maps. By contrast, now we must attend to *weak-to-strong issues*, that is, possible inference of *(strong) holomorphy* from *weak holomorphy.*

Definition: A *Gelfand-Pettis* or *weak* integral of a function $s \to f(s)$ on a measure space (X,μ) with values in a topological vectorspace V is an element $I \in V$ so that for all $\lambda \in V^*$

$$\lambda(I) = \int_X f(s)\, d\mu(s)$$

Definition: A topological vectorspace is *quasi-complete* if every *bounded* (in the topological vectorspace sense, not necessarily the metric sense) Cauchy *net* is convergent.

Proposition: Continuous compactly-supported functions $f : X \to V$ with values in *quasi-complete* (locally convex) topological vectorspaces V have Gelfand-Pettis integrals with respect to finite positive regular Borel measures μ on compact spaces X, and these integrals are *unique.* In particular, for a μ with total measure $\mu(X) = 1$, the integral $\int_X f(x)\, d\mu(s)$ lies in the closure of the convex hull of the image $f(X)$ of the measure space X.

Proof: See Bourbaki's *Integration.* (Thanks to Jacquet for bringing this reference to my attention.) ///

The following property of Gelfand-Pettis integrals is broadly useful in applications, such as justifying differentiation under integrals.

Proposition: Let $T : V \to W$ be a continuous linear map, and let $f : X \to V$ be a continuous compactly supported V-valued function on a topological measure space X with finite positive Borel measure μ. Suppose that V is locally convex and quasi-complete, so that (from above) a Gelfand-Pettis integral of f exists and is unique. Suppose that W is locally convex. Then

$$T\left(\int_X f(x)\, d\mu(x)\right) = \int_X Tf(x)\, d\mu(x)$$

In particular, $T\left(\int_X f(x)\, d\mu(x)\right)$ is a Gelfand-Pettis integral of $T \circ f$.

Proof: First, the integral exists in V, from above. Second, for any continuous linear functional λ on W, $\lambda \circ T$ is a continuous linear functional on V. Thus, by the defining property of the Gelfand-Pettis integral, for every such λ

$$(\lambda \circ T)\left(\int_X f(x)\, d\mu(x)\right) = \int_X (\lambda T f)(x)\, d\mu(x)$$

This exactly asserts that $T\left(\int_X f(x)\, d\mu(x)\right)$ is a Gelfand-Pettis integral of the W-valued function $T \circ f$. Since the two vectors $T\left(\int_X f(x)\, d\mu(x)\right)$ and $\int_X Tf(x)\, d\mu(x)$ give identical values under all $\lambda \in W^*$, and since W is locally convex, these two vectors are equal, as claimed. ///

Corollary: Let V be quasi-complete and locally convex. Then weakly holomorphic V-valued functions are (strongly) holomorphic.

Proof: The Cauchy integral formulas involve continuous integrals on compacta, so these integrals exist as Gelfand-Pettis integrals. Thus, we can obtain V-valued convergent power series expansions for weakly holomorphic V-valued functions, from which (strong) holomorphy follows by an obvious extension of Abel's theorem that analytic functions are holomorphic. See also Rudin's *Functional Analysis* in which the hypothesis that Gelfand-Pettis integrals exist is observed to be sufficient to reach this conclusion.
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Corollary: Give the space $\mathrm{Hom}^o(X, Y)$ of continuous mappings $T : X \to Y$ from an LF space X (strict colimit of Fréchetspaces, e.g., a Fréchetspace) to a quasi-complete space Y the natural *weak operator* topology as follows. For $x \in X$ and $\mu \in Y^*$, define a seminorm $p_{x,\mu}$ on $\mathrm{Hom}^o(X, Y)$ by

$$p_{x,\mu}(T) = |\mu(T(x))|$$

Then with this topology $\mathrm{Hom}^o(X, Y)$ is quasi-complete.

Proof: The collection of finite linear combinations of the functionals

$$T \to \mu(Tx)$$

is exactly the dual space of $\mathrm{Hom}^o(X, Y)$ when it is given the weak operator topology, so we can invoke the previous result. ///

Corollary: Let $s \to T_s$ be a weakly holomorphic $\mathrm{Hom}^o(X, Y)$-valued function, meaning that for every $x \in X$ and $\mu \in Y^*$ the $\mathbf{C}$-valued function

$$s \to \mu(T_s(x))$$

is holomorphic. Then $s \to T_s$ is holomorphic when $\mathrm{Hom}^o(X, Y)$ is given the weak operator topology. ///

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2. A continuation principle

Let V be a topological vector space. Following Bernstein, a *system of linear equations X_o in V* is a collection

$$X_o = \{(V_i, T_i, v_i) : i \in I\}$$

where I is a (not necessarily countable) set of indices, each V_i is a topological vector space,

$$T_i : V \to V_i$$

is a continuous linear map for each index i, and $v_i \in V_i$ are the 'targets'. A *solution* of the system X_o is a vector $v \in V$ such that $T_i(v) = v_i$ for all indices i. The set of solutions is denoted by $\mathrm{Sol}\, X_o$.

Now let the systems of linear equations $X_s = \{V_i, T_{i,s}, v_{i,s}\}$ depend on a parameter s varying in a connected complex manifold, *the parameter space*. Say that the *parametrized linear system* $X = \{X_s : s \in S\}$ is *holomorphic* in s if $T_{i,s}$ and $v_{i,s}$ are weakly holomorphic in s. (Note that $\{V_i\}$ does not depend upon s.)

Definition: Let $X = \{X_s\}$ be a parametrized system of linear equations in a space V, holomorphic in s. Suppose that there is a finite-dimensional space F and a weakly holomorphic family of continuous linear maps $f_s : F \to V$ such that, for each s, $\mathrm{Im}\, f_s \supset \mathrm{Sol}\, X_s$. Then say that $\{f_s\}$ is a *finite holomorphic envelope* for the system X, and that X is of *finite type*.

Definition: Let $U_\alpha, \alpha \in A$ be an open cover of the parameter space. Suppose that for each $\alpha \in A$ we have a finite envelope $\{f_s^{(\alpha)} : s \in U_\alpha\}$ for the system

$$X^{(\alpha)} = \{X_s : s \in U_\alpha\}$$

Then the collection

$$\{f_s^{(\alpha)} : s \in U_\alpha, \alpha \in A\}$$

is said to be a *locally finite holomorphic envelope* of X.

Remark: If there is a meromorphic continuation v_s of a solution, then by taking $F = \mathbf{C}$ and

$$f_s : \mathbf{C} \to V$$

to be

$$f_s(z) = z \cdot v_s$$

we trivially obtain a finite holomorphic envelope for parameter values s away from the poles of v_s. That is, if there is a meromorphic continuation, then for trivial reasons there is a finite holomorphic envelope, and the system is of finite type.

Theorem: (*Bernstein*) *Continuation Principle:* Let $X = \{X_s : s\}$ be a *locally finite* system of linear equations

$$T_{i,s} : V \to V_i$$

for s varying in a connected complex manifold. Suppose that each V_i is (locally convex and) *quasi-complete*. Then the *continuation principle* holds. That is, if for s in some non-empty open subset there is a unique solution v_s, then this solution depends meromorphically upon s, has a meromorphic continuation to s in the whole manifold, and for fixed s off a locally finite set of analytic hypersurfaces inside the complex manifold, the solution v_s is the *unique* solution to the system X_s.

Proof: This reduces to a holomorphically parametrized version of Cramer's rule, in light of comments above about weak-to-strong principles and composition of weakly holomorphic maps.

It is sufficient to check the continuation principle locally, so let $f_s : F \to V$ be a family of morphisms so that $\mathrm{Im}\, f_s \supset \mathrm{Sol}\, X_s$, with F finite-dimensional. We can reformulate the statement in terms of the finite-dimensional space F. Namely, put

$$K_s^+ = \{v \in F : f_s(v) \in \mathrm{Sol}\, X_s\} = \{ \text{ inverse images in } F \text{ of solutions } \}$$

(The set K_s^+ is an affine subspace of F.) By elementary finite-dimensional linear algebra, X_s has a unique solution if and only if

$$\dim K_s^+ = \dim \ker f_s$$

The weak holomorphy of $T_{i,s}$ and f_s yield the weak holomorphy of the composite $T_{i,s} \circ f_s$ from the finite-dimensional space F to V_i, by the corollary of Hartogs' theorem above. The finite-dimensional space F is certainly LF, and V_i is quasi-complete, so by invocation of results above on weak holomorphy the space $\mathrm{Hom}^o(F, V_i)$ is quasi-complete, and a weakly holomorphic family in $\mathrm{Hom}^o(F, V_i)$ is in fact holomorphic.

Take $F = \mathbf{C}^n$. Using linear functionals on V and V_i which separate points we can describe $\ker f_s$ and K_s^+ by systems of linear equations of the forms

$$\ker f_s = \{(x_1, \ldots, x_n) \in F : \sum_j a_{\alpha\, j}\, x_j = 0, \;\; \alpha \in A\}$$

$$K_s^+ = \{ \text{ inverse images of solutions } \} = \{(x_1, \ldots, x_n) \in F : \sum_j b_{\beta\, j}\, x_j = c_\beta, \;\; \beta \in B\}$$

where $a_{\alpha\, j}$, $b_{\beta\, j}$, c_β all depend implicitly upon s, and are holomorphic $\mathbf{C}$-valued functions of s. (The index sets A, B may be of arbitrary cardinality.) Arrange these coefficients into matrices M_s, N_s, Q_s holomorphically parametrized by s, with entries

$$M_s(\alpha, j) = a_{\alpha\, j} \qquad N_s(\beta, j) = b_{\beta\, j} \qquad Q_s(\beta, j) = \begin{cases} b_{\beta\, j} & \text{for } 1 \le j \le n \\ c_\beta & \text{for } j = n \end{cases}$$

of sizes card(A)-by-n, card(B)-by-n, card(B)-by-$(n+1)$. We have

$$\dim\ker f_s = n - \mathrm{rank}\, M_s$$

Certainly for all s

$$\mathrm{rank}\, N_s \le \mathrm{rank}\, Q_s$$

and if the inequality is *strict* then there is *no solution* to the system X_s. By finite-dimensional linear algebra, assuming that $\mathrm{rank}\, N_s = \mathrm{rank}\, Q_s$,

$$\dim K_s^+ = n - \mathrm{rank}\, B_s$$

Therefore, the condition that $\dim K_s^+ = \dim\ker f_s$ can be rewritten as

$$\mathrm{rank}\, M_s = \mathrm{rank}\, N_s = \mathrm{rank}\, Q_s$$

Let S_o be the dense subset (actually, S_o is the complement of an analytic subset) of the parameter space where $\mathrm{rank}\, M_s$, $\mathrm{rank}\, N_s$, and $\mathrm{rank}\, Q_s$ all take their maximum values. Since by hypothesis $S_o \cap \Omega$ is not empty, and since the ranks are equal for $s \in \Omega$, all those maximal ranks are equal to the same number r. Then for all $s \in S_o$ the rank condition holds and X_s has a solution, and the solution is unique.

In order to prove the continuation principle we must show that $X = \{X_s\}$ has a meromorphic solution v_s. Making use of the finite envelope of the system X, to find a meromorphic solution of X it is enough to find a meromorphic solution of the parametrized system

$$Y = \{Y_s\}$$

where

$$Y_s = \{\sum b_{\beta\, i} x_i = c_\beta : \text{ for all } \beta\}$$

with implicit dependence upon s. Let r be the maximum rank, as above. Choose $s_o \in S_o$ and choose an r-by-r minor

$$D_{s_o} = \{b_{\beta,j} : \beta \in \{\beta_1, \ldots, \beta_r\}, j \in \{j_1, \ldots, j_r\}\}$$

of full rank, inside the matrix N_{s_o}, with constraints on the indices as indicated. Let $S_1 \subset S_o$ be the set of points s where D_s has full rank, that is, where $\det D_s \neq 0$. Consider the system of equations

$$Z = \{ \sum_{j \in \{j_1, \ldots, j_r\}} b_{\beta\, j}\, x_j = c_\beta : \beta \in \{\beta_1, \ldots, \beta_r\}\} \quad \text{(with } s \text{ implicit)}$$

By Cramer's Rule, for $s \in S_1$ this system has a unique solution $(x_{1,s}, \ldots, x_{r,s})$. Further, since the coefficients are holomorphic in s, the expression for the solution obtained via Cramer's rule show that the solution is meromorphic in s. Extending this solution by $x_j = 0$ for j not among $j_1, \ldots, j_r$, we see that it satisfies the r equations corresponding to rows $\beta \in \{\beta_1, \ldots, \beta_r\}$ of the system Y_s. Then for $s \in S_1$ the equality $\mathrm{rank}\, N_s = \mathrm{rank}\, Q_s = r$ implies that after satisfying the first r equations of Y_s it will automatically satisfy the rest of the equations in the system Y_s.

Thus, the system has a *weakly* holomorphic solution. Earlier observations on weak-to-strong principles assure that this solution is holomorphic. This proves the continuation principle. ///

Remark: Note that v_s is continued as a *vector-valued* function, rather than simply having a sort of 'weak' meromorphic continuation of some collection of scalar-valued functions $\lambda(v_s)$ for $\lambda \in V^*$. In applications this can be used to obtain growth estimates on the continued v_s, and on its residues. Note that meromorphic continuation of a separating family of coefficient functions is insufficient to obtain meromorphic continuation as a vector-valued function: consider the example

$$s \to (1^{-s}, 2^{-s}, 3^{-s}, 4^{-s}, 5^{-s}, \ldots)$$

This function has values in ℓ^2 for $\mathrm{Re}(s) > 1/2$, for example. Viewed as lying in ℓ^2 or in a variety of other spaces, many coefficient functions have meromorphic continuations (and some do not), but the whole thing does *not* have a meromorphic continuation as an ℓ^2-valued function.

We need some practical criteria to assure the finite envelope property:

Proposition: (*Dominance*) (Called *inference* by Bernstein.) Let $X' = \{X'_s\}$ be another holomorphically parametrized system of equations in a linear space V', with the same parameter space as a system $X = \{X_s\}$ on a space V. We say that X' *dominates* X if there exists a family of morphisms $h_s : V' \to V$, weakly holomorphic in s, so that for each s

$$\mathrm{Sol}\, X_s \subset h_s(\mathrm{Sol}\, X'_s)$$

If X'_s is locally finite then X_s is locally finite.

Proof: The fact that compositions of weakly holomorphic mappings are weakly holomorphic assures that X'_s really meets the definition of 'system'. Granting this, the conclusion is clear. ///

The following criterion has a slightly delicate feature: it requires holomorphy in the *uniform-norm* topology on operators, not in the weak operator topology.

Proposition: (*Banach-space criterion*) Suppose V is a Banach space and X is given by one parametrized homogeneous equation $T_s(v) = 0$, with $T_s : V \to W$, where W is also a Banach space, and where $s \to T_s$ is holomorphic for the uniform-norm Banach-space topology on $\mathrm{Hom}^o(V, W)$. Suppose that for some fixed s_o there exists an operator $A : W \to V$ so that the composite

$$A \circ \lambda_{s_o}$$

has *finite-dimensional kernel* V_o and *closed image* V_1. Then X_s is of finite type in some neighborhood of s_o.

Proof: This is elementary Banach space theory, although there is a little trick involved. Let V_1 be the image of $A \circ T_{s_o}$, and let V_o be the kernel of $A \circ T_{s_o}$. By the Hahn-Banach theorem there exist continuous linear maps $\mathrm{pr}_{V_o} : V \to V_o$ and $\mathrm{pr}_{V_1} : V \to V_1$ which are 'projections' in the weak sense that for $i = 1, 2$

$$\mathrm{pr}_{V_i}\big|_{V_i} = 1_{V_i}$$

Consider a new system X'_s in V, given by the equation

$$T'_s(v) = 0$$

where

$$T'_s = \mathrm{pr}_{V_1} \circ A \circ T_s : V \to V_1$$

Since every solution of X is a solution of X', X' dominates X. Consider the family of maps

$$\varphi_s = \mathrm{pr}_{V_o} \oplus T'_s : V \to V_o \oplus V_1$$

where $V_o \oplus V_1$ is given the natural Banach-space topology. Since the family $s \to A \circ T_s$ is continuous for the uniform-norm topology on $\mathrm{Hom}^o(V, V)$, the same is true of the family $s \to \varphi_s$. By construction, φ_{s_o} is a bijection, and by the open-mapping theorem φ_{s_o} is an isomorphism. The (continuous) inverse $\varphi_{s_o}^{-1}$ has an operator norm $0 < \delta^{-1} < +\infty$. For s sufficiently near s_o so that

$$|\varphi_{s_o} - \varphi_s| < \frac{\delta}{2}$$

we find that

$$|\varphi_s(x)| \geq |\varphi_{s_o}(x)| - |\varphi_s(x) - \varphi_{s_o}(x)| \geq \delta \cdot |x| - \frac{\delta}{2} \cdot |x| \geq \frac{\delta}{2} \cdot |x|$$

This shows that φ_s is an isomorphism for s in a neighborhood of s_o.

Next, we show that the map $s \to \varphi_s^{-1}$ is holomorphic on a neighborhood of s_o. To do this, it suffices to observe that the uniform norm topology on $\mathrm{Hom}^o(V, V_o \oplus V_1)$ when restricted to the subset of invertible elements is the same as the uniform norm topology on $\mathrm{Hom}^o(V_o \oplus V_1, V)$ restricted to invertible elements. And, indeed, this is so, because on a sufficiently small neighborhood of an invertible map T_o the inversion map $T \to T^{-1}$ is continuous, by an elementary Banach-space computation.

Now we can easily see that there is a finite envelope for the parametrized system X':

$$\mathrm{Sol}\, X'_s = \varphi_s^{-1}(V_o \oplus \{0\})$$

and V_o is finite-dimensional by hypothesis. ///

Corollary: (*Compact operator criterion*) Suppose that V is a Banach space and that X is given by one parametrized homogeneous equation $T_s(v) = 0$, with $T_s : V \to W$, where W is also a Banach space, and where $s \to T_s$ is holomorphic for the uniform-norm Banach-space topology on $\mathrm{Hom}^o(V, W)$. Suppose that for some fixed s_o the operator T_{s_o} has a left inverse modulo compact operators, that is, that there exists an operator $A : W \to V$ so that

$$A \circ T_{s_o} = \lambda_o + (\text{compact operator})$$

for a non-zero scalar operator λ_o. Then X_s is of finite type in some neighborhood of s_o. ///

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3. Cuspidal-data Eisenstein series for $GL(n, \mathbf{Z})$

Now we apply this continuation principle to spherical Eisenstein series on $GL(n, \mathbf{R})$.

Let N_o be the upper-triangular unipotent matrices in $G = GL(n, \mathbf{R})$. Let A^+ be the connected component of the identity in the subgroup A of diagonal matrices in G. As usual define a function $G \to A^+$, denoted

$$x \to a_x$$

by an Iwasawa decomposition

$$x \in N_o \cdot a_x \cdot K$$

For a root α, use the standard notation

$$a_x^\alpha = |\alpha(a_x)|$$

and also for complex numbers s write

$$a_x^{s\alpha} = |\alpha(a_x)|^s$$

In this situation *every (rational) parabolic subgroup in G is Γ-conjugate to one of the standard parabolics* containing the standard minimal parabolic P_o consisting of upper-triangular matrices in G.

Let Σ^o be the set of simple positive roots. For a fixed positive constant t, as usual a *Siegel set* $\mathbf{S}_t$ in G is a set of the form

$$\mathbf{S}_t = \{nak : n \in N_o, k \in K, a \in A^+, \alpha(a) \geq t \text{ for all } \alpha \in \Sigma^o\}$$

Our assumption on Γ assures that there exists a Siegel set $\mathbf{S}_t$ so that

$$\Gamma \cdot \mathbf{S}_t = G$$

In fact, by Minkowski reduction we know that this is so with constant $t = \sqrt{3}/2$.

Let P be a standard parabolic and N its unipotent radical. For f an $N \cap \Gamma$-invariant function, the *constant term* $f_P = c_P(f)$ of f along P is defined as usual to be

$$c_P f(g) = \int_{(N\cap\Gamma)\backslash N} f(ng)\, dn$$

We may often normalize the Haar measure dn on N so that the measure of the indicated quotient is 1, which has the convenient side effect that the operation of taking the contant term is idempotent.

We consider only left $GL(n, \mathbf{Z})$-invariant, right $O(n)$-invariant cuspforms on $GL(n, \mathbf{R})$ with trivial central character. We *assume* that cuspforms are square-integrable and generate irreducible unitary representations of G. We also *assume* that cuspforms are spherical Hecke algebra eigenfunctions at all finite primes.

Fix integers n_1, n_2. For $i = 1, 2$ let f_i be *cuspforms* on $GL(n_i, \mathbf{R})$. Let P be the parabolic subgroup of elements of the shape

$$P = \{ \begin{pmatrix} *_{n_1} & *_{n_1 \times n_2} \\ 0_{n_2 \times n_1} & *_{n_2} \end{pmatrix} \}$$

Put

$$\varphi_{s,f}(nmk) = |\det m_1|^{n_2 s} \, |\det m_2|^{-n_1 s} \, f_1(m_1) \, f_2(m_2)$$

where

$$m = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$$

with $m_i \in GL(n_i)$, so that m is in the standard Levi component M of the parabolic subgroup P, N is its unipotent radical, and k is in $O(n_1 + n_2)$. Let $\Gamma_P = \Gamma \cap P$. Then define the associated Eisenstein series by

$$E_{s,f}(g) = \sum_{\gamma \in \Gamma_P \backslash \Gamma} \varphi_{s,f}(\gamma g)$$

For $\mathrm{Re}\,(s)$ sufficiently positive, this series converges absolutely and uniformly on compacta. It is a left $GL(n_1+n_2, \mathbf{Z})$-invariant right $O(n_1+n_2)$-invariant function with trivial central character. Slightly enhancing Godement's argument for convergence, in the region of convergence this Eisenstein series is of moderate growth.

Theorem: (*Langlands et alia*) The Eisenstein series $E_{s,f}$ has a meromorphic continuation.

Proof: We set up a system of parametrized equations in Bernstein's sense, and prove uniqueness and finiteness in that context. As an additional minor element of simplicity we suppose that $n_1 > n_2$. That is, we suppose that the relevant parabolic subgroup is *not* self-associate.

First, let $L^2(Z\Gamma\backslash G/K, \ell)$ be the Hilbert space of locally integrable complex-valued functions f on G so that on any Siegel set

$$f(x) \, a_x^{-\ell}$$

is square-integrable. For $\mathrm{Re}\,(s)$ bounded above but above the bound for convergence, Eisenstein series lie in this space. For a parabolic Q, let N_Q be the unipotent radical of P, and for a left N_Q-invariant function f on G, let $c_Q f$ be the constant term of f along Q. Let $\mathrm{C}_c^\infty(G)^{\mathrm{inv}}$ denote the K-conjugation invariant test functions on G. For $\eta \in \mathrm{C}_c^\infty(G)^{\mathrm{inv}}$, and for a vector v in any representation of G on a quasi-complete locally convex vectorspace V, as usual let

$$\eta \cdot v = \int_G \eta(x) \, x \cdot v \, dx$$

For any η in $\mathrm{C}_c^\infty(G)^{\mathrm{inv}}$, let λ_s be the eigenvalue of $\varphi_{s,f}$, suppressing the reference to η. Let $P^w \neq P$ be the associate standard parabolic to P. The center Z_{P^w} of the standard Levi component M^w of P^w consists of elements

$$z = \begin{pmatrix} z_1 \cdot 1_{n_2} & 0 \\ 0 & z_2 \cdot 1_{n_1} \end{pmatrix}$$

Let w be a Weyl element which conjugates the standard Levi component M of P to the standard Levi component M^w of the associate parabolic P^w: $wMw^{-1} = M^w$. Define a function f^w on M^w by taking the conjugated version of f on M, namely

$$f^w(m') = f(w_1 m' w_1^{-1})$$

Finally, define the conjugated version of $\varphi_{s,f}$ by

$$\varphi^w_s(n'm'k) == |\det m'_1|^{n_1 s}\,|\det m'_2|^{-n_2 s}\, f^w(m')$$

where now (after flipping things by the Weyl element w_1) m'_1 is the $GL(n_2)$ part of m', and m'_2 is the $GL(n_1)$ part of m'.

The system $X = \{X_s\}$ of equations we use is

$$\begin{array}{rcll} (\eta - \lambda_s)v_s &=& 0 & \text{for all } \eta \in C^\infty_c(G)^{\text{inv}} \\ c_Q v_s &=& 0 & \text{for parabolics } Q \text{ other than } P, P^w \\ c_P v_s &=& \varphi_{s,f} & \\ (c_{P^w} v_s(zm') - |z_1/z_2|^{n_1 n_2 (1-s)})\, v_s(g) &=& 0 & \text{for } z \text{ in the center of the Levi component } M^w \text{ of } P^w \\ c_{Q_{M^w}}(c_{P^w} v_s) &=& 0 & \text{for all proper parabolics } Q_{M^w} \text{ of } M^w \\ (\eta^{M^w} - \lambda^{M^w}_s) c_{P^w} v_s &=& 0 & \text{for all } \eta^{M^w} \in C^\infty_c(M^w)^{\text{inv}} \end{array}$$

where $\lambda^{M^w}_s$ is the eigenvalue of η^{M^w} on φ^w_s restricted to M^w.

First, we should clarify the nature of the topological vectorspaces mapped-to in this system of equations. It is important (for later proof of the finiteness property) that the first family maps the Hilbert space $L^2(Z\Gamma\backslash G/K, \ell)$ to itself. By contrast, the linear operators for the other equations need *not* map to Hilbert spaces, and in fact *cannot* do so in just slightly more complicated examples. In the present example, it suffices to say that the target spaces are spaces of locally integrable functions, for example, although in general the target spaces must be spaces of distributions. Indeed, for self-associate parabolics, the equations above that in effect describe the power of determinants that appear must be replaced by differential equations, but since the inputs to the differential operators are merely square integrable we must allow distributions as valued. In summary, the 'main' family of equations attached to $\eta \in C^\infty_c(G)^{\text{inv}}$ maps the Hilbert space $L^2(Z\Gamma\backslash G/K, \ell)$ to itself, so these are elementary. By contrast, the other equations (in general) would involve maps to spaces of distributions left invariant under suitable subgroups of Γ, and right invariant under suitable maximal compact subgroups.

In all cases the target spaces are spaces of distributions or locally integrable functions *right invariant* under the relevant orthogonal group, and *left invariant* under the relevant collection of invertible integral matrices. Thus, the first family of equations uses maps to right $O(n_1+n_2)$-invariant and left $GL(n_1+n_2, \mathbf{Z})$-invariant functions. The conditions on the constant term along P_β (for simple positive root β) involve maps to right $P_\beta \cap O(n_1+n_2)$-invariant and left $P_\beta \cap GL(n_1+n_2, \mathbf{Z})$-invariant functions. Thus, these invariance properties are *built into* the systems of equations, rather than having to be separately asserted.

Remark: At some point, one can consider the point that not all 'conditions' that might be imposed can be couched as 'equations'.

Second, we should explicitly note why and how everything here is well-defined, in particular the apparent integral operator equations attached to $\eta \in C^\infty_c(G)^{\text{inv}}$ and to $\eta^{M^w} \in C^\infty_c(M^w)^{\text{inv}}$. We are supposing that the representation of M generated by $f = f_1 \otimes f_2$ is irreducible, as an implied consequence of the assumption that the f_i's are *cuspforms*. Let $K_M = K \cap M$. Since the convolution algebra $C^\infty_c(K_M\backslash M/K_M)$ is commutative (due to the presence of a suitable involution), the space of K_M-fixed vectors is one-dimensional. Then, via an Iwasawa decomposition, the space of K-fixed vectors in the representation of G generated by $\varphi_{s,f}$ is one-dimensional. Thus, the eigenvalues λ_s in the first family of equations above exist. Further, it is a direct formal computation that they are entire functions.

The other eigenvector conditions, with respect to $\eta^{M^w} \in C^\infty_c(M^w)^{\text{inv}}$, require for their sensibility that the indicated eigenvalues λ^{M^w} exist. When restricted to M^w, φ^w_s is just f^w multiplied by powers of determinants. Thus, again, the irreducibility of the representation generated by f^w, the K_{M^w}-invariance of f^w, and the commutativity of the convolution algebra $C^\infty_c(K_{M^w}\backslash M^w/K_{M^w})$, all combine to imply that φ^w_s restricted to M^w really is an eigenfunction for each one of these operators. The entireness of the eigenvalues as a function of the parameter s is easy.

Thus, we've verified that the indicated eigenvalues actually exist and are (entire) holomorphic functions of the parameter s, so the system makes sense.

Third, we verify that $E_{s,f}$ is a solution of this system, for $\mathrm{Re}(s)$ sufficiently large to have convergence.

Since formation of the Eisenstein series is a sum of left translates, it is a G-intertwining operator (for a right action of G), so the Eisenstein series $E_{s,f}$ (in the region of convergence at least) satisfies any equations that $\varphi_{s,f}$ does as long as the operators involved come from the *right* action of G.

It is a direct computation that, due to the cuspidality of f and the maximality of P, all constant terms of $E_{s,f}$ are zero except those along P and P^w. Thus, the Eisenstein series satisfies the second batch of equations.

Since P is not self-associate, the constant term along P itself is directly computed to be $\varphi_{s,f}$ again, as claimed.

The constant term of $E_{s,f}$ along P^w is more mysterious, since in classical terms it is an an infinite sum of integrals. If we rewrite things in adelic terms, then the P^w constant term of the Eisenstein series is due to a single Bruhat cell, and is

$$\int_{N^w_{\mathbf{A}}} \varphi_{s,f}(wnx)\, dn$$

The assumption that f_i is a Hecke eigenfunction at finite primes (in addition to the irreducibility condition at the archimedean place) assures that this integral is a constant multiple of $\varphi^w_s(m')$ for $m' \in M^w$. Thus, since this constant term is a multiple of $\varphi^w_{s,f}$

Remark: In fact, the constant term of $E_{s,f}$ along P^w differs from φ^w_s by a ratio of values of the tensor product L-function attached to f_1 and the adjoint of f_2. This phenomenon is already visible in the case of $SL(2, \mathbf{Z})$, was studied quite generally by Langlands. But it is important not to use any such information in order to prove properties of Eisenstein series, since in fact we want to use Eisenstein series to prove things about L-functions, as in the Langlands-Shahidi or Rankin-Selberg methods.

The equations describing the equivariance of c_{P^w} with respect to the center of M^w simply reflect the powers of determinant which occur in $\varphi_{s,f}$ and $\varphi^w_{s,f}$.

The equations that assert that the constant term along P^w restricted to M^w is a cuspform certainly are met by $E_{s,f}$, since $c_{P^w}E_{s,f}$ is a constant multiple of $\varphi^w_{s,f}$, and the f_i's are cuspforms by hypothesis.

The last family of equations is *designed* to be satisfied by $E_{s,f}$, once we realize that $c_{P^w}E_{s,f}$ is a constant multiple of $\varphi^w_{s,f}$, since $\varphi^w_{s,f}$ is an eigenfunction for $C^\infty_c(K_{M^w}\backslash M^w/K_{M^w})$, from the fact (implicit in our notion of *cuspform*) that the f_i's generate irreducible representations.

Fourth, we verify *uniqueness*, that for a non-empty open set of parameter values s there is at most one solution to the system. In particular, the non-empty set we take is $\mathrm{Re}(s)$ sufficiently large and off the real line. Let v be the *difference* between two solutions of the system at fixed s with $\mathrm{Re}(s)$ large. Since all but one of the equations are *homogeneous*, the only equation that changes is that the constant term along P is now required to be 0, rather than $\varphi_{s,f}$. We will show that v is square-integrable on $\Gamma\backslash G$, and from this obtain a contradiction. In fact, we prove that v is continuous and is bounded on Siegel sets. The finite volume of Siegel sets then proves the square-integrability.

First, any function $g \to \eta f(g)$ is continuous, since it inherits the continuity of η. Then the fact that $\eta\varphi_{s,f} = \varphi_{s,f}$ for some η (for fixed s) implies that $\varphi_{s,f}$ is continuous.

To prove the boundedness, we use the *theory of the constant term*, of which a convenient and sufficient form for our purposes is the following. First, for a (standard) simple positive root α let

$$A_\alpha = \bigcap_{\beta\neq\alpha,\ \beta \text{ simple}} \ker \beta$$

be (as indicated) the intersection of the kernels (on the torus consisting of all diagonal matrices in G) of all *other* positive simple roots. Let P_α be the standard parabolic whose (standard) Levi component is the centralizer of A_α. Thus, for simple root $\alpha = \alpha_{i,i+1}$ defined as usual by

$$\alpha_{i,i+1} \begin{pmatrix} m_1 & 0 & \dots & & \\ 0 & m_2 & 0 & \dots & \\ 0 & 0 & m_3 & 0 & \dots \\ & & & \ddots & 0 \\ 0 & 0 & \dots & 0 & m_n \end{pmatrix} = m_i/m_{i+1}$$

The corresponding parabolic $P_{\alpha_{i,i+1}}$ is

$$P_\alpha = \{ \begin{pmatrix} i\text{-by-}i & i\text{-by-}(n-i) \\ 0 & (n-i)\text{-by-}(n-i) \end{pmatrix} \}$$

For brevity, let c_α denote the constant term along P_α. Let f be a function on $\Gamma\backslash G$ of moderate growth on Siegel sets, and with some $\eta \in C_c^\infty(G)$ so that $\eta f = f$. Then the basic assertion we need is that, for x in a fixed Siegel set, for every $\ell > 0$,

$$(f - c_\alpha f)(x) = O(a_x^{-\ell\alpha}) \qquad (\text{as } a_x^\alpha \to +\infty)$$

That is, in the 'α direction going to infinity' (inside a Siegel set) f less its α^{th} constant term is of rapid decay. Thus, to estimate f as values of various simple roots go to infinity (inside a Siegel set), it suffices to estimate the corresponding constant terms along maximal proper parabolic subgroups. (Inside a Siegel set, for a sequence of points to 'go to infinity' requires that the values of at least one simple positive root go to infinity.)

Let α be the simple positive root α_{n_1,n_1+1} so that $P = P_\alpha$. Let $\alpha^w = \alpha_{n_2,n_2+1}$. The new system of equations satisfied by the difference v of two solutions of the original system explicitly asserts that $c_\beta v = 0$ for every simple positive root β other than α^w. Thus, by previous remarks, what remains is to verify that $c_{\alpha^w} v$ is bounded on a fixed Siegel set in G as $a_x^\alpha \to +\infty$.

All that we know of $c_{\alpha^w} v$ is that it is of the form

$$c_{\alpha^w} v(nm'k) = |\det m_1'|^{n_1(1-s)}\, |\det m_2'|^{-n_2(1-s)}\, h(m')$$

where

$$m' = \begin{pmatrix} m_1' & 0 \\ 0 & m_2' \end{pmatrix}$$

with $m_1' \in GL(n_2, \mathbf{R})$ and $m_2' \in GL(n_1, \mathbf{R})$, for some cuspform h on $GL(n_2, \mathbf{R}) \times GL(n_1, \mathbf{R})$ (with trivial central character, right invariant under the maximal compact). Further, we know that

$$(\eta^{M^w} - \lambda_s^{M^w})h = 0$$

where $\lambda_s^{M^w}$ is the eigenvalue $\varphi_{s,f}^w$ would have for $\eta^{M^w} \in C_c^\infty(M)^{\text{inv}}$. The basic estimates used to prove that family of theorems, together with the theory of the constant term, *also* show that if $\eta h = h$ for some η and if h is an L^2 cuspform then h is *bounded.* (By assumption it has trivial central character.)

Since $\text{Re}\,(s)$ is large the power $n_2(1-s)$ of the determinant of m_1' has negative real part, and the power $-n_1(1-s)$ of the determinant of m_2' has positive real part. Thus, in a fixed Siegel set as $a_x^{\alpha^w}$ goes to infinity the product of these powers of determinants goes to zero. The cuspform is bounded, so we have decay to 0.

Altogether, this shows that the difference v of two solutions of the original system is bounded on a Siegel set, so is square integrable. Now we prove that any such thing must be 0.

For $\eta \in C_c^\infty(G)^{\text{inv}}$ so that η is real-valued and $\eta(g^{-1}) = \eta(g)$, direct computation shows that the associated operator is self-adjoint on $L^2(Z\Gamma\backslash G)$ (where Z is the center of G). Thus, any (non-zero) eigenvector for such η must have *real* eigenvalue. But the function λ_s attached to such η is a *non-constant* holomorphic function, so cannot be real-valued on any non-empty open set. Thus, for any s with $\text{Re}\,(s)$ sufficiently large and for s

off the real line, there is some η giving a self-adjoint operator and with corresponding eigenvalue λ_s not real, which forces v (the difference of two solutions at that fixed s) to be 0. This finishes the proof of uniqueness.

Fifth, the last condition to verify is the *finiteness* property. By the *dominance* criterion for finiteness, it suffices to prove the finiteness property for the system obtained by replacing the *inhomogeneous* condition $c_P v_s = \varphi_{s,f}$ by a family of homogeneous equations completely analogous to those imposed upon the constant term along P^w, namely

$$\begin{array}{rcll} (c_P v_s(zm) - |z_1/z_2|^{n_1 n_2 s})\, v_s(g) & = & 0 & \text{for } z \text{ in the center of the Levi component } M \text{ of } P \\ c_{Q_M}(c_P v_s) & = & 0 & \text{for all proper parabolics } Q_M \text{ of } M \\ (\eta^M - \lambda_s^M) c_P v_s & = & 0 & \text{for all } \eta^M \in C_c^\infty(M)^{\text{inv}} \end{array}$$

where λ_s^M is the eigenvalue of η^M on $\varphi_{s,f}$ restricted to M. Indeed, to prove finiteness it is essentially unavoidable that we consider this new system, since the target spaces in a linear system *cannot* depend upon the parameter. Thus, we now consider the homogeneous system

$$\begin{array}{rcll} (\eta - \lambda_s) v_s & = & 0 & \text{for all } \eta \in C_c^\infty(G)^{\text{inv}} \\ c_Q v_s & = & 0 & \text{for parabolics } Q \text{ other than } P, P^w \\ (c_{P^w} v_s(zm') - |z_1/z_2|^{n_1 n_2 (1-s)})\, v_s(g) & = & 0 & \text{for } z \text{ in the center of the Levi component } M^w \text{ of } P^w \\ c_{Q_{M^w}}(c_{P^w} v_s) & = & 0 & \text{for all proper parabolics } Q_{M^w} \text{ of } M^w \\ (\eta^{M^w} - \lambda_s^{M^w}) c_{P^w} v_s & = & 0 & \text{for all } \eta^{M^w} \in C_c^\infty(M^w)^{\text{inv}} \\ (c_P v_s(zm) - |z_1/z_2|^{n_1 n_2 s})\, v_s(g) & = & 0 & \text{for } z \text{ in the center of the Levi component } M \text{ of } P \\ c_{Q_M}(c_P v_s) & = & 0 & \text{for all proper parabolics } Q_M \text{ of } M \\ (\eta^M - \lambda_s^M) c_P v_s & = & 0 & \text{for all } \eta^M \in C_c^\infty(M)^{\text{inv}} \end{array}$$

where $\lambda_s^{M^w}$ is the eigenvalue of η^{M^w} on φ_s^w restricted to M^w and where λ_s^M is the eigenvalue of η^M on $\varphi_{s,f}$ restricted to M.

To arrange things so as to meet the compact-operator criterion for finiteness, we must do a little work.

Recall that M_β denotes the standard Levi component and N_β the unipotent radical of the maximal proper parabolic P_β. Also A_β is the center of M_β. We identify

$$N_\beta \backslash P_\beta \approx M_\beta$$

and

$$N_\beta \backslash G / K \approx M_\beta / (M_\beta \cap K)$$

via the Iwasawa decomposition. For a left $\Gamma \cap M_\alpha$ invariant, A_α-invariant, and right $K \cap M_\alpha$ invariant function h on M_α, for complex s, and for a large positive constant t_o, we form the *tail-only* version of the function $\varphi_{s,f}$ used to form the Eisenstein series: define

$$\alpha\text{−tail}\varphi_{s,h}(nmk) = \begin{cases} |\det m_1|^{n_2 s}\, |\det m_2|^{-n_1 s}\, h(m) & (\text{for } a_{nmk}^\alpha \geq t_o\ ) \\ 0 & (\text{otherwise}) \end{cases}$$

where $n \in N_\alpha$, $m \in M_\alpha$ with $m_i \in GL(n_i)$, and $k \in K$. Similarly, for a left $\Gamma \cap M_{\alpha^w}$ invariant, A_{α^w}-invariant, and right $K \cap M_{\alpha^w}$ invariant function h on M_{α^w}, and for complex s, let

$$\alpha^w\text{−tail}\varphi^w_{s,h}(nmk) = \begin{cases} |\det m'_1|^{n_1 s}\, |\det m'_2|^{-n_2 s}\, h(m) & (\text{for } a_{nmk}^{\alpha^w} \geq t_o\ ) \\ 0 & (\text{otherwise}) \end{cases}$$

where $n \in N_{\alpha^w}$, $m \in M_{\alpha^w}$ with $m'_1 \in GL(n_2)$ and $m'_2 \in GL(n_1)$, and $k \in K$.

We can form Eisenstein series from a function α−tail $\varphi_{s,h}$ as above by

$$E^\alpha(\alpha\text{−tail } \varphi_{s,h})(x) = \sum_{\Gamma \cap P_\alpha \backslash \Gamma} \alpha\text{−tail } \varphi_{s,h})(\gamma \cdot x)$$

Similarly,

$$E^{\alpha^w}(\alpha^w\text{−tail } \varphi^w_{s,h_2}) = \sum_{\Gamma\cap P_{\alpha^w}\backslash\Gamma} \alpha^w\text{−tail } \varphi_{s,h})(\gamma \cdot x)$$

Relatively elementary estimate show that these series converge nicely for all complex s. Given a Siegel set S_t, for t_o large enough depending only upon α and S_t, on S_t

$$c_\beta E^\alpha(\alpha\text{−tail } \varphi_{s,h}) = \begin{cases} 0 & (\text{for } \beta \neq \alpha) \\ \alpha\text{−tail}\varphi_{s,h}) & (\text{for } \beta = \alpha) \end{cases}$$

Similarly, given a Siegel set S_t, for t_o large enough depending only upon α^w and S_t, on S_t

$$c_\beta E^{\alpha^w}(\alpha^w\text{−tail}\varphi^w_{s,h}) = \begin{cases} 0 & (\text{for } \beta \neq \alpha^w) \\ \alpha^w\text{−tail}\varphi^w_{s,h}) & (\text{for } \beta = \alpha^w) \end{cases}$$

Similarly, we need a tail-only version of constant terms. For a simple positive root β, and for a large positive constant t_o, the β-tail (at height t_o) of the P_β constant term of h is defined to be

$$\beta\text{−tail}c_\beta f(x) = \begin{cases} c_\beta f(x) & (\text{for } c_\beta(x) \geq t_o) \\ 0 & (\text{otherwise}) \end{cases}$$

With large fixed real constant t_o, let

$$L^2_{(t_o)}(Z\Gamma\backslash G/K, \ell) =$$

$$\{h \in L^2(Z\Gamma\backslash G/K, \ell) : c_\beta h = 0 \quad \text{unless } \beta = \alpha \text{ or } \beta = \alpha^w, \text{ and } \alpha\text{–tail } c_\alpha h = 0,\ \alpha^w\text{–tail}c_{\alpha^w} h = 0\}$$

This space is somewhat larger than the space of cuspforms (with indicated integrability), as it tolerates a fixed amount of non-vanishing of two constant terms. As in the proof that square-integrable cuspforms form a *closed* subspace of $L^2(Z\Gamma\backslash G/K)$, the same is true of $L^2_{(t_o)}(Z\Gamma\backslash G/K, \ell)$ in $L^2(Z\Gamma\backslash G/K, \ell)$. As usual, for a simple positive root β,

$$L^2_o(A_\beta(\Gamma \cap M_\beta)\backslash M_\beta/K \cap M_\beta)$$

is the space of right $K \cap M_\beta$ invariant, left $\Gamma \cap M_\beta$ invariant, trivial central character, square-integrable cuspforms on M_β. That is, for every proper parabolic Q of M_β, the constant term $c_Q h$ is 0 almost everywhere. A technical refinement of the simplest version of the *theory of the constant term* shows that $L^2_{(t_o)}(Z\Gamma\backslash G/K, \ell)$ lies inside $L^2(Z\Gamma\backslash G/K, -\ell)$ for every $\ell > 0$, that is, consists of L^2-rapid-decay functions.

Define the space of cuspforms on M_α of the same *type* as f by

$$V = \left\{h \in L^2_o(A_\alpha(\Gamma \cap M_\alpha)\backslash M_\alpha/K \cap M_\alpha) \text{ so that } \eta^M h = \lambda^M \cdot h\right\}$$

where λ^M is defined by $\eta^M f = \lambda^M \cdot f$. Similarly, the space of cuspforms of the same *type* as f^w on M_{α^w} is

$$V^w = \left\{h \in L^2_o(A_{\alpha^w}(\Gamma \cap M_{\alpha^w})\backslash M_{\alpha^w}/K \cap M_{\alpha^w}) \text{ so that } \eta^{M^w} h = \lambda^{M^w} \cdot h\right\}$$

where λ^{M^w} is defined by $\eta^{M^w} f^w = \lambda^{M^w} \cdot f^w$, with f^w being the w-conjugated version of f as earlier. By the basic theorem of Selberg, Gelfand, Piatetski-Shapiro, Langlands, *et alia*, both V and V^w are finite-dimensional, being non-zero eigenspaces for the compact operators (on cuspforms) $h \to \eta h$.

Again, $L^2(Z\Gamma\backslash G/K, -\ell)$ is the Hilbert space of locally integrable complex-valued functions f on G so that on any Siegel set

$$f(x)\, a_x^{+\ell}$$

is square-integrable. Note the change of sign on ℓ by comparison to $L^2(Z\Gamma\backslash G/K, \ell)$. Let ℓ be sufficiently large and positive. Define a parametrized family T_s of linear maps

$$T_s : L^2(Z\Gamma\backslash G/K, -\ell) \oplus V \oplus V^w \to L^2(Z\Gamma\backslash G/K, \ell)$$

by

$$h \oplus h_1 \oplus h_2 \to h + E^\alpha(\alpha\text{−tail } \varphi_{s,h_1}) + E^{\alpha^w}(\alpha^w\text{−tail } \varphi^w_{s,h_2})$$

Since the tail-only Eisenstein series are entire in s, it is easy to verify that T_s is strongly holomorphic with respect to the uniform operator norm topology on the space of linear operators between these Hilbert spaces.

First, we claim that any possible solution v of the homogeneous system of equations above is expressible in this form. Indeed, the system of equations requires that (up to the desired powers of determinants) $c_\alpha v$ and $c_{\alpha^w} v$ are cuspforms in V, V^w, respectively, and that $c_\beta v = 0$ for all other β. Thus, the difference

$$\mathrm{trunc}\, v = v - E^\alpha(\alpha\text{−tail } c_\alpha v) - E^{\alpha^w}(\alpha^w\text{−tail } c_{\alpha^w} v)$$

has P_β constant terms 0 for β other than α, α^w, and (as noted above) has $c_\alpha \mathrm{trunc}\, v(x) = 0$ for $a_x^\alpha \geq t_o$, and $c_{\alpha^w} \mathrm{trunc}\,(x) = 0$ for $a_x^{\alpha^w} \geq t_o$.

We must check that $\mathrm{trunc}\, v$ does lie in $L^2(Z\Gamma\backslash G/K, -\ell)$. First, by the very set-up of the system of equations, a solution v is in $L^2(Z\Gamma\backslash G/K, \ell)$. Second, Godement-style estimates show that the tail-only Eisenstein series are in $L^2(Z\Gamma\backslash G/K, \ell)$ for large-enough ℓ. Now the constant terms of the tail-only Eisenstein series *exactly* cancel the constant terms outside of compacta (rather than introducing additional, more mysterious parts of constant terms as would happen without the tail-only constraint). Thus, a careful version of *the theory of the constant term* shows that the difference $\mathrm{trunc}\, v$ is actually of *rapid* decay, so is certainly in $L^2(Z\Gamma\backslash G/K, -\ell)$.

To apply the compact-operator criterion for finiteness, at fixed complex s_o, take $\eta \in C_c^\infty(G)^{\mathrm{inv}}$ so that $\lambda_{s_o} \neq 0$. Consider the dominating system X'_s on $L^2(Z\Gamma\backslash G/K, -\ell) \oplus V \oplus V^w$ given by

$$(\eta - \lambda_s) \circ T_s = 0$$

for any such η. Since λ_s is scalar, this family of operators is still holomorphic. The compact-operator criterion says that it is enough to check that $(\eta - \lambda_{s_o}) \circ T_{s_o}$ has a left inverse A modulo compact operators. Define

$$A : L^2(Z\Gamma\backslash G/K, \ell) \to L^2(Z\Gamma\backslash G/K, -\ell) \oplus V \oplus V^w$$

by

$$v \to \Big(v - E^\alpha(c_\alpha^{(t_o)} v) - E^{\alpha^w}(c_{\alpha^w}^{(t_o)} v)\Big) \oplus 0 \oplus 0$$

Keep in mind that V and V^w are finite-dimensional. On the finite-codimension subspace $L^2(Z\Gamma\backslash G/K, -\ell)$ the composite $A \circ (\eta - \lambda_s) \circ T_s$ is

$$h \oplus 0 \oplus 0 \to (\eta - \lambda_{s_o})(\mathrm{trunc}\, h) \oplus 0 \oplus 0$$

Since $\lambda_{s_o} \neq 0$, if η were compact on $L^2(Z\Gamma\backslash G/K, -\ell)$, then the compact-operator criterion for finiteness would be met. And, indeed, for ℓ sufficiently large positive, any map $f \to \eta f$ on $L^2(Z\Gamma\backslash G/K, -\ell)$ is compact, by invocation of a slightly extended form of the classical theorem of Gelfand, Piatetski-Shapiro, Langlands, Selberg, *et alia*.

Remark: An obstacle to the *most* straightforward application of the theory of the constant term to try to conclude that the truncated solution $\mathrm{trunc}\, v$ above is of rapid decay in Siegel sets is that this $\mathrm{trunc}\, v$ seldom has any non-trivial $\eta \in C_c^\infty(G)^{\mathrm{inv}}$ with $\eta \mathrm{trunc}\, v = \mathrm{trunc}\, v$, since (having had tails of its constant terms subtracted) it is not even continuous. (A relation $\eta h = h$ implies continuity, by the continuity of η.)

Remark: There is some fragility to this argument. For example, it would be misguided to imagine that the compactness (and hence finiteness) conclusion is so universal that one might inadvertently appear to prove that truncated Eisenstein series themselves lie in the discrete spectrum.