Math 2263 Fall 2009 Midterm 1, WITH SOLUTIONS October 8, 2009

1. (15 points) The lines given parametrically by

$$(x, y, z) = (1 + t, -2 - 2t, 2t), \quad -\infty < t < \infty$$

and

$$(x, y, z) = (1 + 2s, -2 + 2s, s), \quad -\infty < s < \infty$$

intersect at the point $(x, y, z) = (1, -2, 0)$. Find an equation for the **plane which contains both lines**. Write the equation in the form $ax + by + cz = d$.

SOLUTION: A normal vector $\vec{v}$ is the cross product of the vectors $\vec{i} - 2\vec{j} + 2\vec{k}$ and $2\vec{i} + 2\vec{j} + \vec{k}$, which are vectors in the directions of the two given lines.

$$\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 2 \\ 2 & 2 & 1 \end{vmatrix} = -6\vec{i} + 3\vec{j} + 6\vec{k}.$$

We can divide $\vec{v}$ by 3, so an equation for the plane is $-2(x-1) + (y+2) + 2z = 0$, or simplifying:

$$-2x + y + 2z = -4.$$

2. (13 points) Find an equation for the **hyperbolic cylinder** in (x, y, z)-space containing infinitely many lines parallel to the x-axis, and containing the slanted hyperbola

$$x = y, \quad x^2 + y^2 - z^2 = 4.$$

SOLUTION: Eliminate x from the equations $x = y$, $x^2 + y^2 - z^2 = 4$ to get $2y^2 - z^2 = 4$. This is the equation of the hyperbola which is the projection of the slanted hyperbola into the (y, z)-plane. So the hyperbolic cylinder is given by the same equation:

$$2y^2 - z^2 = 4, \quad -\infty < x < \infty,$$

as an equation for a surface in (x, y, z)-space.

3. (15 points) Evaluate the **limit**

$$\lim_{(x,y)\to(0,0)} \frac{x^4 - xy^3 + y^4}{x^4 + x^2y^2 + y^4},$$

or state that it does not exist, giving reasons.

SOLUTION: First try plugging in $x = 0$ and $y = 0$: the numerator and the denominator of the quotient both converge to 0. So it's not easy to tell what the limit is. Try approaching $(0, 0)$ along the line $y = mx$ of slope m: the function is constant along that line and equal to

$$\frac{x^4 - m^3x^4 + m^4x^4}{x^4 + m^2x^4 + m^4x^4} = \frac{1 - m^3 + m^4}{1 + m^2 + m^4},$$

which depends on m: for example, the quotient equals 1 if $m = 0$, and equals $\frac{1}{3}$ if $m = 1$. So the limit **does not exist**, because you get different limits along lines of different slope m.

4. (15 points) For the function $f(x,y) = (x^2 - y^2)e^{xy}$, find the **second partial derivatives** at $x = 1$, $y = -1$:

$$f_{xx}(1,-1), \quad f_{xy}(1,-1) \quad \text{and} \quad f_{yy}(1,-1).$$

SOLUTION: Compute $f_x = \frac{\partial f}{\partial x} = (2x + x^2y - y^3)e^{xy}$ and $f_y = \frac{\partial f}{\partial y} = (-2y + x^3 - xy^2)e^{xy}$. Then

$$f_{xx} = (2 + 4xy + x^2y^2 - y^4)e^{xy} = -\frac{2}{e},$$
$$f_{xy} = (3x^2 - 3y^2 + x^3y - xy^3)e^{xy} = 0, \quad \text{and}$$
$$f_{yy} = (-2 - 4xy + x^4 - x^2y^2)e^{xy} = \frac{2}{e}.$$

5. (12 points) Suppose $z = f(x,y)$ is a function with first partial derivatives $f_x(1,3) = -2$ and $f_y(1,3) = -1$. If x and y are both functions of t: $x = g(t) = -3 + 4t$ and $y = h(t) = 6 - 3t$, find the **derivative at** $t = 1$:

$$\frac{dz}{dt}(1) = \frac{d}{dt} f\Big(g(t), h(t)\Big)(1).$$

SOLUTION: The **chain rule** says that

$$\frac{dz}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}.$$

Compute $g(1) = -3 + 4 = 1$, $h(1) = 6 - 3 = 3$ so f_x and f_y are to be evaluated at $x = 1$, $y = 3$. Compute $\frac{dx}{dt} = 4$ and $\frac{dy}{dt} = -3$. Finally,

$$\frac{dz}{dt}(1) = (-2)(4) + (-1)(-3) = -5.$$

6. (15 points) A new set of coordinates (r, θ) (polar coordinates) are defined by $x = r\cos\theta$ and $y = r\sin\theta$. The partial derivatives of r and θ are

$$r_x = \cos\theta, \quad r_y = \sin\theta,$$
$$\theta_x = -\frac{\sin\theta}{r}, \quad \theta_y = \frac{\cos\theta}{r}.$$

If $f(x,y)$ is a continuously differentiable function and $r \neq 0$, compute a formula for

$$\left|\vec{\nabla} f\right|^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2$$

in terms of r, θ and the partial derivatives of f with respect to r and θ. Simplify!

SOLUTION: The **chain rule** says that

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x} = f_r \cos\theta - f_\theta \frac{\sin\theta}{r},$$

and

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial y} = f_r \sin\theta + f_\theta \frac{\cos\theta}{r}.$$

So

$$\left|\vec{\nabla} f\right|^2 = \left(f_r \cos\theta - f_\theta \frac{\sin\theta}{r}\right)^2 + \left(f_r \sin\theta + f_\theta \frac{\cos\theta}{r}\right)^2$$

$$= f_r^2 \cos^2\theta - 2f_r f_\theta \frac{\cos\theta \sin\theta}{r} + f_\theta^2 \frac{\sin^2\theta}{r^2} + f_r^2 \sin^2\theta + 2f_r f_\theta \frac{\sin\theta\cos\theta}{r} + f_\theta^2 \frac{\cos^2\theta}{r^2}$$

$$= f_r^2 + \frac{f_\theta^2}{r^2}.$$

7. **(15 points)** **(a)** (7 of 15 points) Find the **gradient** of the function

$$f(x, y, z) = (x + z^2)\sin(xy)$$

at the point $(x, y, z) = (1, \frac{\pi}{2}, -2)$.

(15 points) **SOLUTION:** $f_x = \sin(xy) + (x+z^2)y\cos(xy) = \sin\frac{\pi}{2} + (1+4)\frac{\pi}{2}\cos\frac{\pi}{2} = 1+0 = 1$; $f_y = (x + z^2)x\cos(xy) = (1 + 4)0 = 0$; and $f_z = 2z\sin(xy) = (-4)(1) = -4$. So the gradient of f at $(1, \frac{\pi}{2}, -2)$ is

$$\vec{\nabla} f(1, \frac{\pi}{2}, -2) = \vec{i} - 4\vec{k}.$$

(b) (8 of 15 points) Find the **directional derivative** of f at the point $(1, \frac{\pi}{2}, -2)$ in the direction of the unit vector

$$\vec{u} = \frac{1}{3}\left(2\vec{i} + \vec{j} + 2\vec{k}\right).$$

SOLUTION: We know that

$$D_{\vec{u}} f(1, \frac{\pi}{2}, -2) = \vec{u} \cdot \vec{\nabla} f(1, \frac{\pi}{2}, -2) = \frac{2\vec{i} + \vec{j} + 2\vec{k}}{3} \cdot \left(\vec{i} - 4\vec{k}\right) = \frac{2-8}{3} = -2.$$