(*October 6, 2004*)

Volume of $SL_n(\mathbf{Z})\backslash SL_n(\mathbf{R})$ and $Sp_n(\mathbf{Z})\backslash Sp_n(\mathbf{R})$

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We prove by induction that, reasonably normalized,

$$\mathrm{vol}\,(SL(n,\mathbf{Z})\backslash SL(n,\mathbf{R})) = \zeta(2)\,\zeta(3)\,\zeta(4)\,\zeta(5)\ldots\zeta(n)$$

Note that mysterious $\zeta(\mathrm{odd})$ values appear. By contrast, for symplectic groups

$$\mathrm{vol}\,(Sp(n,\mathbf{Z})\backslash Sp(n,\mathbf{R})) = \zeta(2)\,\zeta(4)\,\zeta(6)\,\zeta(8)\ldots\zeta(2n)$$

In particular, for symplectic groups the values of zeta at odd integers do not appear.

In both cases, Poisson summation plays a seemingly critical role. To compute volumes of other classical groups, Poisson summation must be replaced by subtler devices.

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1. Volume of $SL(2,\mathbf{Z})\backslash SL(2,\mathbf{R})$

Let $G = SL(2,\mathbf{R})$ and $\Gamma = SL(2,\mathbf{Z})$. To describe a right G-invariant measure on $\Gamma\backslash G$, it suffices to tell how to integrate compactly-supported continuous functions on $\Gamma\backslash G$. One first proves that, given a compactly-supported continuous function F on $\Gamma\backslash G$, there is a compactly-supported continuous function f on G so that

$$F(g) = \sum_{\gamma\in\Gamma} f(\gamma\cdot g)$$

Then define

$$\int_{\Gamma\backslash G} F(g)\,dg = \int_G f(g)\,dg$$

(and verify that this is well-defined, meaning that it is independent of the choice of f).

Then, to describe the measure on G, let K be the usual special orthogonal group

$$K = SO(2) = \{g\in G : g^\top g = 1_2\}$$

and let P be the standard parabolic subgroup

$$P = \{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in G\}$$

with subgroup

$$P^+ = \{\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a > 0,\ b\in\mathbf{R}\}$$

Recall the Iwasawa decomposition

$$G = P^+\cdot K \approx P^+\times K$$

The normalization of the Haar measure on G can be chosen so that for any absolutely integrable function φ on G

$$\int_G \varphi(g)\,dg = \int_{P^+}\int_K \varphi(pk)\,dk\,dp$$

where the Haar measure on K gives it total measure 2π, and where the *left* Haar measure dp on P^+ is normalized as follows. Let $p = na$ where

$$n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \qquad a = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$$

and take

$$dp = d(na) = \frac{dn\, da}{t^2}$$

Choose an auxiliary Schwartz function f on $\mathbf{R}^2$) and define a function F on G by

$$F(g) = \sum_{v \in \mathbf{Z}^2} f(xg)$$

By design, this function F is left Γ-invariant. By evaluating

$$\int_{\Gamma\backslash G} F(g)\, dg$$

in two different ways we will determine the volume of $\Gamma\backslash G$.

For a fixed positive integer ℓ, the set $\{(c,d) : \gcd(c,d) = \ell\}$ is an orbit of Γ in $\mathbf{Z}^2$. We choose $(0,1)$ as a convenient base point and observe that

$$\mathbf{Z}^2 - \{0\} = \{\ell \cdot (0,1) \cdot \gamma :\ \gamma \in \Gamma, \ell > 0\}$$

The stabilizer of $(0,1)$ in Γ is

$$N_{\mathbf{Z}} = \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbf{Z}\}$$

so we have a bijection

$$\mathbf{Z}^2 - \{0\} \longleftrightarrow \{\ell > 0\} \times N_{\mathbf{Z}}\backslash\Gamma$$

given by

$$\ell \cdot (0,1)\gamma \leftarrow \ell \times N_{\mathbf{Z}}\gamma$$

Let

$$N = \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbf{R}\} \subset G$$

By unwinding the iterated integral

$$\int_{\Gamma\backslash G} F(g)\, dg = \int_{\Gamma\backslash G} f(0)\, dg + \int_{\Gamma\backslash G} \sum_{x \neq 0} f(xg)\, dg = \int_{\Gamma\backslash G} f(0)\, dg + \sum_{\ell > 0} \int_{N_{\mathbf{Z}}\backslash G} f(\ell \cdot (0,1)g)\, dg$$

where $N_{\mathbf{Z}} = N \cap \Gamma = P^+ \cap \Gamma$. By expressing the Haar integral on G in terms of an iterated integral on P^+ and K

$$\int_{\Gamma\backslash G} f(0)\, dg + \sum_{\ell > 0} \int_{N_{\mathbf{Z}}\backslash P} \int_K f(\ell \cdot (0,1)pk)\, dg$$

We choose the function f on $\mathbf{R}^2$ to be rotation invariant. Then

$$f(\ell(0,1)pk) = f(\ell(0,1)p)$$

and the integral becomes

$$\int_{\Gamma\backslash G} f(0)\, dg + 2\pi \cdot \sum_{\ell > 0} \int_{N_{\mathbf{Z}}\backslash P} f(\ell(0,1)p)\, dp$$

since the total measure of K is 2π. Write the Haar measure on P in terms of $N = \{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\}$ and $M = \{\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}\}$, to obtain

$$\int_{\Gamma\backslash G} f(0)\,dg + 2\pi \sum_\ell \int_M \int_{N_{\mathbf{Z}}\backslash N} f(\ell(0,1)nm)\,dn\,t^{-2}\,dm$$

where $m = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$. Note that

$$f(\ell(0,1)nm) = f(\ell(0,1)m)$$

so the integral over $n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ in N is just

$$\int_{N_{\mathbf{Z}}\backslash N} 1\,dx = \int_{\mathbf{R}/\mathbf{Z}} 1\,dx = 1$$

Thus, the whole integral is

$$\int_{\Gamma\backslash G} F(g)\,dg = \int_{\Gamma\backslash G} f(0)\,dg + 2\pi\cdot\sum_\ell \int_M f(\ell(0,1)m)\,\frac{dm}{t^2} = \int_{\Gamma\backslash G} f(0)\,dg + 2\pi\cdot\sum_\ell \int_0^\infty f(\ell(0,t^{-1}))\,t^{-2}\,\frac{dt}{t}$$

$$= f(0)\cdot\mathrm{vol}\,(\Gamma\backslash G) + 2\pi\cdot\sum_\ell \int_0^\infty f(0,\ell t)\,t^2\,\frac{dt}{t}$$

upon replacing t by t^{-1}. Replacing t by t/ℓ gives

$$\int_{\Gamma\backslash G} F(g)\,dg = f(0)\cdot\mathrm{vol}\,(\Gamma\backslash G) + 2\pi\cdot\sum_\ell \ell^{-2}\int_0^\infty f(0,t)\,t^2\,\frac{dt}{t} = f(0)\cdot\mathrm{vol}\,(\Gamma\backslash G) + 2\pi\,\zeta(2)\cdot\int_0^\infty f(0,t)\,t^2\,\frac{dt}{t}$$

Further, using again the rotation invariance of f,

$$\int_0^\infty f(0,t)\,t^2\,\frac{dt}{t} = \int_0^\infty f(0,t)\,t\,dt = \frac{1}{2\pi}\int_{\mathbf{R}^2} f(x)\,dx = \frac{1}{2\pi}\,\hat{f}(0)$$

Thus, the factors of 2π cancel, and altogether

$$\int_{\Gamma\backslash G} F(g)\,dg = \int_{\Gamma\backslash G} \sum_{x\in\mathbf{Z}^2} f(xg)\,dg = f(0)\cdot\mathrm{vol}\,(\Gamma\backslash G) + \zeta(2)\,\hat{f}(0)$$

On the other hand, via Poisson summation,

$$\sum_{x\in\mathbf{Z}^2} f(xg) = \frac{1}{|\det g|}\sum_{x\in\mathbf{Z}^2} \hat{f}(x\,{}^\top g^{-1}) = \sum_{x\in\mathbf{Z}^2} \hat{f}(x\,{}^\top g^{-1})$$

(since $\det g = 1$). The group Γ is stable under transpose-inverse, so we can do a completely analogous computation with the roles of f and $\hat{f}$ reversed, finally obtaining

$$f(0)\cdot\mathrm{vol}\,(\Gamma\backslash G) + \zeta(2)\,\hat{f}(0) = \int_{\Gamma\backslash G} F(g)\,dg = \hat{f}(0)\cdot\mathrm{vol}\,(\Gamma\backslash G) + \zeta(2)\,f(0)$$

from which follows

$$(f(0) - \hat{f}(0))\cdot\mathrm{vol}\,(\Gamma\backslash G) = (f(0) - \hat{f}(0))\cdot\zeta(2)$$

Take f such that $f(0) \neq \hat{f}(0)$ to obtain

$$\mathrm{vol}\,(\Gamma\backslash G) = \zeta(2)$$

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2. Comparison with $SL(2,\mathbf{Z})\backslash\mathbf{H}$

We now reconcile the previous computation with the computation, in a somewhat different normalization, of the volume of $SL(2,\mathbf{Z})\backslash\mathbf{H}$, where $\mathbf{H}$ is the upper half-plane with the usual linear fractional transformation action of $SL(2,\mathbf{R})$. Integrating the traditional measure $dx\,dy/y^2$ on the usual fundamental domain

$$\mathbf{F} = \{z = x+iy \in \mathbf{H} : |x| \le \frac{1}{2} \ \ |z| \ge 1\}$$

one obtains $\pi/3$. It is worthwhile to see that this value is compatible with the group-theoretic value $\zeta(2) = \pi^2/6$ obtained above.

First, $\mathbf{H} \approx G/K$ by $g(i) \leftarrow g$, since K is the isotropy group of the point $i \in \mathbf{H}$. But at the same time the center $\{\pm 1_2\}$ of G, which also lies inside K, acts trivially on $\mathbf{H}$. This effectively gives $\{\pm 1\}\backslash K$ total measure 1, thus giving K total measure 2, rather than 2π.

Second, the usual coordinates $z = x+iy$ on $\mathbf{H}$ correspond to coordinates

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix}$$

rather than

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix}$$

as above. The change of coordinates has the effect of doubling the measure in the y-coordinate by comparison to the t-coordinate.

Thus, based on the $\Gamma\backslash G$ computation above, we would expect the measure of $\mathbf{F}$ to be

$$\mathrm{vol}\,(\Gamma\backslash G) \times \frac{2}{2\pi} \times 2 = \frac{\pi}{3} = \frac{\pi^2}{6} \times \frac{2}{2\pi} \times 2 = \frac{\pi}{3}$$

This does match the direct computation in the $z = x+iy$ coordinates.

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3. Volume of $SL(n,\mathbf{Z})\backslash SL(n,\mathbf{R})$ by induction

Now we prove by induction that, reasonably normalized,

$$\mathrm{vol}\,(SL_n(\mathbf{Z})\backslash SL_n(\mathbf{R})) = \zeta(2)\zeta(3)\zeta(4)\zeta(5)\ldots\zeta(n)$$

The normalization of measure needs explanation. First, let $G = SL(n,\mathbf{R})$ and $\Gamma = SL(n,\mathbf{Z})$. Given a compactly-supported continuous function F on $\Gamma\backslash G$, there is a compactly-supported continuous function f on G so that

$$F(g) = \sum_{\gamma\in\Gamma} f(\gamma\cdot g)$$

Then define

$$\int_{\Gamma\backslash G} F(g)\,dg = \int_G f(g)\,dg$$

This is well-defined, meaning that it is independent of the choice of f. Thus, a right G-invariant measure on the quotient $\Gamma\backslash G$ is completely specified by choice of a Haar measure on G.

We reduce the normalization of a Haar measure on G to measures on subgroups. Let $K = SO(n)$, and let P^+ be the collection of upper-triangular real matrices with positive diagonal entries. Then $K \cap P^+ = 1_n$

and by the Iwasawa decomposition $G = P^+ \cdot K$. For a choice of Haar measure on K and choice of *left* Haar measure on P^+, for f compactly supported and continuous on G, the integral

$$f \to \int_{P^+} \int_K f(pk)\, dk\, dp$$

is a Haar integral on G. The normalization of the left Haar measure on P^+ is completely elementary, given by

$$d\begin{pmatrix} p_{11} & p_{12} & \dots & & p_{1n} \\ 0 & p_{22} & \dots & & \\ & & \ddots & & \\ & & & p_{n-1,n-1} & \\ & & & & \frac{1}{p_{11}p_{22}\cdots p_{n-1,n-1}} \end{pmatrix} = \prod_{1\le i<n} p_{ii}^{2n-2i+2} \cdot \prod_{1\le i<n} \frac{dp_{ii}}{p_{ii}} \cdot \prod_{i<j} dp_{ij}$$

where the leading factor is the modular function on P^+. The measure on K is normalized so that

$$\mathrm{vol}\,(K) = \mathrm{vol}\,(S^1)\mathrm{vol}\,(S^2)\,\mathrm{vol}\,(S^3)\,\mathrm{vol}\,(S^4)\dots\mathrm{vol}\,(S^{n-1})$$

where S^k is the standard k-sphere in $\mathbf{R}^{k+1}$ with the usual total measure $\mathrm{vol}\,(S^k)$. In particular, for suitable functions f on $[0,\infty)$ we have the integration formula

$$\mathrm{vol}\,S^k \cdot \int_0^\infty f(r) r^{k-1}\, dr = \int_{\mathbf{R}^k} f(|x|)\, dx$$

Let f be a Schwartz function on $\mathbf{R}^n$ and define a function F on G by

$$F(g) = \sum_{v\in\mathbf{Z}^n} f(xg)$$

This function is left Γ-invariant. Consider

$$\int_{\Gamma\backslash G} F(g)\, dg$$

Let

$$Q = \{\begin{pmatrix} h & * \\ 0 & 1 \end{pmatrix} :\ h \in SL_{n-1}(\mathbf{R})\}$$

This is the subgroup of G fixing $e = (0, 0, \dots, 0, 1)$ under right multiplication. By linear algebra over $\mathbf{Z}$,

$$\mathbf{Z}^n - \{0\} = \sum_{\ell>0} \sum_{\gamma\in Q_{\mathbf{Z}}\backslash\Gamma} \ell\cdot e\cdot\gamma$$

where ℓ ranges over positive integers and $Q_{\mathbf{Z}} = Q \cap \Gamma$. Then

$$\int_{\Gamma\backslash G} F(g)\, dg = \int_{\Gamma\backslash G} f(0)\, dg + \sum_\ell \int_{\Gamma\backslash G} \sum_{\gamma\in(Q_{\mathbf{Z}}\backslash\Gamma)} f(\ell e\gamma g)\, dg$$

where $Q_{\mathbf{Z}} = Q \cap \Gamma$. By unwinding, this is

$$\mathrm{vol}\,(\Gamma\backslash G) f(0) + \sum_\ell \int_{Q_{\mathbf{Z}}\backslash G} f(\ell e g)\, dg$$

Let

$$P^+ = \{\begin{pmatrix} h & * \\ 0 & \frac{1}{\det h} \end{pmatrix} :\ \det h > 0\}$$

$$M = \{\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} : h \in SL_{n-1}(\mathbf{R})\}$$

$$A^+ = \{\begin{pmatrix} t^{\frac{1}{n-1}} \cdot 1_{n-1} & 0 \\ 0 & t^{-1} \end{pmatrix} : t > 0\}$$

$$N = \{\begin{pmatrix} 1_{n-1} & v \\ 0 & 1 \end{pmatrix} : v \in \mathbf{R}^{n-1}\}$$

Then $P^+ = NMA^+$ and $Q = NM$. Let $N_{\mathbf{Z}} = N \cap \Gamma$ and $M_{\mathbf{Z}} = M \cap \Gamma$. Via the Iwasawa decomposition $G = P^+ \cdot K$, and using induction, we normalize the invariant integral so that for left-invariant functions Φ

$$\int_{Q_{\mathbf{Z}}\backslash G} \Phi(g)\, dg = \mathrm{vol}\,(S^{n-1}) \cdot \int_{A^+} \int_{Q_{\mathbf{Z}}\backslash NM} \int_K \Phi(nmak)\, t^{-n}\, dk\, dn\, dm\, da$$

where $\mathrm{vol}\,(S^{n-1})$ is the natural measure of the $(n-1)$-sphere S^{n-1}, where $\mathrm{vol}\,(K) = 1$, and where

$$a = \begin{pmatrix} t^{\frac{1}{n-1}} \cdot 1_{n-1} & 0 \\ 0 & t^{-1} \end{pmatrix} : t > 0\}$$

The measure given by $t^{-n} dn\, dm\, da$ is a left Haar measure on P^+. (One should verify that this normalization in this induction step produces the aggregate effect indicated at the beginning. We comment on this at the end of the computation.) Then the integral above becomes

$$\mathrm{vol}\,(\Gamma\backslash G) f(0) + \mathrm{vol}\,(S^{n-1}) \cdot \sum_\ell \int_{A^+} \int_{Q_{\mathbf{Z}}\backslash NM} f(\ell \cdot e \cdot nmak)\, dk\, dn\, dm\, da$$

The integrand is invariant under NM, and the volume of $N_{\mathbf{Z}}\backslash N_{\mathbf{R}}$ is 1, so the whole becomes

$$\mathrm{vol}\,(\Gamma\backslash G) f(0) + \mathrm{vol}\,(S^{n-1}) \cdot \mathrm{vol}\,(SL_{n-1}(\mathbf{Z})\backslash SL_{n-1}(\mathbf{R})) \cdot \sum_\ell \int_{A^+} f(\ell \cdot e \cdot ak)\, t^{-n}\, dk\, da$$

Take f to be right K-invariant. Then this becomes

$$\mathrm{vol}\,(\Gamma\backslash G) f(0) + \mathrm{vol}\,(S^{n-1}) \cdot \mathrm{vol}\,(SL_{n-1}(\mathbf{Z})\backslash SL_{n-1}(\mathbf{R})) \cdot \sum_\ell \int_{A^+} f(\ell e a)\, t^{-n}\, da$$

$$= \mathrm{vol}\,(\Gamma\backslash G) f(0) + \mathrm{vol}\,(S^{n-1}) \cdot \mathrm{vol}\,(SL_{n-1}(\mathbf{Z})\backslash SL_{n-1}(\mathbf{R})) \cdot \sum_\ell \int_0^\infty f(\ell e t^{-1})\, t^{-n}\, \frac{dt}{t}$$

$$= \mathrm{vol}\,(\Gamma\backslash G) f(0) + \mathrm{vol}\,(S^{n-1}) \cdot \mathrm{vol}\,(SL_{n-1}(\mathbf{Z})\backslash SL_{n-1}(\mathbf{R})) \cdot \sum_\ell \frac{1}{\ell^n} \int_0^\infty f(et)\, t^n\, \frac{dt}{t}$$

where we replace t by t^{-1}. Using the rotation-invariance of f,

$$\mathrm{vol}\,(S^{n-1}) \cdot \int_0^\infty f(et)\, t^n\, \frac{dt}{t} = \int_{\mathbf{R}^n} f(x)\, dx = \hat{f}(0)$$

Thus, altogether,

$$\int_{\Gamma\backslash G} F(g)\, dg = \mathrm{vol}\,(\Gamma\backslash G) f(0) + \mathrm{vol}\,(SL_{n-1}(\mathbf{Z})\backslash SL_{n-1}(\mathbf{R})) \cdot \zeta(n) \cdot \hat{f}(0)$$

By Poisson summation,

$$F(g) = \sum_{x\in\mathbf{Z}^n} f(xg) = \sum_{x\in\mathbf{Z}^n} \hat{f}(x\, {}^\top g^{-1}) = F({}^\top g^{-1})$$

The automorphism $g \to {}^\top g^{-1}$ preserves measure on G and stabilizes Γ. Thus, as $f^{\hat{}\hat{}}(0) = f(0)$,

$$\mathrm{vol}\,(\Gamma\backslash G) f(0) + \mathrm{vol}\,(SL_{n-1}(\mathbf{Z})\backslash SL_{n-1}(\mathbf{R})) \cdot \zeta(n) \cdot \hat{f}(0) = \int_{\Gamma\backslash G} F(g)\, dg$$

$$= \mathrm{vol}\,(\Gamma\backslash G)\hat{f}(0) + \mathrm{vol}\,(SL_{n-1}(\mathbf{Z})\backslash SL_{n-1}(\mathbf{R})) \cdot \zeta(n) \cdot f(0)$$

Taking f such that $f(0) \neq \hat{f}(0)$,

$$\mathrm{vol}\,(SL_n(\mathbf{Z})\backslash SL_n(\mathbf{R})) = \mathrm{vol}\,(SL_{n-1}(\mathbf{Z})\backslash SL_{n-1}(\mathbf{R})) \cdot \zeta(n)$$

and by induction

$$\mathrm{vol}\,(SL_n(\mathbf{Z})\backslash SL_n(\mathbf{R})) = \zeta(2)\zeta(3)\zeta(4)\zeta(5)\ldots\zeta(n)$$

Last, we should be sure that the normalizations of measure appearing in the induction matches the normalization described at the outset. For the contribution of K this is explicitly clear. The contribution of the subgroups P^+ is less clear. The normalization effectively given by the induction would put coordinates on diagonal matrices of determinant 1 by

$$\begin{pmatrix} t_1 t_2^{\frac{1}{2}} t_3^{\frac{1}{3}} \ldots t_{n-1}^{\frac{1}{n-1}} & & & & & & \\ & t_1^{-1} t_2^{\frac{1}{2}} t_3^{\frac{1}{3}} \ldots t_{n-1}^{\frac{1}{n-1}} & & & & & \\ & & t_2^{-1} t_3^{\frac{1}{3}} \ldots t_{n-1}^{\frac{1}{n-1}} & & & & \\ & & & t_3^{-1} t_4^{\frac{1}{4}} \ldots t_{n-1}^{\frac{1}{n-1}} & & & \\ & & & & t_4^{-1} \ldots t_{n-1}^{\frac{1}{n-1}} & & \\ & & & & & \ddots & \\ & & & & & & t_{n-1}^{-1} \end{pmatrix}$$

versus the coordinates

$$\begin{pmatrix} t_1 & & & & \\ & t_2 & & & \\ & & \ddots & & \\ & & & t_{n-1} & \\ & & & & \frac{1}{p_{11}p_{22}\cdots p_{n-1,n-1}} \end{pmatrix}$$

The lower right $(n-1)$-by-$(n-1)$ minor of the former has exponents

$$\begin{pmatrix} -1 & \frac{1}{2} & \frac{1}{3} & \ldots & \frac{1}{n-1} \\ 0 & -1 & \frac{1}{3} & \ldots & \frac{1}{n-1} \\ 0 & 0 & -1 & \ldots & \frac{1}{n-1} \\ & & & \ddots & \\ & & & & -1 \end{pmatrix}$$

which has determinant ± 1, so the change-of-measure going from one set of coordinates to the other is trivial. Thus, the measure used in the induction match the measure described at the beginning. ///

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4. Symplectic groups

Let $G = Sp(n, \mathbf{R})$ be the usual symplectic group of $2n$-by-$2n$ matrices, and $\Gamma = Sp(n, \mathbf{Z})$. With reasonably normalized measure,

$$\mathrm{vol}\,(Sp(n,\mathbf{Z})\backslash Sp(n,\mathbf{R})) = \zeta(2)\,\zeta(4)\,\zeta(6)\,\zeta(8)\ldots\zeta(2n)$$

The measure on $\Gamma\backslash G$ is determined from a Haar measure on G by the requirement that

$$\int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} \varphi(\gamma\cdot g)\, dg = \int_G \varphi(g)\, dg$$

for compactly-supported continuous φ on G. To specify a Haar measure on G, let

$$K = \{\begin{pmatrix} A & B \\ -B & A \end{pmatrix} : A + iB \in U(n)\} \approx U(n)$$

with the usual unitary group

$$U(n) = \{h \in GL(n,\mathbf{C}) : h^* h = 1_n\}$$

where h^* is h-conjugate-transpose. And let P^+ be the subgroup of $Sp_n(\mathbf{R})$ consisting of elements of the form

$$\begin{pmatrix} t_1 & * & * & \ldots & * & \ldots & & * \\ 0 & t_2 & * & \ldots & \vdots & & & \\ \vdots & & \ddots & & & & & \vdots \\ 0 & \ldots & 0 & t_n & * & & \ldots & * \\ 0 & \ldots & & 0 & t_1^{-1} & 0 & \ldots & 0 \\ \vdots & & & & * & t_2^{-1} & & \vdots \\ & & & \vdots & \vdots & & \ddots & 0 \\ 0 & & \ldots & 0 & * & & & t_n^{-1} \end{pmatrix} = \begin{pmatrix} A & * \\ 0 & {}^\top A^{-1} \end{pmatrix} \quad (A \text{ upper-triangular})$$

Let N be the unipotent radical of P^+ (consisting of unipotent matrices in P^+). In these coordinates, a left Haar measure on P^+ is

$$t_1^{-2n}\, t_2^{-2n+2} \ldots t_{n-1}^{2n-2}\, t_n^{2n}\, dn\, \frac{dt_1}{t_1} \ldots \frac{dt_n}{t_n}$$

where dn is a Haar measure on N. Give K the Haar measure so that it has total measure

$$\mathrm{vol}\,(S^1)\mathrm{vol}\,(S^3)\mathrm{vol}\,(S^5)\ldots\mathrm{vol}\,(S^{2n-3})\mathrm{vol}\,(S^{2n-1})$$

where $\mathrm{vol}\,(S^k)$ is the standard volume of the k-sphere in $\mathbf{R}^{k+1}$. Then

$$\varphi \to \int_{P^+}\int_K f(pk)\, dp\, dk$$

is a Haar integral on G.

Let f be a Schwartz function on $\mathbf{R}^{2n}$, and define

$$F(g) = \sum_{x\in\mathbf{Z}^{2n}} f(x\cdot g)$$

viewing $x \in \mathbf{Z}^{2n}$ as a row vector. Evaluating $\int_{\Gamma\backslash G} F(g)\, dg$ in two different ways will allow evaluation of the volume of $\Gamma\backslash G$.

First, Γ is transitive on primitive elements in $\mathbf{Z}^{2n}$ (those whose entries have greatest common divisor 1), so

$$\mathbf{Z}^{2n} - \{0\} = \{\ell\cdot e\cdot\gamma :\ \ell > 0,\ \gamma\in\Gamma\}$$

where

$$e = (\underbrace{0,\ldots,0}_{n}, 1, \underbrace{0,\ldots,0}_{n-1})$$

that is, with the lone 1 at the $(n+1)^{\text{th}}$ place. The isotropy group of e in G is

$$Q = \{\begin{pmatrix} 1 & * & * & * \\ 0 & a & * & b \\ 0 & 0 & 1 & 0 \\ 0 & c & * & d \end{pmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp_{n-1}(\mathbf{R})\}$$

and the other entries are of suitable sizes. Then

$$\int_{\Gamma\backslash G} F(g)\,dg = \mathrm{vol}\,(\Gamma\backslash G)\cdot f(0) + \sum_{\ell>0}\int_{\Gamma\backslash G}\sum_{\gamma\in Q_{\mathbf{Z}}\backslash\Gamma} f(\ell\cdot e\cdot\gamma g)\,dg$$

Let

$$P^+ = \{\begin{pmatrix} t & * & * & * \\ 0 & a & * & b \\ 0 & 0 & t^{-1} & 0 \\ 0 & c & * & d \end{pmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(n-1,\mathbf{R})\},\ t>0\}$$

$$M = \{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & t^{-1} & 0 \\ 0 & c & 0 & d \end{pmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(n-1,\mathbf{R}),\ t>0\}$$

$$A^+ = \{\begin{pmatrix} t & 0 & 0 & 0 \\ 0 & 1_{n-1} & 0 & 0 \\ 0 & 0 & t^{-1} & 0 \\ 0 & 0 & 0 & 1_{n-1} \end{pmatrix} \in P^+\} \qquad N = \{\begin{pmatrix} 1 & * & * & * \\ 0 & 1_{n-1} & * & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & * & 1 \end{pmatrix} \in Q\}$$

Then

$$P^+ = N\cdot M\cdot A^+ \qquad Q = N\cdot M$$

Note that

$$t^{-2n}\cdot dn\,dm\,da$$

is a left Haar measure on P^+, with coordinate $a\in A^+$ as just above.
Unwinding the integral, it is

$$\int_{\Gamma\backslash G} F(g)\,dg = \mathrm{vol}\,(\Gamma\backslash G)\cdot f(0) + \sum_{\ell>0}\int_{Q_{\mathbf{Z}}\backslash G} f(\ell\cdot e\cdot g)\,dg$$

$$= \mathrm{vol}\,(\Gamma\backslash G)\cdot f(0) + \mathrm{vol}\,(S^{2n-1})\cdot\sum_{\ell>0}\int_{A^+}\int_{M_{\mathbf{Z}}\backslash M}\int_{N_{\mathbf{Z}}\backslash N}\int_K f(\ell\cdot e\cdot nmak)\,t^{-2n}\ dk\,dm\,dn\,da$$

which for right K-invariant f is

$$\mathrm{vol}\,(\Gamma\backslash G)\cdot f(0) + \mathrm{vol}\,(S^{2n-1})\cdot\mathrm{vol}\,(Sp_{n-1}(\mathbf{Z})\backslash Sp_{n-1}(\mathbf{R}))\cdot\sum_{\ell>0}\int_{A^+} f(\ell\cdot e\cdot a)\,t^{-2n}\,da$$

$$= \mathrm{vol}\,(\Gamma\backslash G)\cdot f(0) + \mathrm{vol}\,(S^{2n-1})\cdot\mathrm{vol}\,(Sp_{n-1}(\mathbf{Z})\backslash Sp_{n-1}(\mathbf{R}))\cdot\sum_{\ell>0}\int_0^\infty f(\ell\cdot t^{-1}\cdot e)\,t^{-2n}\,\frac{dt}{t}$$

Replacing t by t/ℓ gives

$$= \mathrm{vol}\,(\Gamma\backslash G)\cdot f(0) + \mathrm{vol}\,(S^{2n-1})\cdot\mathrm{vol}\,(Sp_{n-1}(\mathbf{Z})\backslash Sp_{n-1}(\mathbf{R}))\cdot\zeta(2n)\int_0^\infty f(t\cdot e)\,t^{-2n}\,\frac{dt}{t}$$

$$= \mathrm{vol}\,(\Gamma\backslash G)\cdot f(0) + \mathrm{vol}\,(Sp_{n-1}(\mathbf{Z})\backslash Sp_{n-1}(\mathbf{R}))\cdot\zeta(2n)\int_{\mathbf{R}^{2n}} f(x)\,dx$$

$$= \mathrm{vol}\,(\Gamma\backslash G) \cdot f(0) + \mathrm{vol}\,(Sp_{n-1}(\mathbf{Z})\backslash Sp_{n-1}(\mathbf{R})) \cdot \zeta(2n)\,\hat{f}(0)$$

On the other hand, by Poisson summation

$$\int_{\Gamma\backslash G} F(g)\,dg = \int_{\Gamma\backslash G} \sum_{x\in\mathbf{Z}^{2n}} f(xg) = \int_{\Gamma\backslash G} \sum_{x\in\mathbf{Z}^{2n}} \hat{f}(x\,{}^\top g^{-1}) = \int_{\Gamma\backslash G} \sum_{x\in\mathbf{Z}^{2n}} \hat{f}(xg)$$

since the involution $g \to^\top g^{-1}$ preserves the Haar measure, and preserves Γ. Thus,

$$\mathrm{vol}\,(\Gamma\backslash G) \cdot f(0) + \mathrm{vol}\,(Sp_{n-1}(\mathbf{Z})\backslash Sp_{n-1}(\mathbf{R})) \cdot \zeta(2n)\,\hat{f}(0)$$

$$= \mathrm{vol}\,(\Gamma\backslash G) \cdot \hat{f}(0) + \mathrm{vol}\,(Sp_{n-1}(\mathbf{Z})\backslash Sp_{n-1}(\mathbf{R})) \cdot \zeta(2n)\,f(0)$$

For f such that $f(0) \neq \hat{f}(0)$, solve for the volume

$$\mathrm{vol}\,(Sp_n(\mathbf{A})\backslash Sp_n(\mathbf{R})) = \zeta(2n) \cdot \mathrm{vol}\,(Sp_{n-1}(\mathbf{Z})\backslash Sp_{n-1}(\mathbf{R}))$$

Since $Sp(1) = SL(2)$, by induction, as claimed

$$\mathrm{vol}\,(Sp(n,\mathbf{Z})\backslash Sp(n,\mathbf{R})) = \zeta(2)\,\zeta(4)\,\zeta(6)\,\zeta(8)\ldots\zeta(2n)$$

Verification that the measure used in the induction agree with the measure specified at the outset is straightforward. ///