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A stunt using traces

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Insightful specific choices of physical objects on which to do standard analytical things can be enlightening.

Consider a very simple Sturm-Liouville problem: to solve

$$\Delta u = f \text{ on } [a,b] \text{ with } u(a) = u(b) = 0$$

A *Green's function* [1] for this problem is [2]

$$G(x,y) = \begin{cases} (y-a)(x-b)/(b-a) & (\text{for } a \le y < x \le b) \\ (x-a)(y-b)/(b-a) & (\text{for } a \le x < y \le b) \end{cases}$$

The associated eigenvalue problem,

$$(\Delta - \lambda)u = 0$$

with the same boundary conditions, specialized to $a = 0$ and $b = 1$, is easily solved directly, yielding eigenvectors [3]

$$u_n(x) = \sin(\pi n x) \quad (\text{for } n \ge 1)$$

The *trace* of the *inverse* mapping

$$T : f \to \int_0^1 G(x,y)\, f(y)\, dy$$

can be evaluated two ways: sum the inverses of the eigenvalues for the differential operator, and as the integral along the diagonal, [4] $\int_a^b G(x,x)dx$ of the kernel

$$G(x,y) = \begin{cases} y(x-1) & (\text{for } 0 \le y < x \le 1) \\ x(y-1) & (\text{for } 0 \le x < y \le 1) \end{cases}$$

Thus,

$$\sum_{n \ge 1} \frac{1}{(\pi n)^2} = \int_0^1 x(x-1)\, dx = \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_0^1 = \frac{1}{3} - \frac{1}{2} = \frac{1}{6}$$

One can also evaluate $\zeta(2k)$ by computing the iterated kernel and taking its trace. For $\zeta(4)$ this is still not too unpleasant.

Naturally, one should have some care about taking traces of operators.

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[1] No, we are certainly *not* appealing to any apocryphal existence argument for Green's functions [sic] in general.

[2] To be annihilated by Δ (in x) away from $x = y$, for fixed y $f(x) = G(x,y)$ is piecewise linear, say $f(x) = A(x-a)$ for $a \le x < y$ and $f(x) = B(x-b)$ for $y < x \le b$, where A and B depend upon y. So that $f(x)$ is continuous at $x = y$ these two linear fragments must match at $x = y$, so $A(y-a) = B(y-b)$. The first derivative in x is then A to the left of y and B to the right. For the second derivative to be δ, $B - A = 1$. Solving for A and B, we obtain the indicated $G(x,y)$. Note that Δ applied to $G(x,y)$ in x gives a δ at y as desired, but also yields multiples of δ at the endpoints a, b. Thus, we pose the problem on the space of functions vanishing at the endpoints. However, vanishing at endpoints does not make sense in $L^2(0,1)$. A minor conundrum for the reader.

[3] Indeed, not 2π in the argument, just π.

[4] Yes, integrate the integral kernel along the diagonal. Yes, a person should worry about whether this is legitimate. Probably expressing the operator T as a limit of finite-rank operators allowing an analogous computation of trace yields the most robust argument that this trace exists and that the diagonal integral correctly computes the trace.