Math 2263 Spring 2008 Midterm 2, WITH SOLUTIONS April 3, 2008

1. (10 points) Let R be the rectangle $0 \le x \le \ln 7$, $0 \le y \le \ln 3$. Find the double integral

$$\iint_R e^{x+2y}\, dA.$$

SOLUTION: $e^{x+2y} = e^x e^{2y}$, so $\iint_R e^{x+2y}\, dA = \int_0^{\ln 3} e^{2y}\, dy \int_0^{\ln 7} e^x\, dx =$
$= \left[\frac{1}{2}e^{2y}\right]_{y=0}^{\ln 3} \left[e^x\right]_{x=0}^{\ln 7} = \frac{1}{2}(9-1)(7-1) = 24.$

2. (10 points) Let R be the square $-2 \le x \le 2$, $-2 \le y \le 2$ in the (x, y)-plane. If a continuous function $f(x, y)$ satisfies

$$0 \le f(x, y) \le |x|,$$

what does this tell you about the value of $\iint_R f(x, y)\, dA$?

SOLUTION: In general, if $f(x, y) \le g(x, y)$, then $\iint_R f(x, y)\, dA \le \iint_R g(x, y)\, dA$. So use this twice, first with $f(x, y) = 0$ and $g(x, y) = f(x, y)$, and second with $f(x, y) = f(x, y)$ and $g(x, y) = |x|$. You compute $\iint_R |x|\, dA = \int_{-2}^{2}\int_{-2}^{2} |x|\, dxdy = (4)(2)\int_0^2 x\, dx = 16$, and of course $\iint_R 0\, dA = 0$. So

$$0 \le \iint_R f(x, y)\, dA \le 16.$$

3. (10 points) Let R be the triangle $0 \le x \le 1$, $0 \le y \le x$ in the (x, y)-plane. Find the double integral

$$\iint_R e^{x^2}\, dA.$$

SOLUTION: $\iint_R e^{x^2}\, dA = \int_0^1 \int_0^x e^{x^2} dy\, dx = \int_0^1 xe^{x^2}\, dx =$
$-\frac{1}{2}\int_0^{-1} e^u\, du = -\frac{1}{2}(e^{-1} - 1) = \frac{e-1}{2e}.$

4. (20 points) A plate is in the shape of the triangle D: $x \ge 0, y \ge 0, x + y \le 1$. The plate has mass density at the point (x, y) equal to $\rho(x, y) = 1 - x - y$.
(a) (10 points) Find the **total mass** m of the plate.

(b) (10 points) Find the **center of mass** $(\overline{x}, \overline{y})$ of the plate.

SOLUTION: **(a)** $m = \iint_D \rho\, dA = \int_0^1 \int_0^{1-y} (1 - x - y)\, dx\, dy = \int_0^1 \left[x(1-y) - \frac{(1-y)^2}{2}\right]_0^{1-y} dy =$
$\int_0^1 \frac{(1-y)^2}{2}\, dy = \frac{1}{6}.$
(b) $m\overline{y} = \iint_D y\rho\, dA = \int_0^1 \int_0^{1-y} y(1 - x - y)\, dxdy = \int_0^1 y\frac{(1-y)^2}{2}\, dy = \frac{1}{2}\int_0^1 [y - 2y^2 + y^3]\, dy = \frac{1}{24}.$
So $\overline{y} = \frac{6}{24} = \frac{1}{4}$. By symmetry, $\overline{x} = \overline{y} = \frac{1}{4}$. So the **center of mass** is $(\frac{1}{4}, \frac{1}{4})$.

5. **(20 points)** **(a)** (5 points) Let D be the circular disk of radius 2 and center $(0,0)$ in the (x,y)-plane. Write a double integral over the domain D with respect to area which represents the **volume** bounded above by the hemisphere $z = \sqrt{4 - x^2 - y^2}$ and below by the (x,y)-plane.

   (20 points) **(b)** (5 points) Convert this integral to **polar coordinates.**

   (20 points) **(c)** (10 points) Evaluate this integral.

   (20 points) **SOLUTION:** **(a)** $V = \iint_D \sqrt{4 - x^2 - y^2}\, dA$.

   (b) $\int_0^{2\pi} \int_0^2 \sqrt{4 - r^2}\, r\, dr\, d\theta$.

   (c) $\int_0^{2\pi} \int_0^2 \sqrt{4 - r^2}\, r\, dr\, d\theta = 2\pi(-\frac{1}{2}) \int_4^0 \sqrt{u}\, du = -\pi \left[\frac{u^{3/2}}{3/2}\right]_4^0 = \frac{2}{3}\pi(8 - 0) = \frac{16\pi}{3}$.

6. (25 points) Let $f(x,y) = 2x^3 + xy^2 + 5x^2 + y^2$.
   (a) (5 points) Compute the first and second partial derivatives of $f(x,y)$.

   (b) (10 points) Find all the **critical points** of $f(x,y)$.

   (c) (10 points) For each critical point, state whether it is a **local minimum point**, a **local maximum point** or a **saddle point**.

   SOLUTION: **(a)** The first partial derivatives $f_x = 6x^2 + y^2 + 10x$ and $f_y = 2xy + 2y$; the second partial derivatives $f_{xx} = 12x + 10$, $f_{xy} = 2y$ and $f_{yy} = 2x + 2$.

   (b) $f_y = 2xy + 2y = 0$ requires either $y = 0$ or $x = -1$. If $y = 0$, then $f_x = 6x^2 + 10x = 0$ requires either $x = 0$ or $x = -\frac{5}{3}$. If $x = -1$, then $f_x = y^2 - 4 = 0$ requires $y = \pm 2$. So there are four critical points: $(x,y) = (0,0)$; $(x,y) = (0, -\frac{5}{3})$; $(x,y) = (-1,2)$; and $(x,y) = (-1,-2)$.

   (c) The second partial derivatives at the four critical points are:
   $(f_{xx}, f_{xy}, f_{yy}) = (10, 0, 2)$ at $(x,y) = (0,0)$, so the Hessian determinant $f_{xx}f_{yy} - f_{xy}^2 = 20 > 0$, while $f_{xx} = 10 > 0$, indicating $(x,y) = (0,0)$ is a **local minimum** point;
   $(f_{xx}, f_{xy}, f_{yy}) = (10, -\frac{10}{3}, 2)$ at $(x,y) = (0, -\frac{5}{3})$, so $f_{xx}f_{yy} - f_{xy}^2 = 20 - 100/3 < 0$, indicating $(0, -\frac{5}{3})$ is a **saddle** point;
   $(f_{xx}, f_{xy}, f_{yy}) = (-2, 4, 0)$ at $(x,y) = (-1,2)$, so $f_{xx}f_{yy} - f_{xy}^2 = 0 - 16 < 0$, indicating $(-1,2)$ is a **saddle** point; and
   $(f_{xx}, f_{xy}, f_{yy}) = (-2, -4, 0)$ at $(x,y) = (-1,-2)$, so $f_{xx}f_{yy} - f_{xy}^2 = 0 - 16 < 0$, indicating a third **saddle** point at $(-1,-2)$.