BOUNDARY CONTROL OF PDES VIA CURVATURE FLOWS: THE VIEW FROM THE BOUNDARY II

SANTIAGO BETELÚ, ROBERT GULLIVER AND WALTER LITTMAN

Abstract. We describe some results on the exact boundary controllability of the wave equation on an orientable two-dimensional Riemannian manifold with nonempty boundary. If the boundary has positive geodesic curvature, we show that the problem is controllable in finite time if (and only if) there are no closed geodesics in the interior of the manifold. This is done by solving a parabolic problem to construct a convex function. We exhibit an example for which control from a subset of the boundary is possible, but cannot be proved by means of convex functions. We shall also describe a numerical implementation of this method.

Dedicated to the memory of J. L. Lions

J. L. Lions was a pioneer in many areas of mathematics. His strong leadership role, both as a creator of and advocate for mathematics, will be sorely missed. "He beat a powerful drum."

1. Introduction

On a Riemannian manifold $(\overline{\Omega}, g)$, for a linear hyperbolic partial differential equation such as

$$u_{tt} = \Delta_g u + \text{lower order terms} \tag{1}$$

where $u : \overline{\Omega} \times [0,T] \to I\!R$, the problem of boundary controllability is of widespread interest. By this, we mean the question whether, for any given initial data (u_0, u_1) on $\overline{\Omega}$, there is a choice of boundary data $U : \partial\Omega \times [0,T] \to I\!R$ such that the solution of (1) satisfying initial conditions $u(x,0) = u_0(x)$ and $u_t(x,0) = u_1(x)$ for all $x \in \overline{\Omega}$ and boundary conditions $u(x,t) = U(x,t)$ for all $(x,t) \in \partial\Omega \times [0,T]$ vanishes identically for $t \geq T$.

In this paper, we shall assume that all manifolds are connected and orientable.

Recently, the last two authors established the following result [7]:

Theorem 1.1. *Let $\overline{\Omega}$ be a smooth, compact Riemannian manifold with boundary equipped with the metric g, whose boundary $\partial\Omega$ has positive second fundamental form. Then the wave equation (1) can be controlled from the boundary in finite time, provided any two points on $\partial\Omega$ can be connected by a unique chord which is nondegenerate. The time of control is any $T > T_0 = \mathrm{diam}_{\overline{\Omega}}\, \partial\Omega$, that is, T_0 is the maximum distance between points of $\partial\Omega$ as measured by curves in $\overline{\Omega}$.*

Here, a *chord* is a geodesic in $\overline{\Omega}$ of minimum length between two points of $\partial\Omega$.

The conditions for this theorem are far from necessary, and indeed, examples are given in [7] where controllability holds without the condition of the uniqueness of chords being satisfied.

One necessary condition for controllability is well known (see e.g. [7], p. 151]): there can be no closed geodesic in Ω. On the other hand, if $\partial\Omega$ has positive second fundamental form and there is no closed geodesic in Ω, then Ω must be simply connected (see [7], Corollary 3.3). The question thus arises whether the wave equation (1) is controllable from $\partial\Omega$ if Ω has no closed geodesics and $\partial\Omega$ has positive second fundamental form. The question has a positive answer if Ω is two-dimensional:

Theorem 1.2. Let $\overline{\Omega}$ be a smooth compact Riemannian surface whose boundary $\partial\Omega$ is a smooth embedded curve with nonzero geodesic curvature vector pointing into Ω. Assume there are no closed geodesics in Ω. Then the wave equation with lower order terms

$$u_{tt} = \Delta_g u + \sum_{i=1}^{2} b^i(x,t)u_{x_i} + a(x,t)u \tag{2}$$

is controllable from $\partial\Omega$ in finite time.

To be more precise, given initial data $u_0 \in H_{-1}(\Omega)$ and $u_1 \in H_0(\Omega)$, there exist Dirichlet controls $U \in L_2\left([0,T];\, L_2(\partial\Omega)\right)$ achieving $u \equiv 0$ after finite time T.

Remark 1. It thus becomes very important to be able to discern whether a given Ω, with a given Riemannian metric (or with a given time-independent second-order elliptic operator) contains closed geodesics. In Section 5 we shall describe a computational method which implements the parabolic flow of Proposition 2 below and will enable us to either find geodesics homotopic to a component of $\partial\Omega$, or to conclude that there are no closed geodesics.

2. TOOLS NEEDED IN THE PROOF

An essential tool in this paper will be the construction of an appropriate convex function (with respect to the Riemannian metric) on Ω. The importance of convex functions for control of hyperbolic equations is seen in the following result of Lasiecka, Triggiani and Yao.

Proposition 1. ([8].) Suppose there is a C^2 function v uniformly convex on $\overline{\Omega}$ with respect to the Riemannian metric g:

$$\nabla^2_{X,X}\, v = \mathrm{Hess}_g v(X,X) \geq 2\rho |X|_g^2 \quad \forall X \in T_x\Omega,$$

where ρ is a positive constant and $0 \leq v(x) \leq K$. Then equation (2) is controllable in time $T > T_0 := 2\sqrt{K/\rho}$. (See also [3].)

The condition on the Hessian tensor of v has the following geometric interpretation: for any geodesic $\gamma(s)$ parameterized by arc length, $v(\gamma(s))'' \geq 2\rho$. The uniqueness theorem needed according to [8] for the validity of Proposition 1 follows from [10]. The theorem in [8] is stated for the case when Ω is topologically a domain in $I\!R^n$; the proof holds for a Riemannian manifold Ω with minor modifications.

The following proposition is an excellent illustration the unity of mathematics. In attacking a control problem for a linear *hyperbolic* equation, we are forced to use a nonlinear *parabolic* equation which arises in a quite unrelated geometric problem of curve-shortening flows.

Proposition 2. ([5]; see also [2].) Let F be a smooth compact Riemannian surface without boundary. Let $C(0) : S^1 \to F$ be a smooth curve embedded in F. Then there exists a unique family of curves $C(t) : S^1 \to F$ for $t \in [0, t_\infty)$ satisfying flow by geodesic curvature:

$$\frac{\partial C}{\partial t} = k\vec{N}$$

where $k\vec{N}$ is the geodesic curvature vector of $C(t)$, k the signed geodesic curvature, and $\vec{N}$ a unit normal vector whose orientation is induced from the orientation of S^1 and of Ω. The curves $C(t)$ are embedded. If t_∞ is finite, $C(t)$ converges to a point and its length approaches 0 as $t \to t_\infty$. If $t_\infty = \infty$, a subsequence of $C(t)$ converges to a geodesic (see Remark 2(b).) Furthermore, the "evolving curve map" $C(u,t) : S^1 \times [0, t_\infty) \to F$ is smooth. (*Note:* We will later see that it is a smooth diffeomorphism onto a subset of F).

The following *Strong Maximum Principle* is a well-known tool for problems in analysis and geometry.

Proposition 3. Suppose

$$w_t - w_{ss} - h(s,t)\, w \geq 0$$

in $a < s < b, 0 < t < T$, with h bounded. If w takes on the minimum value of zero at an interior point, then $w \equiv 0$.

Proposition 4. ([2].) For the family of curves $C(t)$ satisfying the flow of Proposition 2, the curvature $k(s,t)$ of $C(t)$ at arc-length position s satisfies the following parabolic partial differential equation:

$$k_t = k_{ss} + k^3 + Rk$$

where s is arclength along $C(t)$ and R is the Gaussian curvature of F. (Observe that (s,t) is not the system of coordinates arising from the flow.)

Proposition 5. If the initial curve $C(0)$ has positive geodesic curvature, then $C(t)$ has positive geodesic curvature for all $t \geq 0$.

Proof: That $k \geq 0$ follows from Proposition 4 and the weak maximum principle for parabolic equations. That $k > 0$ follows from the strong maximum principle (Proposition 3). ■

Our aim is to construct the convex function v needed in Proposition 1. Our method is to construct the one-parameter family $C(t)$ of smooth Jordan curves as in Proposition 2, and show that these are the level curves of a convex function. Unfortunately, Proposition 2 assumes that F is a compact surface without boundary, while in Theorem 1.2 we have a compact domain $\overline{\Omega}$ with boundary. What we do is to adjoin to $\overline{\Omega}$ a "cap", that is, a topological disc, at each connected component of $\partial\Omega$, so as to enlarge $\overline{\Omega}$ to a closed surface F without boundary, and to choose a Riemannian metric on F which extends the metric on $\overline{\Omega}$, via a partition of unity argument. Then $C(0)$ is an embedded curve in F whose curvature vector points into Ω. It follows that the flow $C(t)$ of Proposition 2 will avoid the caps and not cross $C(0)$ for $t > 0$.

Remark 2. It follows from Proposition 5 that **(a)** two different curves $C(t_1)$ and $C(t_2)$ cannot intersect. In fact, for $t > t_1$, $C(t_1)$ acts as a barrier for the flow of Proposition 2; $C(t)$ remains disjoint from $C(t_1)$ for all $t > t_1$ by Proposition 3. Further, **(b)** if $t_\infty = \infty$ in Proposition 2, then $C(t)$ itself converges to a geodesic (not merely a subsequence).

The following lemma holds for an n-dimensional Riemannian manifold Ω, although only the case $n = 2$ is used in the present paper. Given $f : \Omega \to I\!R$, we write $\nu := \nabla f / |\nabla f|$ for the unit normal vector to the level sets $\Sigma_s := f^{-1}(s)$. The second fundamental form (with a non-traditional sign) may be defined as $B_{\Sigma_s}(X, Y) := \langle \nabla_X \nu, Y \rangle$.

Lemma 1. Suppose that $f : \Omega \to I\!R$ has convex level sets and nonvanishing gradient, and assume at points of each level surface Σ_s, for all unit vectors X tangent to Σ_s, that

$$|\nabla f|\, B_{\Sigma_s}(X, X) \geq c_1 > 0. \tag{3}$$

Further let $-c_2 \in I\!R$ be a lower bound for $\nabla^2_{Y,\nu} f$ among all unit vectors Y, and suppose that $|\nabla f|^2 \geq c_3 > 0$. Write $v(x) := e^{\Lambda f(x)}$. If Λ is chosen large enough that $c_1(c_3\Lambda - c_2) > {c_2}^2$, then for all vectors Y,

$$\nabla^2_{Y,Y}\, v \geq c\, \Lambda\, v\, |Y|^2,$$

where $c > 0$. Specifically, $0 < c < c_1$ is small enough that $(c_3\Lambda - c_2 - c)(c_1 - c) > {c_2}^2$.

Proof: Write any vector $Y \in T_x\Omega$ as $Y = \langle Y, \nu \rangle \nu + X$, where $\langle X, \nu \rangle = 0$. Then the Hessian tensor

$$\begin{aligned}
\nabla^2_{Y,Y} v &= \langle \nabla_Y(\nabla v), Y \rangle = \langle \nabla_Y(\Lambda v \nabla f), Y \rangle \\
&= \Lambda^2 v \langle Y, \nabla f \rangle^2 + \Lambda v \nabla^2_{Y,Y} f \\
&= \Lambda^2 v \langle Y, \nu \rangle^2 |\nabla f|^2 + \Lambda v \left\{ \langle Y, \nu \rangle^2 \nabla^2_{\nu,\nu} f + 2\langle Y, \nu \rangle \nabla^2_{X,\nu} f + \langle \nabla_X(\nu |\nabla f|), X \rangle \right\} \\
&= \Lambda v \left\{ \langle Y, \nu \rangle^2 \left(\Lambda |\nabla f|^2 + \nabla^2_{\nu,\nu} f \right) + 2\langle Y, \nu \rangle \nabla^2_{X,\nu} f + |\nabla f| B_{\Sigma_s}(X, X) \right\} \\
&\geq \Lambda v \left\{ \langle Y, \nu \rangle^2 (c_3\Lambda - c_2) - 2c_2 \langle Y, \nu \rangle |X| + c_1 |X|^2 \right\} \\
&\geq c\Lambda v \left\{ \langle Y, \nu \rangle^2 + |X|^2 \right\} \\
&= c\Lambda v |Y|^2,
\end{aligned}$$

provided $0 < c < c_1$ is less than or equal to the (positive) smaller eigenvalue of the positive definite 2×2 matrix

$$\begin{pmatrix} c_3\Lambda - c_2 & c_2 \\ c_2 & c_1 \end{pmatrix}.$$

■

3. CONSTRUCTION OF THE CONVEX FUNCTION v

We are now ready to prove Theorem 1.2. Let $\overline{\Omega}$ be a smooth compact Riemannian surface with boundary, extended to a closed two-dimensional Riemannian manifold F, and choose the initial curve $C(0) = \partial\Omega \subset F$ for the flow of Proposition 2. We assume that $\partial\Omega$ has nonzero geodesic curvature vector pointing into Ω, and that there are no closed geodesics in Ω. Our plan of action is to begin by using the curves $C(t)$ to construct an initial version

$w(x)$ of the convex function $v(x)$. Observe that for $t > 0$, $C(t) \subset \Omega$ by the remarks following Proposition 5. According to Proposition 2, the time of existence t_∞ must be finite (otherwise $C(t)$ would converge to a geodesic in Ω); $C(t)$ must converge to a point $x_0 \in \Omega$ as $t \to t_\infty$; and the length of $C(t)$ approaches zero. Note that $C(t)$ crosses every point of $\overline{\Omega} \backslash \{x_0\}$ as $0 \le t < t_\infty$, since otherwise, its length could not tend to zero. Also, since $C(0)$ has curvature $k(0, s) > 0$, the curvature $k(t, s)$ of $C(t)$ is uniformly positive for $0 \le t \le t_1 < t_\infty$ by Proposition 5. Note that because of the absence of closed geodesics, Ω must be simply connected (see proof of Corollary 3.3 of [7]).

Now every $x \in \overline{\Omega}$ has a unique value of $t = t(x) \in [0, t_\infty]$ associated with it, the "time" at which the moving curve $C(t)$ crosses x, so that $x \in C(t)$ (see Remark 2(a)). Now let

$$f(x) = t_\infty - t(x),$$

so that $f(x_0) = 0,\ f \ge 0$ and $\lim_{t \to t_\infty} C(t) = \{x_0\}$.

In case t_∞ is finite, the map $C(u, t)$ enables us to introduce (u, t) as orthogonal coordinates in $\overline{\Omega} \backslash \{x_0\}$. Since the map $C(u, t)$ has nonzero Jacobian there, the map is a smooth diffeomorphism onto a subset of F. In particular, $t(x)$ and $f(x)$ are smooth away from x_0.

It is clear that away from x_0, $|\nabla f| = 1/k(s, t)$ is bounded away from 0, as is $B_{\Sigma_s}(X, X)$ for unit vectors X tangent to the level curves Σ_s. Thus, by Lemma 1, in the complement of a small convex neighborhood of x_0, the function

$$w(x) := e^{\Lambda f(x)}$$

will be a uniformly convex function, provided Λ is taken sufficiently large. Note that

$$1 = w(x_0) \le w(x) \le e^{\Lambda t_\infty} =: \alpha_\infty.$$

Since the level sets of the function w are nothing but the curves $C(t)$, we can relable them so that for $1 < \alpha \le \alpha_\infty$,

$$C_\alpha := \{x : w(x) = \alpha\}$$

and denote their interiors by

$$S_\alpha := \{x : w(x) < \alpha\}.$$

We pick α_1 sufficiently close to 1 so that S_{α_1} is contained in a single coordinate patch

$$\xi = (\xi_1, \xi_2), \quad |\xi| < 1,$$

with $\xi(x_0) = 0$ and so that $|\xi|^2$ is a uniformly convex function (relative to the metric g) in S_{α_1}. Now choose α_2 with $1 < \alpha_2 < \alpha_1$. Then for $\beta > 0$ the function

$$z(x) = \alpha_2 + \beta\,|\xi|^2$$

will also be uniformly convex in S_{α_1}. Now letting $0 < \beta < \alpha_1 - \alpha_2$, we have $\alpha_2 \le z(x) \le \alpha_1$. If we further make β sufficiently small, the set

$$C_* := \{x : z(x) = w(x)\},$$

being a C^2 perturbation of the curve C_{α_2}, will be another Jordan curve (with positive curvature), located between the curves C_{α_2} and C_{α_1}, i.e.,

$$C_* \subset S_{\alpha_1} \backslash \overline{S_{\alpha_2}}.$$

Definition 1. A continuous function $v : \Omega \to I\!R$ is *uniformly convex* if, for some $k_0 > 0$, along every geodesic γ the function $v(\gamma(s)) - k_0\, s^2/2$ is convex as a function of arc length s along γ.

Note that for C^2 functions, the term "uniformly convex" was already defined in the statement of Proposition 1. In the C^2 case, as is not difficult to show, the two definitions are equivalent.

It is easily checked (using the classical definition of a convex function of one variable) that the maximum of two uniformly convex functions is again uniformly convex.

Hence if we define

$$v(x) := \left\{ \begin{array}{c} w(x), x \in \overline{\Omega} \setminus S_{\alpha_1}, \\ \max[w(x), z(x)], x \in S_{\alpha_1} \end{array} \right\},$$

we see that v is uniformly convex in Ω.

As before, let the compact Riemannian surface-with-boundary $\overline{\Omega}$ be extended to a compact Riemannian surface F without boundary. Extend v to F so that it remains uniformly convex on a neighborhood of $\overline{\Omega}$.

Greene and Wu show ([6], p. 214):

Proposition 6. Let $v : F \to I\!R$ be continuous and uniformly convex, with convexity constant $k_0 > 0$. Then v is the uniform limit, as $\varepsilon \to 0$, of smooth functions $v_\varepsilon : F \to I\!R$ with $\nabla^2_{Y,Y}\, v_\varepsilon > (k_0 - \varepsilon)|Y|^2$ for all vectors Y.

The proof of Proposition 6 is local in the sense that the uniform convexity of v_ε on a closed subset $\overline{\Omega}$ requires only the uniform convexity of v on a neighborhood of $\overline{\Omega}$. The function v_ε in the conclusion of Theorem 6 is the function required for the hypothesis of Proposition 1 (and called v there). This concludes the proof of Theorem 1.2. ■

Remark 3. Even in the event that Ω contains a closed geodesic, all is not lost. Suppose that Ω has only one boundary component. We choose a second boundary component Γ_0 to be one of the curves $C(t_1)$, t_1 close to t_∞, on which zero Dirichlet boundary conditions are imposed. $\overline{\Omega}$ is replaced by the subdomain swept out by $C(t)$, $0 \le t \le t_1$. Then with the help of Proposition 7 below, we get controllability from $\Gamma_1 := C(0)$ in the new (reduced) manifold, which is a topological annulus.

4. CONTROL FROM PART OF THE BOUNDARY: AN EXAMPLE

In many interesting situations, control of a partial differential equation is carried out only on a closed subset $\Gamma_1 \subset \partial\Omega$, rather than on all of $\partial\Omega$. On the remaining open subset $\Gamma_0 := \partial\Omega \backslash \Gamma_1$, we suppose (for example) that homogeneous Dirichlet conditions are imposed. In this situation, [8] prove the following generalization of Proposition 1 above:

Proposition 7. Suppose there is a C^2 function v on $\overline{\Omega}$ which is uniformly g-convex:

$$\nabla^2_{X,X} v \ge 2\rho |X|^2$$

for all $X \in T_x\Omega$, all $x \in \overline{\Omega}$, where ρ is a positive constant. Along the uncontrolled boundary portion Γ_0, suppose that v has nonpositive outward g-normal derivative. If $0 \le v(x) \le K$ for all $x \in \overline{\Omega}$, then the hyperbolic PDE (2) is controllable from Γ_1 in time $T > T_0 := 2\sqrt{K/\rho}$.

The paper [3] provides another proof of Proposition 7 in the C^∞ case, assuming that no geodesic has infinite-order contact with $\partial\Omega$ (which is always true when $\partial\Omega$ has positive definite second fundamental form), by verifying the geometric-optics hypothesis of [1].

The example we shall treat in this section involves the Euclidean wave equation on a bounded domain $\Omega \subset I\!R^n$:

$$u_{tt} = \sum_{i=1}^{n} u_{x_i x_i} \tag{4}$$

with control from a proper subset $\Gamma_1 \subset \partial\Omega$. In this example, control in a finite time may be proved via the result of [1], although there is no convex function satisfying the hypotheses of Proposition 7.

We begin with a regular n-simplex $W \subset I\!R^n$, that is the convex hull of $n+1$ points $p_0, \dots p_n$ on the unit sphere ($|p_i| = 1$), at pairwise distance $\sqrt{2(n+1)/n}$. Choose $0 < a < 1/6$, and let V be obtained from W by removing all points at distance $< a$ from the vertices $p_i,\ 0 \leq i \leq n$; and all points at distance $< 2a$ from the edge midpoints $(p_i + p_j)/2,\ 0 \leq i < j \leq n$. The domain Ω will be a C^∞ approximation to V in the uniform norm: we require that the balls of radius $2a$ centered at the edge midpoints $(p_i + p_j)/2$ remain outside Ω, and that neighborhoods of the points $q_i := (1-a)p_i$ be unchanged. In particular, $\partial\Omega$ is concave in a neighborhood $S_i \subset \partial\Omega$ of q_i, that is, the line segment joining any two points of S_i lies outside Ω. Now replace each S_i by the intersection of S_i with the ball of radius $2a$ centered at p_i. We may now choose Γ_0 to be the union of $S_0, \dots, S_n$.

We claim that there is no C^2, strictly convex function $v : \overline{\Omega} \to I\!R$ satisfying $\partial v/\partial \vec{N} \leq 0$ on Γ_0, as required in the hypothesis of Proposition 7. Here $\vec{N}$ is the outward unit normal vector to $\partial\Omega$. For suppose there were such a function v. Then for $i = 0, \dots, n$, $f_i(t) := v(tq_i)$ satisfies $f_i''(t) > 0$, $0 \leq t \leq 1$, and $f_i'(1) \leq 0$. It follows that $f_i'(0) < 0$. But $f_i'(0)$ is the inner product of $\nabla v(0)$ with q_i. Since the vectors $q_0, \dots, q_n$ do not lie in any closed half-space of $I\!R^n$, we arrive at a contradiction.

According to [1], nonetheless, the wave equation (4) may be controlled from Γ_1 in time $T > T_0$, provided that every geodesic in $\overline{\Omega}$, continued after meeting Γ_0 by reflection with equal angles and in the same normal plane, reaches Γ_1 after distance at most T_0. In our example, a geodesic is a straight line segment σ. If $\sigma(t_0) \in \Gamma_0$, then in forward and backward time, the next meeting $\sigma(t_1)$ of σ with $\partial\Omega$ will be inside the controlled set Γ_1. Namely, $\Gamma_0 = S_0 \cup \cdots \cup S_n$ lies in the union of the balls $B_{2a}(p_i)$ of radius $2a$ centered at p_i, $0 \leq i \leq n$. If $\sigma(t_0) \in S_i \subset B_{2a}(p_i)$, then at the next time t_1 when $\sigma(t_1) \in \partial\Omega$, $\sigma(t_1)$ cannot lie in $B_{2a}(p_i)$ by the concavity of S_i; and $\sigma(t_1)$ cannot lie in $B_{2a}(p_j)$, $j \neq i$, since the line segment $\sigma([t_0, t_1])$ resp. $\sigma([t_1, t_0])$ would pass through $B_{2a}(\frac{p_i + p_j}{2})$, which lies outside of Ω by construction. This shows that a reflecting unit-speed geodesic $\sigma : [0, T] \to \overline{\Omega}$ travels a distance T at most equal to twice the diameter of Ω, before crossing Γ_1; hence, the time of control is at most

$$T_0 \leq 2 \ \mathrm{diam}_{I\!R^n} \Omega := \sup\{|x - y| : x, y \in \Omega\}.$$

Remark 4. A non-Euclidean example, where control is active on all of $\partial\Omega$ but there is no strictly convex function whatever, has been given by Galbraith [4]. In the case of the Euclidean wave equation (4) on a bounded domain $\Omega \subset I\!R^n$, in contrast, uniformly convex functions are always available, such as $v(x) = |x - x_0|^2$. The Euclidean example above is therefore possible only because of the uncontrolled boundary Γ_0.

5. Numerical Approximation to Geodesic Curvature Flows on Parametric Surfaces

We need to approximate numerically the evolution of a curve $C(t)$ on a surface $\Omega = \mathbf{x}(D) \subset I\!R^3$, $D \subset I\!R^2$, which is given parametrically as

$$\mathbf{x}(u^1, u^2) = (x^1(u^1, u^2), x^2(u^1, u^2), x^3(u^1, u^2)) \in I\!R^3. \tag{5}$$

Note that the notation used in this section, for a surface in $I\!R^3$, is slightly different from the previous sections. The curve is defined by the *level set function* $\phi(u^1, u^2, t)$ as

$$C(t) = \{\mathbf{x}(u^1, u^2) : \phi(u^1, u^2, t) = 0\} \tag{6}$$

and evolves according to the *geodesic curvature flow*,

$$\frac{\partial C}{\partial t} = k\vec{N} \tag{7}$$

where $k\vec{N}$ is the geodesic curvature vector. We think of the curve $C(0)$ as being given, and choose $\phi(\cdot, \cdot, 0)$, for example, close to the signed distance from $C(0)$. The evolving curve $C(t)$ depends only on $C(0)$, and not otherwise on the choice of $\phi(\cdot, \cdot, 0)$. Then we choose the level set function to satisfy

$$\phi_t = |\nabla\phi| \ \nabla \cdot \left(\frac{\nabla\phi}{|\nabla\phi|}\right) \tag{8}$$

where $\nabla\cdot$ and ∇ are the divergence and gradient operators on the surface S. Then (7) will be satisfied. More precisely, equation (8) is equivalent to the validity of (7) for each of the level sets of ϕ.

We are ready to reformulate the flow by curvature as an intrinsic problem in the geometry of the surface Ω (thus arriving at the context of the first four sections of this paper). In fact, the equation (8) can also be written in terms of the surface Ω itself (rather than $I\!R^3$) using the metric tensor

$$g_{\mu\nu} = \frac{\partial \mathbf{x}}{\partial u^\mu} \cdot \frac{\partial \mathbf{x}}{\partial u^\nu} \qquad g_{\mu\sigma} g^{\sigma\nu} = \delta_\mu^\nu \tag{9}$$

as

$$\phi_t = |\nabla\phi| \frac{1}{\sqrt{g}} \frac{\partial}{\partial u^\nu} \left(\frac{\sqrt{g}}{|\nabla\phi|} g^{\mu\nu} \frac{\partial\phi}{\partial u^\mu}\right) \tag{10}$$

where

$$|\nabla\phi|^2 = g^{\mu\nu} \frac{\partial\phi}{\partial u^\mu} \frac{\partial\phi}{\partial u^\nu} \tag{11}$$

and

$$g = det(g_{\mu\nu}) \neq 0. \tag{12}$$

Note that the summation convention is used in (10) and (11) with implicit summation $\mu, \nu = 1, 2$ over repeated indices. The degenerate parabolic PDE (10) exhibits diffusion only along of the level sets of the solution, so that the zero level set $C(t)$ of $\phi(\cdot,\cdot,t)$ evolves independently of the other level sets.

We compute a second order numerical solution to Eq. (10) with a finite difference scheme. A square domain domain of integration on the variables (u^1, u^2) is uniformly discretized with $N \times N$ gridpoints, where the values of the approximate solution ϕ_{ij} are stored, $1 \le i, j \le N$. The coordinate axis u^1, u^2 run at 45 degrees with respect to the grid as shown in Fig. 1. The temporal part of the integration is done with a simple second order Runge-Kutta scheme

$$\phi_{ij}\left(t+\frac{\delta}{2}\right) = \phi_{ij}(t) + \frac{\delta}{2}H_{ij}(t) \tag{13}$$

$$\phi_{ij}(t+\delta t) = \phi_{ij}(t) + \delta H_{ij}\left(t+\frac{\delta}{2}\right)$$

where H_{ij} is the discretization of the right hand side of Eq. (10) and δ is the time step. The components of the metric tensor are precomputed at the center of the square $(i,j), (i+1,j), (i+1,j+1), (i,j+1)$ of diagonal h (circles in Fig. 1); for this we compute the centered derivatives $\partial \mathbf{x}/\partial u^1 \approx (\mathbf{x}_{i+1,j+1} - \mathbf{x}_{i,j})/h$ and $\partial \mathbf{x}/\partial u^2 \approx (\mathbf{x}_{i,j+1} - \mathbf{x}_{i+1,j})/h$ and then compute $g^{\mu\nu}$ and g directly with Eqs. (9) and (12).

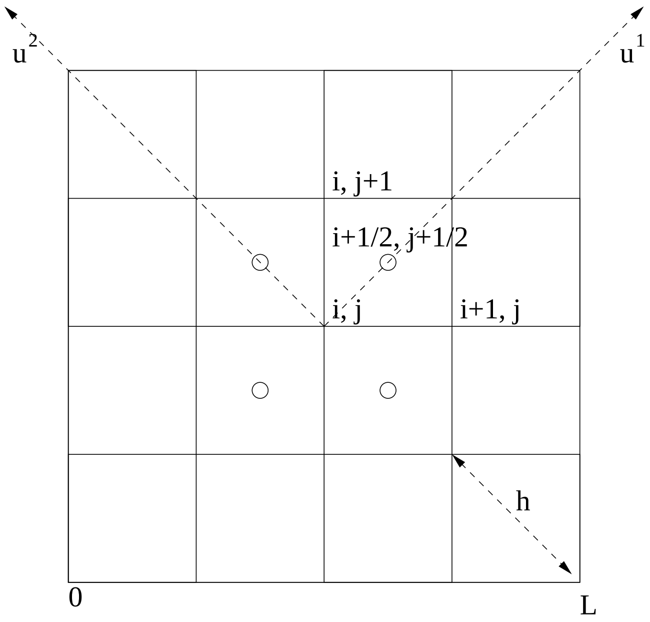

FIGURE 1. Numerical grid and discretization

During the time integration we compute $|\nabla\phi|$ at $(i+1/2, j+1/2)$ with Eq. (11) where we approximate the derivatives as $\partial\phi/\partial u^1 \approx (\phi_{i+1,j+1} - \phi_{i,j})/h$ and $\partial\phi/\partial u^2 \approx (\phi_{i,j+1} - \phi_{i+1,j})/h$. Then we compute the following quantities:

$$P^{\nu}_{i+1/2,j+1/2} = \frac{\sqrt{g}}{|\nabla\phi|} g^{\mu\nu} \frac{\partial\phi}{\partial u^{\mu}} \tag{14}$$

where we are using the summation convention on μ but not in i or j, and all the quantities on the right hand side are computed at the center $(i+1/2, j+1/2)$. Finally,

$$H_{ij} = |\nabla\phi|_c \frac{1}{\sqrt{g_c}} \frac{1}{h} \left(P^1_{i+1/2,j+1/2} - P^1_{i-1/2,j-1/2} + P^2_{i-1/2,j+1/2} - P^2_{i+1/2,j-1/2} \right) \tag{15}$$

where g_c is centered at (i,j) and is given by the averaged value on the four surrounding squares $4g_c = g_{i+1/2,j+1/2} + g_{i+1/2,j-1/2} + g_{i-1/2,j+1/2} + g_{i-1/2,j-1/2}$. A similar expression is used to compute $|\nabla\phi|_c$ at (i,j).

Here we use periodic boundary conditions on ϕ and on the surface S, however this is not essential in this numerical scheme provided the curve remains bounded inside the domain of integration during its evolution; this is ensured for these examples by the approximate vanishing of geodesic curvature of the boundary of the computational domain. For stability considerations the time step is constrained as $\delta < Ch^2$.

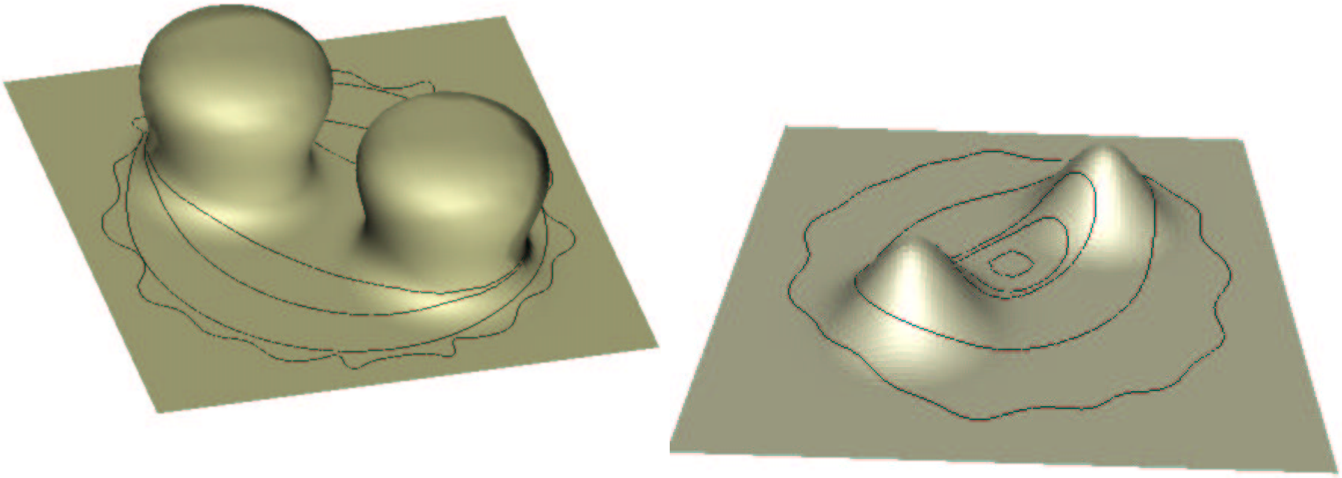

FIGURE 2. Evolution of a curve towards a closed geodesic (2A, left) and towards a point (2B, right)

In Fig. 2A we show the evolution of a curve surrounding a surface S that has a closed geodesic, in fact several. Asymptotically, the curve approaches the outermost geodesic. In Fig. 2B the surface does not have a closed geodesic, and the curve collapses into a small circle. In order to visualize the curves and the surfaces we used *Geomview*. The curves are computed as the union of the zero crossings of the linearly interpolated function $\phi(u^1, u^2)$ on each discretization square. If the curve collapses into a point, then the sign of the function ϕ becomes uniform on the domain of integration. If the curve converges to a geodesic then the length of the curve will converge to a finite value.

References

[1] Bardos, Claude, Gilles Lebeau and Jeffrey Rauch: Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. *SIAM J. Control Optim.* **30** (1992), 1024–1065.

[2] Chou, Kai-Seng and Xi-Ping Zhu: *The Curve Shortening Problem*, Chapman and Hall, 2001.

[3] Galbraith, Michael: A geometric-optics proof of a theorem on boundary control given a convex function. IMA Preprint # 1769, 2001.

[4] Galbraith, Michael: Geometric Optics and Convex Functions in the Boundary Control of the Wave Equation. Proceedings 3rd ISAAC Conference (Berlin, 2001).

[5] Grayson, M. A.: Shortening embedded curves, *Annals of Math.* **129** (1989), 71–111.

[6] Greene, Robert E. and Hung-hsi Wu: C^∞ convex functions and manifolds of positive curvature. *Acta Math.* **137** (1976), no. 3-4, 209–245.

[7] Gulliver, Robert and Walter Littman: Chord Uniqueness and controllability: the view from the boundary I. *Contemporary Mathematics* **268,** Differential Geometric Methods in the Control of Partial Differential Equations, AMS, 2000, pp. 145-175.

[8] Lasiecka, Irene, Roberto Triggiani and Peng-Fei Yao: Inverse/observability estimates for second-order hyperbolic equations with variable coefficients. *J. Math. Anal. Applications* **235** (1999), 13-57.

[9] Littman, Walter: Near-optimal-time boundary controllability for a class of hyperbolic equations. *Control Problems for Systems Described by Partial Differential Equations and Applications* (Gainesville, Fla., 1986), 307–312, *Lecture Notes in Control and Inform. Sci.* **97,** Springer, Berlin-New York, 1987.

[10] Littman, Walter: Remarks on global uniqueness theorems for partial differential equations. *Contemporary Mathematics* **26** (2000), 363–371.

[11] Ralston, James: Gaussian beams and the propagation of singularities. *Studies in Partial Differential Equations* (W. Littman, ed.), *MAA Studies in Mathematics* **23** (1982), 206–248.

[12] Yao, Peng-Fei: On the observability inequalities for exact controllability of wave equations with variable coefficients. *SIAM J. Control Optim.* **37** (1999), 1568–1599.

School of Mathematics & Institute for Mathematics and its Applications, University of Minnesota, Minneapolis, MN 55455, U.S.A

E-mail address: `betelu@ima.umn.edu gulliver@math.umn.edu littman@math.umn.edu`