Solutions for midterm 1

1. Since $\mathbb{Q}$ is a field, every non-zero polynomial in a single variable with coefficients in $\mathbb{Q}$ has a finite number of roots in $\mathbb{Q}$ (in fact the number of roots is at most the degree of the polynomial). But the set of all integers is infinite. Therefore a non-zero polynomial cannot vanish at every integer.

Thus the only polynomial that vanishes at every integer is the zero polynomial. But the zero polynomial vanishes at every point of $\mathbb{Q}^1$. Therefore the set of all integers in $\mathbb{Q}^1$ is not an algebraic variety.

Answer: No.

2. Since g and h are polynomials in a single variable over a field, the ideal (g, h) is generated by the GCD of g and h. To find this GCD we divide g by h with a remainder:

$$\begin{array}{rl} & \quad x+2 \\ \cline{2-2} x^2-x-2 & \big) \; x^3+x^2-x-1 \\ & - \; \underline{x^3-x^2-2x} \\ & \qquad\quad 2x^2+x-1 \\ & \qquad - \; \underline{2x^2-2x-4} \\ & \qquad\qquad\quad 3x+3 \end{array}$$

Thus the remainder is $r = 3x + 3$. The GCD of g and h equals the GCD of h and r. Since $r = 3(x + 1)$ and 3 is a non-zero element of the field $\mathbb{Q}$, the GCD of h and r equals the GCD of h and $x + 1$. To find the GCD of h and $x + 1$ we divide h by $x + 1$ with remainder:

$$\begin{array}{rl} & \quad x-2 \\ \cline{2-2} x+1 & \big) \; x^2-x-2 \\ & - \; \underline{x^2+x} \\ & \qquad -2x-2 \\ & \quad - \; \underline{-2x-2} \\ & \qquad\qquad 0 \end{array}$$

Thus the remainder is zero, i.e. the GCD of g and h is $x+1$. Therefore the ideal (g, h) equals the ideal $(x+1)$. The polynomial f belongs to the ideal $(g, h) = (x+1)$ if and only if f is divisible by $(x + 1)$. We divide f by $x + 1$:

$$\begin{array}{rl} & \quad x-1 \\ \cline{2-2} x+1 & \big) \; x^2-2 \\ & - \; \underline{x^2+x} \\ & \qquad -x-2 \\ & \quad - \; \underline{-x-1} \\ & \qquad\quad -1 \end{array}$$

Thus the remainder is $-1 \neq 0$, i.e. f is not divisible by $x + 1$. Therefore the polynomial f is not in the ideal (g, h).

Answer: No.

3. (a) lex ordering: $x^3y^4z^2 > x^3y^3z^3 > y^5z^6$

(b) grlex ordering: $y^5z^6 > x^3y^4z^2 > x^3y^3z^3$

(c) grevlex ordering: $y^5z^6 > x^3y^4z^2 > x^3y^3z^3$.

4.

$$
\begin{array}{ll}
 & a_1 = 2y \\
 & a_2 = x^2 + 2xy^2 - 2y \\
g_1 = -xy^3 + x & \\
g_2 = x + y^2 & \left[\; x^3 + 3x^2y^2 + y^4 \right. \\
 & \underline{x^3 + x^2y^2} \\
 & \quad 2x^2y^2 + y^4 \\
 & \quad \underline{2x^2y^2 + 2xy^4} \\
 & \qquad -2xy^4 + y^4 \\
 & \qquad \underline{-2xy^4 + 2xy} \\
 & \qquad\quad -2xy + y^4 \\
 & \qquad\quad \underline{-2xy - 2y^3} \\
 & \qquad\qquad y^4 + 2y^3
\end{array}
$$

Answer: the remainder equals $r = y^4 + 2y^3$

5. A polynomial f belongs to a monomial ideal I if and only if every monomial appearing in f is divisible by some monomial generator of I. But the monomial x^4y^2z appearing in f is divisible by neither xyz^2, nor x^5y, nor yz^3.

Answer: No.

6a. The leading monomials of f and g are x^2y and x^3. Their LCM is x^3y. The leading terms of f and g are $3x^2y$ and $-5x^3$.

$S(f,g) = \frac{x^3y}{3x^2y}(3x^2y - 4y^3) - \frac{x^3y}{-5x^3}(-5x^3 + 2xy^2) = -\frac{4}{3}xy^3 + \frac{2}{5}xy^3 = -\frac{14}{15}xy^3.$

Answer: $S(f,g) = -\frac{14}{15}xy^3$.

6b The leading monomial of $S(f,G)$ is xy^3 and it not divisible by the leading monomial of f nor by the leading monomial of g. Therefore the remainder of $S(f,g)$ upon division by the basis $G = \{f,g\}$ is non-zero. By the Buchberger criterion, $\{f,g\}$ is not a Groebner basis of the ideal $I = (f,g)$.

Answer: No.