Math 2263 Fall 2008 Midterm 2, WITH SOLUTIONS October 30, 2008

1. (10 points) Let R be the rectangle $-2 \le x \le 2$, $0 \le y \le 1$. Find the double integral $\iint_R \frac{x^2}{y+1}\,dA$.

   SOLUTION: $\iint_R \frac{x^2}{y+1}\,dA = \int_{y=0}^1 \int_{x=-2}^2 \frac{x^2}{y+1}\,dx\,dy =$
   $\int_{y=0}^1 \left[\frac{x^3}{3(y+1)}\right]_{x=-2}^2 dy = \frac{16}{3}\Big[\ln(y+1)\Big]_{y=0}^1 = \frac{16}{3}\ln 2.$

2. (10 points) Let R be the rectangle $0 \le x \le 5$, $0 \le y \le 2$ in the (x,y)-plane. If a continuous function $f(x,y)$ satisfies $-1 \le f(x,y) \le 3 - y^2$, what does this tell you about the value of $\iint_R f(x,y)\,dA$?

   SOLUTION: Since $-1 \le f(x,y) \le 3 - y^2$, the only information we have is that

   $$\iint_R (-1)\,dA \le \iint_R f(x,y)\,dA \le \iint_R 3 - y^2\,dA.$$

   So since $\iint_R 3 - y^2\,dA = \int_{-2}^2 \int_0^5 (3-y^2)\,dx\,dy = 5\Big[3y - \frac{y^3}{3}\Big]_{-2}^2 = \frac{100}{3}$, and since R has area 10, this only tells us that

   $$-10 \le \iint_R f(x,y)\,dA \le \frac{100}{3}.$$

3. (10 points) Let D be the region $0 \le x \le 2$, $0 \le y \le x^2$ in the (x,y)-plane. Find the double integral $\iint_D xy\,dA$.

   SOLUTION: $\iint_D xy\,dA = \int_0^2 x \int_0^{x^2} y\,dy\,dx =$
   $\int_0^2 x\frac{x^4}{2}\,dx = \left[\frac{x^6}{12}\right]_0^2 = \frac{16}{3}.$

4. (15 points) A plate is in the shape of the triangle D: $0 \le y \le 1 - |x|$, with corners $(-1,0)$, $(0,1)$ and $(1,0)$. The plate has mass density at the point (x,y) equal to $\rho(x,y) = y$ per unit area.

   (a) (5 points) Find the **total mass** m of the plate.

   SOLUTION: $m = \iint_D \rho(x,y)\,dA = \int_0^1 \int_{-(1-y)}^{1-y} y\,dx\,dy =$
   $\left[y^2 - \frac{2y^3}{3}\right]_0^1 = \frac{1}{3}.$

   (b) (10 points) Find the **center of mass** $(\overline{x}, \overline{y})$ of the plate.

   SOLUTION: $m\overline{x} = \iint_D x\rho(x,y)\,dA = \int_0^1 \int_{-(1-y)}^{1-y} xy\,dx\,dy =$
   $\int_0^1 y\Big[\frac{(1-y)^2}{2} - \frac{(1-y)^2}{2}\Big]\,dy = 0$. So $\overline{x} = 0$. Meanwhile, $m\overline{y} = \iint y\rho(x,y)\,dA = \int_0^1 \int_{-(1-y)}^{1-y} y^2\,dx\,dy =$
   $\int_0^1 2y^2(1-y)\,dy = \left[\frac{2}{3}y^3 - \frac{1}{2}y^4\right]_0^1 = \frac{1}{6}$. So $\overline{y} = \frac{1}{2}$, Thus the center of mass of the plate is $(0, \frac{1}{2})$.

5. (10 points) Let D be the circular disk of radius R and center $(0,0)$ in the (x,y)-plane. Find $\iint_D \sin(x^2+y^2)\,dA$. (*Hint:* polar coordinates.)

SOLUTION: $\iint_D \sin(x^2+y^2)\,dA = \int_0^{2\pi}\int_0^R \sin(r^2)r\,dr\,d\theta = 2\pi\int_0^{R^2} \sin u(\frac{1}{2})\,du = \pi[1-\cos(R^2)]$.

6. (15 points) Find the **maximum** and **minimum** values of $f(x,y) = xy - x$ subject to the **constraint** $g(x,y) = x^2+4y^2 = 4$. (*Hint:* Lagrange multipliers.)

SOLUTION: $\vec{\nabla} f = (y-1)\vec{i} + x\vec{j} = \lambda\vec{\nabla} g = \lambda(2x\vec{i} + 8y\vec{j}$. This says that $y-1 = 2\lambda x$ and $x = 8\lambda y$, so $4y(y-1) = 8\lambda xy = x^2$. Substituting in $g(x,y) = 4$, we get $8y^2 - 4y = 4$, so $y = 1$ or $y = -\frac{1}{2}$ which implies $x = 0$ or $x = \pm\sqrt{3}$, respectively: there are three points $(x,y) = (0,1)$, $(\sqrt{3}, -\frac{1}{2})$ or $(-\sqrt{3}, -\frac{1}{2})$ which satisfy the Lagrange multiplier condition. Evaluate $f(0,1) = 0$; $f(\sqrt{3}, -\frac{1}{2}) = -\frac{3\sqrt{3}}{2}$ which is the minimum; and $f(-\sqrt{3}, -\frac{1}{2}) = +\frac{3\sqrt{3}}{2}$ which is the maximum.

7. (25 points) Let $f(x,y) = 3y^3 - x^2y + x^2 + \frac{9}{2}y^2$.
(a) (5 points) Compute the **first** and **second** partial derivatives of $f(x,y)$.

SOLUTION: $\frac{\partial f}{\partial x} = f_x = -2xy + 2x$; $f_y = 9y^2 - x^2 + 9y$; differentiating again, $\frac{\partial^2 f}{\partial x^2} = f_{xx} = -2y + 2$; $f_{xy} = -2x$; and $f_{yy} = 18y$.

(b) (10 points) Find all the **critical points** of $f(x,y)$.

SOLUTION: $f_x = -2xy + 2x = 0$ when $x = 0$ or $y = 1$; if $x = 0$, then $f_y = 9y^2 + 9y = 0$ when $y = 0$ or $y = -1$, so $(0,0)$ and $(0,-1)$ are critical points. if $y = 1$ then $f_y = 18 - x^2$, so there are two more critical points $(3\sqrt{2}, 1)$ and $(-3\sqrt{2}, 1)$.

(c) (10 points) For each critical point, state whether it is a **local minimum point**, a **local maximum point** or a **saddle point**.

SOLUTION: The four critical points are $(0,0)$, $(0,-1)$, $(3\sqrt{2},1)$ and $(-3\sqrt{2},1)$; the respective values of $f_{xx}f_{yy} - f_{xy}^2$ are 18 (positive),-36 (negative), -72 (negative) and -72 (negative), so since $f_{xx}(0,0) = 2$ (positive), they are respectively a local minimum, a saddle point, a saddle point and a saddle point.