Math 2243	**Name (Print):** ____________________
Fall 2006	**Student ID:** ____________________
Midterm 2, WITH SOLUTIONS	**Section Number:** __________
October 24, 2006	**Teaching Assistant:** ____________________
Time Limit: 50 minutes	**Signature:** ____________________

This exams contains. 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated. You are allowed to take one-half of one (single - sided) 8.5 inch $\times$ 11 inch sheet of notes into the exams.

Do not give numerical approximations to quantities such as $\sin 5$, π, or $\sqrt{2}$. However, you should simplify $\cos\frac{\pi}{2} = 0$, $e^0 = 1$, and so on.

The following rules apply:

- **Show your work**, in a reasonably neat and coherent way, in the space provided. **All answers must be justified by valid mathematical reasoning, including a brief justification of the evaluation of definite and indefinite integrals**.
- **Mysterious or unsupported answers will not receive full credit**. Your work should be mathematically correct and carefully and legibly written.
- **A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit**; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- Full credit will be given only for work that is presented neatly and logically; work scattered all over the page without a clear ordering will receive very little credit.
- You may use a **crib sheet** which you have prepared in advance. The crib sheet may only be a half page ($8\frac{1}{2} \times 5\frac{1}{2}$ inches), on one side.
- No **calculators** are allowed, nor will they be needed.

1	15 pts	
2	25 pts	
3	15 pts	
4	10 pts	
5	10 pts	
6	25 pts	
TOTAL	100 pts	

1. (15 points) Find all solutions of this system of three linear equations in five unknowns x, y, x, u, v. (**Hint:** the system is in reduced row-echelon form already!)

$$\begin{aligned} x + 2z + v &= 0 \\ y - 3z &= 2 \\ u + 5v &= 4. \end{aligned}$$

SOLUTION: x, y and u are pivot variables: each appears in only one equation. Use the first equation to solve for x, the second equation to solve for y and the third equation to solve for u. z and v are not pivot variables, so they are arbitrary. **Answer:** for any real numbers s $(= z)$ and t $(= v)$, a solution is given by

$$x = -2s - t, \quad y = 2 + 3s, \quad z = s, \quad u = 4 - 5t, \quad v = t.$$

2. (25 points) Find all solutions to this system of three linear equations in three unknowns (x, y, z):

$$\begin{aligned} x + y + z &= 5 \\ 2x - 3y + 7z &= 0 \\ 4x - y + 9z &= 10. \end{aligned}$$

Show your work! (Partial credit for row-echelon form.)

SOLUTION: It's easiest to write the system as an augmented matrix (imagine there's a vertical bar between the third and fourth columns):

$$\begin{bmatrix} 1 & 1 & 1 & 5 \\ 2 & -3 & 7 & 0 \\ 4 & -1 & 9 & 10 \end{bmatrix}.$$

Subtract 2 times the first row from the second row, and 4 times the first row from the third row:

$$\begin{bmatrix} 1 & 1 & 1 & 5 \\ 0 & -5 & 5 & -10 \\ 0 & -5 & 5 & -10 \end{bmatrix}.$$

Next subtract the second row from the third row:

$$\begin{bmatrix} 1 & 1 & 1 & 5 \\ 0 & -5 & 5 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The last row says that $0 = 0$, so the system of equations is consistent. Now divide the last row by -5, and subtract the result from the first row to get reduced row-echelon form:

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So for each real number s, a solution is

$$x = 3 - 2s, \quad y = 2 + s, \quad z = s.$$

3. (15 points) (True or False) $\mathcal{C}(-\infty, \infty)$ is the vector space of all continuous functions on the real line $(-\infty, \infty)$. Which of the following sets are **subspaces** of $\mathcal{C}(-\infty, \infty)$?

(a) The set of functions $y(t)$ with $y(0) = 0$ and $\frac{dy}{dt}(0) = 0$.

SUBSPACE	
NOT A SUBSPACE	

SOLUTION: A subspace. If $y = c_1 y_1 + c_2 y_2$, then $y(0) = c_1 y_1(0) + c_2 y_2(0)$, and $\frac{dy}{dt}(0) = c_1 \frac{dy_1}{dt}(0) + c_2 \frac{dy_2}{dt}(0)$. So if y_1 and y_2 are in this set and c_1, c_2 are any real constants, then y is in this set.

(b) The set of functions $y(t)$ with the product $y(1)y(-1) = 0$.

SUBSPACE	
NOT A SUBSPACE	

SOLUTION: Not a subspace: if $y_1(1) = 0$, $y_1(-1) = 3$, $y_2(1) = 4$ and $y_2(-1) = 0$, then y_1 and y_2 are in the set but $y_1(1) + y_2(1) = 4$ and $y_1(-1) + y_2(-1) = 3$, so $[y_1(1) + y_2(1)][y_1(-1) + y_2(-1)] = 12$, and $y_1 + y_2$ is not in the set.

(c) The set of solutions $y(t)$ of the differential equation

$$\frac{dy}{dt} + \big(y(t)\big)^2 = 0.$$

SUBSPACE	
NOT A SUBSPACE	

SOLUTION: Not a subspace: if $y(t) \neq 0$ is in the set, then compute $\frac{d(2y)}{dt} + \big((2y)(t)\big)^2 = -2y(t)^2 + 4y(t)^2$, which doesn't equal zero, so $2y(t)$ is not in the set.

4. (10 points) Determine whether or not the three vectors $\vec{u}, \vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$ form a **basis** for $\mathbb{R}^3$. Why or why not?

$$\vec{u} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \qquad \vec{v} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \qquad \vec{w} = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}.$$

SOLUTION: Do row reduction on the matrix

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & -2 & -2 \\ -1 & 2 & 0 \end{bmatrix}.$$

You get

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The row of zeroes shows that the three vectors are not independent, so they are **not a basis** for $\mathbb{R}^3$. Another solution is to compute the determinant, for example by expanding in the second row: the determinant is

$$\begin{vmatrix} 1 & 1 & 3 \\ 0 & -2 & -2 \\ -1 & 2 & 0 \end{vmatrix} = -2 \begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix} - (-2) \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} = -6 - (-6) = 0.$$

5. (10 points) The 3×3 matrix A has one entry $a_{23} = t$ which depends on t, so the determinant of A is a function of t. Find its derivative

$$\frac{d}{dt} \begin{vmatrix} 2 & 4 & 9 \\ -1 & -2 & t \\ 3 & 5 & 7 \end{vmatrix}.$$

(**Hint:** try expanding the determinant in the second row).

SOLUTION: Expanding the determinant in the second row gives

$$\begin{vmatrix} 2 & 4 & 9 \\ -1 & -2 & t \\ 3 & 5 & 7 \end{vmatrix} = -(-1) \begin{vmatrix} 4 & 9 \\ 5 & 7 \end{vmatrix} + (-2) \begin{vmatrix} 2 & 9 \\ 3 & 7 \end{vmatrix} - (+t) \begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix}.$$

The first two terms are constants, so the derivative of the determinant is

$$- \begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} = +2.$$

6. (25 points) Consider the system of differential equations:

$$\frac{dx}{dt}(t) = 2x(t) + y(t)$$
$$\frac{dy}{dt}(t) = x(t) - 2y(t).$$

(a) Sketch the v-nullclines and the h-nullclines in the (x, y)-plane.

DESCRIPTION OF SOLUTION: The sketch shows two orthogonal nullclines passing through $(0, 0)$: $2x + y = 0$ is a v-nullcline, with arrows pointing down where $y > 0$ and arrows pointing up where $y < 0$. The orthogonal line $x - 2y = 0$ is an h-nullcline with arrows pointing to the right where x and y are positive, and pointing to the left where both are negative.

(b) On the same sketch, draw a few arrows on and off the nullclines, indicating where $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are positive or negative.

DESCRIPTION OF SOLUTION: Above both nullclines, arrows will point down and to the right; in the region including the positive x-axis, arrows will point up and to the right; below both nullclines, arrows will point up and to the left; in the region to the left of both nullclines, arrows point down and to the left.

(c) Is the equilibrium point $(0, 0)$ **stable** or **unstable?**

SOLUTION: Unstable, since a solution with its initial point $(x(0), y(0))$ in the region to the right or left of both nullclines will stay in that region. But $\frac{dx}{dt}$ is positive in the region to the right of both nullclines, and negative in the region to the left of both nullclines. In either of these regions, $\frac{d|x|}{dt}$ will always be positive, so the point $(x(t), y(t))$ will keep moving away from $(0, 0)$.