

Math 2243  
Fall 2006  
Midterm 3, WITH SOLUTIONS  
November 30, 2006  
Time Limit: 50 minutes

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1. (20 points) The matrix

$$A = \begin{bmatrix} -5 & -4 \\ 12 & 9 \end{bmatrix}$$

has eigenvalues  $\lambda_1 = 1$  and  $\lambda_2 = 3$ . Find corresponding **eigenvectors**  $\vec{v}_1$  and  $\vec{v}_2$ .

**SOLUTION:**  $\lambda_1 = 1$ : The augmented matrix for the system satisfied by  $\vec{v}_1$  (imagine there's a vertical bar between the last two columns) is:  $[A - I | \vec{0}] = \begin{bmatrix} -6 & -4 & 0 \\ 12 & 8 & 0 \end{bmatrix}$ , which reduces to  $\begin{bmatrix} 3 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ , so we may choose

$$\vec{v}_1 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}.$$

$\lambda_2 = 3$ :  $[A - 3I | \vec{0}] = \begin{bmatrix} -8 & -4 & 0 \\ 12 & 6 & 0 \end{bmatrix}$ , which reduces to  $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ , so we may choose

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

2. (15 points) Let  $T$  be the linear transformation from  $\mathbb{R}^3$  to  $\mathbb{R}^3$  defined by  $T(\vec{v}) = A\vec{v}$ , where the  $3 \times 3$  matrix  $A$  has eigenvalues  $\lambda_1 = 2$ ,  $\lambda_2 = 0$ , and  $\lambda_3 = -1$ ; with corresponding eigenvectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ and } \vec{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}.$$

(a) (4 points) What is the **kernel** (or nullspace) of  $T$ ?

**SOLUTION:** Since the eigenvectors are a basis of  $\mathbb{R}^3$ , the kernel of  $T$  is the set of vectors  $a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3$  with  $T(a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3) = 2a_1\vec{v}_1 + 0a_2\vec{v}_2 - a_3\vec{v}_3 = \vec{0}$ . So the kernel is the

span of  $\vec{v}_2$ : all vectors of the form  $\begin{bmatrix} s \\ s \\ s \end{bmatrix}$ .

(b) (4 points) Find the **rank** of  $T$ .

**SOLUTION:** By the dimension theorem, the rank of  $T$  = the dimension of  $\mathbb{R}^3$  – the nullity of  $T$  =  $3 - 1 = 2$ .

(c) (7 points) What subspace of  $\mathbb{R}^3$  is the **image** of  $T$ ? Describe the image of  $T$  in the form

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : ax_1 + bx_2 + cx_3 = 0 \right\}.$$

**SOLUTION:** Since the eigenvectors are a basis of  $\mathbb{R}^3$ , the image of  $T$  is the set of all vectors  $T(\vec{v}) = T(a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3) = 2a_1\vec{v}_1 + 0a_2\vec{v}_2 - a_3\vec{v}_3$  with scalars  $a_1, a_2, a_3$ . (This is the span of  $\vec{v}_1$  and  $\vec{v}_3$ .) These are all vectors of the form  $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2a_1 + 2a_3 \\ -a_3 \\ 2a_1 - a_3 \end{bmatrix}$ , thus, after computation,

$$x_1 + 3x_2 - x_3 = 0.$$

3. (20 points) Find a **particular solution**  $y_p(t)$  for the nonhomogeneous differential equation

$$y''(t) - 4y'(t) = 9te^t.$$

**SOLUTION:** Try  $y_p(t) = (At+B)e^t$ . Then  $y_p'(t) = (At+A+B)e^t$  and  $y_p''(t) = (At+2A+B)e^t$ . The differential equation says  $y_p'' - 4y_p' = [(At+2A+B) - 4(At+A+B)]e^t = 9te^t$ . We need  $At - 4At = -3At = 9t$  so  $A = -3$ ; and  $(2A+B) - 4(A+B) = -2A - 3B = 0$  so  $B = -\frac{2}{3}A = 2$ . The particular solution is

$$y_p(t) = (-3t + 2)e^t.$$

4. (25 points) The matrix

$$A = \begin{bmatrix} 5 & 3 & 6 \\ -10 & -8 & -10 \\ 4 & 4 & 5 \end{bmatrix}$$

has eigenvalues  $\lambda_1 = -1$ ,  $\lambda_2 = 2$ , and  $\lambda_3 = -3$ ; with corresponding eigenvectors  $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ ,

$\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$  and  $\vec{v}_3 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$ . Consider the nonhomogeneous system of first-order DEs

$$\vec{x}'(t) = A\vec{x}(t) + \vec{b},$$

where  $\vec{b} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ . A particular solution is  $\vec{x}_p(t) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$  (a constant solution).

(a) (15 points) Find the **general solution**.

**SOLUTION:** The general solution of the **homogeneous** system is  $\vec{x}_h(t) = c_1e^{-t}\vec{v}_1 + c_2e^{2t}\vec{v}_2 + c_3e^{-3t}\vec{v}_3$ . The general solution of this nonhomogeneous problem has the same arbitrary constants  $c_1, c_2, c_3$  :

$$\vec{x}(t) = c_1e^{-t} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2e^{2t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3e^{-3t} \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

(b) (10 points) Find the solution which satisfies the **initial condition**  $\vec{x}(0) = \begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix}$ .

**SOLUTION:** We need  $c_1, c_2, c_3$  to satisfy the system of equations

$$c_1 - c_2 + 0 = 1 - 0 = 1$$

$$0 + c_2 - 2c_3 = -5 - 1 = -6 \text{ and}$$

$$-c_1 + 0 + c_3 = 2 - (-1) = 3.$$

The system reduces to  $c_1 = 0$ ,  $c_2 = -1$ , and  $c_3 = 2$ . So the solution is

$$\vec{x}(t) = -e^{2t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + 2e^{-3t} \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

5. (20 points) The  $2 \times 2$  matrix

$$A = \begin{bmatrix} 1 & 9 \\ -1 & -5 \end{bmatrix}$$

has a double eigenvalue  $\lambda_1 = \lambda_2 = -2$  and only one independent eigenvector  $\vec{v} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ .

(a) (10 points) Find a **generalized eigenvector**.

**SOLUTION:** A generalized eigenvector  $\vec{u}$  satisfies  $(A - \lambda I)\vec{u} = \vec{v}$ . The augmented matrix of this system of equations is  $\begin{bmatrix} 3 & 9 & -3 \\ -1 & -3 & 1 \end{bmatrix}$ , which reduces to  $\begin{bmatrix} 1 & 3 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ . We may choose

$$\vec{u} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

(b) (10 points) Find two independent solutions of the system of DEs

$$\vec{x}'(t) = A\vec{x}(t).$$

**SOLUTION:** The first solution is based on the eigenvector and the eigenvalue  $\lambda = -2$ :

$$\vec{x}_1(t) = e^{-2t} \begin{bmatrix} -3 \\ 1 \end{bmatrix}.$$

A second, independent, solution uses the generalized eigenvector  $\vec{u} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ :

$$\vec{x}_2(t) = e^{\lambda t}(t\vec{v} + \vec{u}) = e^{-2t} \begin{bmatrix} -3t - 1 \\ t \end{bmatrix}.$$