Math 2243
Fall 2006
Midterm 1, WITH SOLUTIONS
September 28, 2006
Time Limit: 50 minutes

Name (Print): ____________________
Student ID: ____________________
Section Number: __________
Teaching Assistant: ____________________
Signature: ____________________

This exams contains 6 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated. You are allowed to take one-half of one (single - sided) 8.5 inch $\times$ 11 inch sheet of notes into the exams.

Do not give numerical approximations to quantities such as $\sin 5$, π, or $\sqrt{2}$. However, you should simplify $\cos\frac{\pi}{2} = 0$, $e^0 = 1$, and so on.

The following rules apply:

- **Show your work**, in a reasonably neat and coherent way, in the space provided. **All answers must be justified by valid mathematical reasoning, including a brief justification of the evaluation of definite and indefinite integrals**.
- **Mysterious or unsupported answers will not receive full credit**. Your work should be mathematically correct and carefully and legibly written.
- **A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit**; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- Full credit will be given only for work that is presented neatly and logically; work scattered all over the page without a clear ordering will receive very little credit.
- You may use a **crib sheet** which you have prepared in advance. The crib sheet may only be a half page ($8\frac{1}{2} \times 5\frac{1}{2}$ inches), on one side.
- No **calculators** are allowed, nor will they be needed.

1	15 pts	
2	20 pts	
3	15 pts	
4	25 pts	
5	25 pts	
TOTAL	100 pts	

1. (15 points) Find the general solution of the differential equation

$$\frac{dy}{dt} = \frac{t^2 - 3}{y + 4} \ .$$

If possible, write your answer in the form, "$y(t) = \ldots$" with an arbitrary constant.

SOLUTION: This DE is separable:

$$(y + 4)\, dy = (t^2 - 3)\, dt.$$

Integrate:

$$\frac{y^2}{2} + 4y = \frac{t^3}{3} - 3t + C,$$

which can be solved using the quadratic formula:

$$y(t) = -4 \pm \sqrt{\widetilde{C} - 6t + \frac{2}{3}t^3},$$

where the constant $\widetilde{C} = 16 + C$ and the sign $\pm$ are arbitrary. (CORRECTED!)

2. (20 points) (Check yes or no) **Picard's Theorem** states that under certain conditions, the differential equation

$$\frac{dy}{dt} = f(t, y(t)),$$

with the initial conditions

$$y(t_0) = y_0,$$

has a unique solution on some interval $t_1 < t < t_2$ around $t = t_0$. But Picard's Theorem requires the right-hand side $f(t, y)$ to have certain nice properties. For each of the four parts, answer the question: does $f(t, y)$ satisfy the conditions required by Picard's Theorem at (t_0, y_0)?

(a) $f(t, y) = \dfrac{y - 2t}{|y - 2t|}$ at $t_0 = 1,\ y_0 = 2$.

YES	
NO	

SOLUTION: NO. $f(t, y)$ is not continuous at $t = 1,\ y = 2$, where $y - 2t = 0$.

(b) $f(t, y) = \sqrt{|y|} + \dfrac{1}{\sqrt{|y|}}$ at $t_0 = 1,\ y_0 = 2$.

YES	
NO	

SOLUTION: YES: $f(t, y)$ and $\frac{\partial f}{\partial y}$ are continuous at $t = 1,\ y = 2$.

(c) $f(t, y) = y^2 \tan t$ at $t_0 = \pi/2,\ y_0 = 2$.

YES	
NO	

SOLUTION: NO: $\tan t = \frac{\sin t}{\cos t}$ is not continuous at $t = \pi/2$, since $\cos \pi/2 = 0$.

(d) $f(t, y) = \sqrt{|t^2 - 2ty + 3|}$ at $t_0 = 1,\ y_0 = 2$.

YES	
NO	

SOLUTION: NO: $t_0^2 - 2t_0y_0 + 3 = 0$, and $\frac{\partial f}{\partial y}$ is not continuous at that point.

3. (15 points) Here are two facts that you may use for this question but do not need to justify:

(i) The general solution of $\frac{d^2y_h}{dt^2} + 4y_h = 0$ is

$$y_h(t) = A\sin 2t + B\cos 2t.$$

(ii) The function $u(t) = e^t$ satisfies $\frac{d^2u}{dt^2} + 4u = 5\,e^t$.

Here is the question: Find the solution $y(t)$ of the (different!) differential equation

$$\frac{d^2y}{dt^2} + 4y = 10\,e^t$$

which satisfies the **initial conditions**

$$\frac{dy}{dt}(0) = 0, \qquad y(0) = -1.$$

SOLUTION: The general solution of the non-homogeneous linear DE is $y(t) = y_p(t) + y_h(t)$, where $y_p(t)$ is a particular solution, and $y_h(t)$ is the general solution to $\frac{d^2y_h}{dt^2} + 4y_h = 0$. But one nonhomogeneous DE has $5\,e^t$ on the right-hand side and the one we want to solve has $10\,e^t$ on the right-hand side. So $y_p(t) = 2u(t) = 2e^t$. We get

$$y(t) = 2e^t + A\sin 2t + B\cos 2t,$$

where A and B are constants to be determumined by the initial conditions:

$$0 = \frac{dy}{dt}(0) = 2e^0 + 2A\cos 0 = 2A + 2, \quad A = -1;$$

and

$$-1 = y(0) = 2e^0 + B\cos 0 = B + 2, \quad B = -3.$$

So the answer is

$$y(t) = 2e^t - \sin 2t - 3\cos 2t.$$

4. (25 points) Use the substitution $v(t) = \dfrac{y(t)}{t}$ to find the solution of the differential equation

$$t^2 \frac{dy}{dt} = y^2 - ty$$

which satisfies the **initial condition**

$$y(1) = 3.$$

SOLUTION: $v(t) = \frac{y(t)}{t}$, so $y(t) = tv(t)$ and $\frac{dy}{dt} = t\frac{dv}{dt} + v(t)$. The DE now says that

$$t\,dv + v\,dt = \frac{y^2 - ty}{t^2}\,dt$$

which reduces to the separated form

$$\frac{dv}{v^2 - 2v} = \frac{dt}{t}.$$

This requires the method of partial fractions: for some constants A and B,

$$\frac{1}{v^2 - 2v} = \frac{A}{v} + \frac{B}{v-2}.$$

Multiplying by v and plugging in $v = 0$ gives $A = -\frac{1}{2}$; and multiplying by $v - 2$ and plugging in $v = 2$ gives $B = \frac{1}{2}$. So

$$\ln|t| + C = \frac{1}{2}\int \Big(-\frac{1}{v} + \frac{1}{v-2}\Big)\,dv = \frac{1}{2}\ln\Big|\frac{v-2}{v}\Big|,$$

and further computation gives the general solution

$$y(t) = tv(t) = \frac{2t}{1 + Dt^2}.$$

The initial conditions require that $D = -\frac{1}{3}$, and

$$y(t) = \frac{6t}{3 - t^2}.$$

5. (25 points) Find the solution of the differential equation

$$\frac{dy}{dt} - 4t^3 y = t^3$$

which satisfies the **initial condition**

$$y(1) = 0.$$

(**Hint:** You might want to multiply the ODE by the appropriate integrating factor. Or, use another technique.)

SOLUTION: $p(t) = -4t^3$, so the integrating factor is $F(t) := e^{-t^4}$. Compute using the DE that

$$\frac{d}{dt}\Big(F(t)y(t)\Big) = F(t)t^3,$$

and the indefinite integral is

$$F(t)y(t) = \int t^3 F(t)dt + C = -\frac{1}{4}F(t) + C.$$

So the general solution is

$$y(t) = -\frac{1}{4} + \frac{C}{F(t)} = -\frac{1}{4} + Ce^{t^4}.$$

The initial condition gives $C = \frac{1}{4e}$, and

$$y(t) = -\frac{1}{4} + \frac{1}{4e}\, e^{t^4}.$$