von Neumann algebras

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The point here is to define and describe some basic ideas about von Neumann algebras, without proofs, but including some relevant background material in compact form.

- Topologies on operators on Hilbert spaces
- Commutants, Schur's lemma, central characters
- The von Neumann density theorem
- Definition of von Neumann algebras, factor algebras
- Finite and infinite von Neumann algebras
- Basic classification of von Neumann algebras

Topologies on operators on Hilbert spaces

Let V be a (complex) Hilbert space with inner product $\langle,\rangle$ and norm $||$. Let $\mathcal{B}(V)$ be the algebra of continuous linear operators $T : V \to V$. There are at least 3 important topologies on $\mathcal{B}(V)$.

The **uniform** or **norm** topology is the strongest topology we will consider, and gives $\mathcal{B}(V)$ the structure of *Banach space*. This topology is defined via the **operator norm**

$$|T| = \sup_{|v|=1} |Tv|$$

where as indicated v ranges over unit vectors (in V).

The **strong topology** on $\mathcal{B}V$ is defined by a collection of *semi-norms*

$$\nu_v(T) = |Tv|$$

as v ranges over V. Note that it is unlikely that there is a *countable* collection of semi-norms giving this topology, so it is therefore *not* obviously metrizable.

The **weak topology** on $\mathcal{B}V$ is defined by a collection of *semi-norms*

$$\nu_{v,w}(T) = |\langle Tv, w\rangle|$$

as v, w range over V.

There are also ultra-strong and ultra-weak topologies, and others besides, but we don't need them here.

Let $\mathcal{B}(V)^\times$ be the group of continuous linear operators on a Hilbert space V having continuous *inverses*.

Commutants, Schur's lemma

Let A be a subalgebra of the algebra $\mathcal{B}(V)$ of continuous linear operators on a Hilbert space V. Then the **commutant** A' of A is defined to be

$$A' = \{T \in \mathcal{B}(V) : T \circ \alpha = \alpha \circ T, \text{ for all } \alpha \in A\}$$

Schur's Lemma asserts that, if (π, V) is an irreducible unitary Hilbert space representation of a topological group G, then the commutant $\rho(G)'$ of $\rho(G)$ consists just of the scalar operators $\mathbf{C} \cdot 1$ on V. This is an immediate consequence of elementary spectral theory for bounded operators on Hilbert spaces.

The von Neumann density theorem

Let A be a subalgebra of the algebra $\mathcal{B}(V)$ of continuous linear operators on a Hilbert space V. Suppose that A contains the scalar operators, and is stable under taking *adjoints*. Then von Neumann's **Density Theorem** asserts that

$$A'' = \text{ strong-topology closure of } A$$

This result is also called the **double commutant theorem**. It has analogues in other situations, as well.

Definition of von Neumann algebras, factor algebras

A **von Neumann algebra** A is an adjoint-closed subalgebra of the algebra $\mathcal{B}(V)$ of bounded operators of a Hilbert space V, closed in the strong topology on operators.

Some sources require that a von Neumann algebra contain the scalar operators, or that it have a unit. We do not require the former condition, and note below that the latter property is *provable*.

In light of the density theorem, *if the scalar operators lie in the algebra*, then closedness in the strong operator topology is equivalent to the *purely algebraic* condition

$$A'' = A$$

Also, for any $*$-closed algebra A in $\mathcal{B}(V)$ containing the scalars, since A' is readily checked to be strong-topology closed, *the commutant A' is a von Neumann algebra.*

A von Neumann algebra $A \subset \mathcal{B}(V)$ is a **factor** or **factor algebra** if

$$A \cap A' = \mathbf{C} \cdot 1$$

Finite and infinite von Neumann algebras

Perhaps surprisingly, *every non-zero von Neumann algebra has a unit* (although this unit certainly may not be the identity map in the algebra $\mathcal{B}(V)$ in which the von Neumann algebra lies).

A **projection** p in $\mathcal{B}(V)$ is a self-adjoint element so that $p^2 = p$. The **rank** $rk(p)$ of a projection $p \in \mathcal{B}(V)$ is the dimension of its image (so is ∞ if not finite). Recall that a self-adjoint operator T on V is **positive**, written $T \geq 0$, if

$$\langle Tv, v \geq 0\rangle$$

for all $v \in V$. A projection p is **finite** if for any other projection q so that $p - q \geq 0$ and $rk(q) = rk(p)$ we have $p = q$. Otherwise, p is said to be **infinite**. A projection p in a von Neumann algebra A is **abelian** if the subalgebra pAp is abelian.

A von Neumann algebra is **finite** if its unit is a *finite* projection. A von Neumann algebra is **infinite** if its unit is an infinite projection.

Basic classification of von Neumann (factor) algebras

The classification of von Neumann algebras has a terminology involving *Types I,II,III* which is *not* directly related to the similar-sounding terminology for topological groups and C^*-algebras. At the level indicated here, these are really just definitions. The decomposition theory via Hilbert integrals is only slightly more substantial.

The classification is actually in terms of *factor algebras*, since one would prove the basic theorem that any von Neumann algebra can be decomposed as a Hilbert integral of factors.

A von Neumann algebra is

- **Type I** if every non-zero central projection in majorizes a non-zero *abelian* projection.
- **Type II** if is has *no* non-zero abelian projections and if every non-zero central projection majorizes a non-zero *finite* projection.
- **Type III** if it contains no non-zero finite projections.
- **properly infinite** if it has no non-zero finite central projections.
- **Type II_∞** if it is Type II and properly infinite.
- **Type II_1** if it is Type II and finite.