Solutions for Midterm 2

1. The first elimination ideal is generated by y^2-1. Since $y^2-1=0$ if and only if $y=1$ or $y=-1$, the partial soultions are $y=1$ and $y=-1$. The ideal I contains the polynomial $f=x^2y^2+2xy^3-1$ which is of degree 2 in x and the coefficient at x^2 in this polynomial is $g(y)=y^2$. Since $g(y)$ does not vanish for $y=1$, by the Extension Theorem the partial solution $y=1$ can be extended to a solution, i.e. $V(I)$ contains a point whose y-coordinate equals 1. Since $g(y)$ does not vanish for $y=-1$, by the Extension Theorem the partial solution $y=-1$ can be extended to a solution, i.e. $V(I)$ contains a point whose y-coordinate equals -1.

2a. The idea is to eliminate x using the lex ordering $x>y$. We set

$f_1=x^2+xy+y^2$

$f_2=x^3-x^2y+y^3$

$f_3=S(f_1,f_2)=f_2-xf_1=x^3-x^2y+y^3-x^3-x^2y-xy^2=-2x^2y-xy^2+y^3$.

$f_4=S(f_1,f_3)=f_3+2yf_1=-2x^2y-xy^2+y^3+2yx^2+2y^2x+2y^3=xy^2+3y^3$.

$f_5=S(f_1,f_4)=xf_4-y^2f_1=x^2y^2+3xy^3-x^2y^2-xy^3-y^4=2xy^3-y^4$.

$f_6=S(f_4,f_5)=f_5-2yf_4=2xy^3-y^4-2xy^3-6y^4=-7y^4$.

Thus the ideal $I=(f_1,f_2)$ contains the polynomial $f_6=-7y^4$. This polynomial must vanish at every point of $V(I)$. But this polynomial vanishes only at $y=0$. Therefore every point of $V(I)$ has zero y-coordinate. Plugging $y=0$ into f_1 we get the equation $x^2=0$ which has only one solution, $x=0$. Therefore $(0,0)$ is the only point on $V(I)$.

2b. The radical of the ideal $I=(f_1,f_2)$ is the defining ideal of the variety $V(I)$. Since $V(I)$ consists of a single point $(0,0)$, the defining ideal is (x,y).

3. Set $f=x^2+2xy-y^2-2x+2y-2$. We compute the partial derivatives of f with respect to x and y and set them equal to 0:

$\frac{\partial f}{\partial x}=2x+2y-2=0$

$\frac{\partial f}{\partial y}=2x-2y+2=0$

This system of equations has just one solution: $(0,1)$. But this point is not on the variety $V(f)$ because $f(0,1)\neq 0$.

Answer: the variety $V(f)$ has no singular points.

4. The variety of the ideal $I=(x^5,(x-y)^3(x+y+2)^4)$ consists of the solutions of the system of equations

$x^5=0$

$(x-y)^3(x+y+2)^4=0$.

The first equation implies $x=0$. Plugging this into the second equation yields $y^3(y+2)^4=0$, i.e. either $y=0$ or $y=-2$. Thus $V(I)$ consists of two points, $(0,0)$ and $(0,-2)$. The polynomial $x+y$ vanishes only at $(0,0)$ but not at $(0,-2)$. Since it does not vanish at every point of $V(I)$, it does not belong to the radical of I.

Answer: No.

5a. The intersection of two principal ideals in a polynomial ring is generated by the LCM of the two polynomials, i.e. a basis of the intersection is the polynomial $h = (x+y+1)^3(x-y)^5(2x+4)^6(x+y)^4$.

5b One has to compute a Groebner basis in the ring $\mathbb{C}[t,x,y]$ of the ideal $(tf_1, tf_2, (1-t)g_1, (1-t)g_2)$ with respect to the lex ordering $t > x > y$. The elements of this Groebner basis that do not contain the parameter t form a basis of $I \cap J$ in $\mathbb{C}[x,y]$.

6. The idea is to compute a basis of the first elimination ideal of $I = (f_1, f_2)$ where $f_1 = t^2+t-x$ and $f_2 = t^3-y$ with respect to the lex ordering $t > x > y$.

$f_3 = S(f_1, f_2) = f_2 - tf_1 = t^3 - y - t^3 - t^2 + tx = -t^2 + tx - y$.

$f_4 = S(f_3, f_1) = f_3 + f_1 = -t^2 + tx - y + t^2 + t - x = tx + t - x - y$.

$f_5 = S(f_4, f_1) = tf_4 - xf_1 = t^2x + t^2 - tx - ty - t^2x - tx + x^2 = t^2 - 2tx - ty + x^2$.

$f_6 = S(f_5, f_1) = f_5 + f_1 = t^2 - 2tx - ty + x^2 + t^2 + t - x = -2tx - ty + t + x^2 - x$.

At this point one could use a shortcut by noticing that both f_4 and f_6 are linear in t. Since $f_4 = t(x+1) - x - y$ and $f_6 = t(-2x-y+1) + x^2 - x$, one can completely eliminate t by computing

$f_7 = (x+1)f_6 - (-2x-y+1)f_4 = x^3 - 3yx - y^2 - y$.

Answer: $I(V) = (x^3 - 3yx - y^2 - y)$.

7a. The resultant equals the following determinant

$$\det \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & c & 1 \\ 1 & 1 & 2 & c \\ 0 & 1 & 0 & 2 \end{bmatrix} = \det \begin{bmatrix} 1 & c & 1 \\ 1 & 2 & c \\ 1 & 0 & 2 \end{bmatrix} + \det \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & c \\ 0 & 1 & 2 \end{bmatrix}$$

which equals

$$\det \begin{bmatrix} c & 1 \\ 2 & c \end{bmatrix} + 2\det \begin{bmatrix} 1 & c \\ 1 & 2 \end{bmatrix} + \det \begin{bmatrix} 1 & c \\ 1 & 2 \end{bmatrix} - \det \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = c^2 - 3c + 3.$$

Answer: $c^2 - 3c + 3$.

7b. The polynomial $c^2 - 3c + 3$ has no roots in $\mathbb{Q}$. Therefore the polynomials f and g have no common factors in $\mathbb{Q}[x]$ for any value of the parameter c. The polynomial $c^2 - 3c + 3$ has roots $c_1 = \frac{3}{2} + \frac{\sqrt{3}}{2}i$ and $c_2 = \frac{3}{2} - \frac{\sqrt{3}}{2}i$ in $\mathbb{C}$. For these values of c the polynomials f and g have a common factor in $\mathbb{C}[x]$.