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Fujisaki's lemma, units theorem, class number

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Fujisaki's lemma asserts the compactness of \mathbb{J}^1/k^\times , where \mathbb{J}^1 is the ideles of idele-norm 1 of a number field k . This is basic in the harmonic analysis of automorphic forms. It is a corollary of existence and uniqueness of Haar measure. It *implies* finiteness of class numbers and the units theorem. We recall the standard argument, apparently first enunciated by Iwasawa. ^[1]

Let i be the *ideal map* from ideles to non-zero fractional ideals of the integers \mathfrak{o} of k . That is,

$$i(\alpha) = \prod_{v < \infty} \mathfrak{p}_v^{\text{ord}_v \alpha} \quad (\text{for } \alpha \in \mathbb{J})$$

where \mathfrak{p}_v is the prime ideal in \mathfrak{o} attached to the place v . The subgroup \mathbb{J}^1 of \mathbb{J} still surjects to the group of non-zero fractional ideals. The kernel in \mathbb{J} of the ideal map is

$$H = \prod_{v | \infty} k_v^\times \times \prod_{v < \infty} \mathfrak{o}_v^\times$$

and the kernel on \mathbb{J}^1 is $H^1 = H \cap \mathbb{J}^1$. The principal ideals are the image $i(k^\times)$. The map of \mathbb{J}^1 to the ideal class group factors through the idele class group \mathbb{J}^1/k^\times , noting that the product formula implies that $k^\times \subset \mathbb{J}^1$.

The intersection $H^1 = H \cap \mathbb{J}^1$ is open in \mathbb{J}^1 , so its image K in the quotient \mathbb{J}^1/k^\times is open. The cosets of K cover \mathbb{J}^1/k^\times , and by compactness there is a finite subcover. Thus, the quotient $\mathbb{J}^1/k^\times K$ is finite, and this finite group is the absolute ideal class group.

Since K is open, its cosets are open. Thus, K is closed. Since \mathbb{J}^1/k^\times is Hausdorff and compact, K is compact. That is, we have compactness of

$$K = (H^1 \cdot k^\times)/k^\times \approx H^1/(k^\times \cap H^1) = H^1/\mathfrak{o}^\times$$

with the global units \mathfrak{o}^\times imbedded on the diagonal. Since

$$\prod_{v < \infty} \mathfrak{o}_v^\times$$

is compact, its image U under the continuous map to H^1/\mathfrak{o}^\times is compact. By Hausdorff-ness, the image U is closed. Thus, we can take a further (Hausdorff) quotient by U ,

$$H^1/(U \cdot \mathfrak{o}^\times) = (\text{compact})$$

Let

$$k_\infty^1 = \{\alpha \in \prod_{v | \infty} k_v^\times : \prod_v |\alpha_v|_v = 1\}$$

Then

$$k_\infty^1/\mathfrak{o}^\times \approx H^1/(U \cdot \mathfrak{o}^\times) = (\text{compact})$$

This compactness is the units theorem. ///

Obviously, the same arguments prove finiteness of generalized ideal class groups and prove generalized units theorems. One might want to prove the accompanying result that a discrete (closed) additive subgroup L of \mathbb{R}^n such that \mathbb{R}^n/L is *compact* is a free \mathbb{Z} -module on n generators.

<http://www.math.umn.edu/~garrett/m/v/fujisaki.pdf>

[Lang 1970] S. Lang, *Algebraic Number Theory*, Addison-Wesley, 1970.

[Weil 1968] A. Weil, *Basic Number Theory*, Springer-Verlag, 1968.

[1] The compactness can be made a corollary of finiteness of class number and the units theorem.