

**SHARP GROWTH RATE FOR GENERALIZED SOLUTIONS
EVOLVING BY MEAN CURVATURE
PLUS A FORCING TERM**

ROBERT GULLIVER, YONGHOI KOO

ABSTRACT. When a hypersurface $\Sigma(t)$ evolves with normal velocity equal to its mean curvature plus a forcing term $g(x, t)$, the generalized (viscosity) solution may be “fattened” at some moment when $\Sigma(t)$ is singular. This phenomenon corresponds to nonuniqueness of codimension-one solutions. A specific type of geometric singularity occurs if $\Sigma(t)$ includes two smooth pieces, at the moment $t = 0$ when the two pieces touch each other. If each piece is strictly convex at that moment and at that point, then we show that fattening occurs at the rate $t^{1/3}$. That is, for small positive time, the generalized solution contains a ball of \mathbb{R}^n of radius $ct^{1/3}$, but its complement meets a ball of a larger radius $\kappa_0 t^{1/3}$. In this sense, the sharp rate of fattening of the generalized solution is characterized. We assume that the smooth evolution of the two pieces of $\Sigma(t)$, considered separately, do not cross each other for small positive time.

1. INTRODUCTION

Consider the problem of a hypersurface $\Sigma(t)$ in \mathbb{R}^n which flows in time with normal velocity given by its mean curvature plus, perhaps, a continuous forcing term $g(x, t)$. When singularities develop in this problem, the smooth solutions may cease to exist and the weak solutions may become nonunique. This has been observed in a number of recent papers; see [BSS] and [BP]. A weak solution as defined by Brakke [B], in particular, is not unique (see [I]). However, uniqueness holds for the *generalized solution* defined as follows. A real-valued function u on $\mathbb{R}^n \times [t_0, T]$ is constructed so that at the initial time t_0 , $u(\cdot, t_0)$ is positive on one side of the oriented initial hypersurface $\Sigma(t_0)$ and negative on the other side. $u(x, t)$ is then required to be continuous and to satisfy the degenerate parabolic partial differential equation

$$(1.1) \quad \frac{\partial u}{\partial t} = |\nabla u| \left(\operatorname{div} \frac{\nabla u}{|\nabla u|} - g(x, t) \right),$$

in the viscosity sense, with the the initial condition $u(x, t_0)$. We shall sketch the definition of “viscosity” solutions at the end of this section. The significance of this equation is as follows (see [ES]): if all level hypersurfaces of $u(x, t)$ were smooth, then each of the level sets $\{(x, t) \mid u(x, t) = \lambda\}$, for various real values of λ , would

evolve with normal velocity equal to its mean curvature plus $g(x, t)$. We write this motion of the level sets as $V = H + g(x, t)$. We assume here that the level sets are oriented in the direction of increasing u ; compare Proposition 2.1 below. The level set for $\lambda = 0$ is a closed subset of \mathbb{R}^n which evolves uniquely in time, and does not depend on the choice of the initial function $u(x, t_0)$. This solution is known as the *generalized solution* to the problem since it need not be smooth, need not have Hausdorff codimension one, and may even have a nonempty interior as a subset of \mathbb{R}^n . The phenomenon of an initially smooth hypersurface which later develops a nonempty interior is known as *fattening* or *ballooning*. This phenomenon occurs precisely when Brakke's weak solution is nonunique [I].

In 1994, Belletini and Paolini [BP] worked out some interesting examples of fattening in \mathbb{R}^2 which involved two circles meeting externally at a certain time $t = 0$. In 1999, the second author [K] extended the results of [BP] and showed that their examples were manifestations of a general principle, valid for hypersurfaces in \mathbb{R}^n evolving by mean curvature plus a forcing term, which guaranteed that the generalized solution begins to have positive Lebesgue measure as soon as two components $\Sigma^\pm(t)$ of an immersed solution touch from the outside at time $t = 0$, without crossing each other immediately before or after the critical time $t = 0$.

An examination of the proof in [K] shows that the size of the fat level set grows at least as fast as \sqrt{t} , *i.e.*, at the rate suggested by parabolic scaling. In the present paper, we shall show that in fact, the lower bound $c\sqrt{t}$ on growth of the fat level set may be replaced by the much faster growth $ct^{1/3}$ (Theorem 3.5 below). This improves the estimate of [K]. Moreover, this estimate is sharp. In fact, with the additional assumption of strict convexity at the touching point, Theorem 4.4 below shows that the region outside $\Sigma^\pm(t)$ and outside the fat level set is at a distance at most $\kappa_0 t^{1/3}$ from the touching point, for a larger constant κ_0 .

More precisely, combining Theorems 3.5 and 4.4 below, we have the

Theorem 1.1. *Let $\Sigma^\pm(t)$, $|t| < T$, be two smooth, oriented hypersurfaces of \mathbb{R}^n which move with normal velocity $V = H + g(x, t)$ for some continuous forcing term $g : \mathbb{R}^n \times \mathbb{R} \rightarrow [0, \infty)$. Suppose $\Sigma^+(t)$ and $\Sigma^-(t)$ are disjoint for $t \neq 0$ but that they meet at one point x_0 at time $t = 0$, at which each is strictly convex. Then there are c, κ_0 and $\delta > 0$ so that for all $0 < t < \delta$, the region outside $\Sigma^\pm(t)$ and outside the generalized solution $\Gamma(t)$ includes some points at distance $\kappa_0 t^{1/3}$ from x_0 but does not intersect the ball $B_{ct^{1/3}}(x_0)$.*

It should be observed that fattening of a specific level set cannot happen in most circumstances. More precisely, if $\Sigma_\lambda(t)$ is a disjoint one-parameter family of generalized solutions evolving according to the same function of curvature, then fattening does not occur for almost all λ . In fact, at any time t , the set of real numbers $\{\lambda \mid u(\cdot, t)^{-1}(\lambda) \text{ has positive measure}\}$ has measure zero in \mathbb{R} , by the additivity of

Lebesgue measure. This observation is consistent with Koo's principle, since Koo's result only applies to the first time fattening occurs, and requires touching to occur from the outside. Assuming the family $\Sigma_\lambda(t)$ is real-analytic, as might follow from parabolicity, the set of λ to which Koo's principle applies is discrete.

The intuition behind the distinction between [K] and the present paper may be understood in the following way. Koo's proof relies on comparison with a *self-similar* solution of the degenerate parabolic partial differential equation

$$(1.2) \quad \frac{\partial v}{\partial t} = |\nabla v| \left(\operatorname{div} \frac{\nabla v}{|\nabla v|} \right);$$

and the parabolic scaling $x \propto \sqrt{t}$ follows from parabolic self-similarity. However, *the spatial aspect of self-similarity is homothetic scaling*. Homothetic scaling is adapted to manifold-like geometries, such as Euclidean space, and more generally to cone-like geometries. In the problem considered by [BSS], [BP] and [K], however, the region exterior to $\Sigma^\pm(0)$ is not a cone but a sort of cusp. The region rescales in a small neighborhood of the touching point to small neighborhoods of a hyperplane. In particular, the homothetic scaling of [K] occurs independently of this cusp geometry and in a certain sense replaces it by a cone. This replacement of the region given in the problem by a larger and very different region might lead one to suspect that the $c\sqrt{t}$ estimate cannot be sharp. Thus, as Theorem 1.1 shows, for the analysis of behavior inside a cusp, *self-similar solutions are not enough*.

We conjecture that if the strict convexity of $\Sigma^\pm(0)$ in Theorem 1.1 is replaced by contact of order m , then the generalized solution $\Gamma(t)$ grows like $ct^{\frac{1}{m+1}}$.

An interface which moves by mean curvature plus a forcing term is a simple, although perhaps suggestive, model for solidification of isotropic materials. It would be of interest to understand the phenomena discussed in the present paper, and analogous phenomena, in the context of a more realistic system of equations incorporating temperature as a dependent variable along with one or more order parameters of the material. Anisotropic materials would also be of interest.

The definition of *viscosity solution* $u : \mathbb{R}^n \times [t_0, T] \rightarrow \mathbb{R}$ of (1.1) is well known; see e.g. pp. 753–754 of [CGG]. For completeness, we shall sketch the definition here. A continuous function u is a *viscosity subsolution* of (1.1) if, whenever a function ψ is smooth, $\psi \geq u$ and $\psi(x_0, t_0) = u(x_0, t_0)$ for some $(x_0, t_0) \in \mathbb{R}^n \times [t_0, T]$, then *at the point* (x_0, t_0) ,

$$\frac{\partial \psi}{\partial t} \leq |\nabla \psi| \left(\operatorname{div} \frac{\nabla \psi}{|\nabla \psi|} - g(x_0, t_0) \right).$$

The definition of viscosity supersolution is analogous, with both inequalities reversed. A continuous function u is called a *viscosity solution* of (1.1) if it is both a viscosity subsolution and a viscosity supersolution.

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2. LEVEL-SET FORMULATION OF HYPERSURFACE FLOW

In this section, we shall establish the relationship between the level-set equation (1.1) and the flow of its level sets by $V = H \pm g$. Smooth cases of Proposition (2.1) below have appeared frequently in the literature, *e.g.* in [ES].

For a function r of one space variable y and of time t , we write $r'(y, t) = \frac{\partial r}{\partial y}(y, t)$. For $x \in \mathbb{R}^n$, we will use the potentially confusing notation $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$. We trust that, in context, the reader will be able to distinguish this use of the notation $x' = (x_2, \dots, x_n)$ for a point $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ from the notation for the space derivative $r'(y, t)$ of a function $r(y, t)$ of two variables.

For convenience, in this section, the forcing term will be cylindrically symmetric: $g(x, t) = g(x_1, |x'|, t)$. When the following proposition is applied to our main results, an arbitrary continuous forcing term $g(x, t)$ will be estimated above by a function $g_{\max}(t)$ or below by 0.

Proposition 2.1. *Let $\{r_a(y, t)\}$ be a one-parameter family of viscosity subsolutions to*

$$(2.1) \quad \frac{\partial r_a}{\partial t} = \frac{r_a''}{1 + (r_a')^2} - \frac{n-2}{r_a} + g(y, r_a, t) \sqrt{1 + (r_a')^2}$$

satisfying $\frac{\partial}{\partial a} r_a(y, t) > 0$. Choose a continuous, locally monotone function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$, and let a function v be defined on \mathbb{R}^n by

$$v(x_1, x', t) := \varphi(a)$$

whenever $|x'| = r = r_a(y, t) = r_a(x_1, t)$. If $\varphi(a)$ is locally nondecreasing, then v is a viscosity supersolution of

$$(2.2) \quad \frac{\partial v}{\partial t} = |\nabla v| \left(\operatorname{div} \frac{\nabla v}{|\nabla v|} - g(x_1, |x'|, t) \right);$$

where $\varphi(a)$ is locally nonincreasing, v is a viscosity subsolution of

$$(2.3) \quad \frac{\partial v}{\partial t} = |\nabla v| \left(\operatorname{div} \frac{\nabla v}{|\nabla v|} + g(x_1, |x'|, t) \right).$$

If, instead, each r_a is a viscosity supersolution to equation (2.1), then v is a viscosity subsolution of equation (2.2) or a viscosity supersolution of equation (2.3), wherever $\varphi(a)$ is locally nondecreasing resp. locally nonincreasing.

Proof. Write $\Gamma_a(t)$ for the hypersurface obtained by rotating the graph $r = r_a(y, t)$ about the x_1 -axis, that is, $\Gamma_a(t) := \{(x_1, x') \mid |x'| = r_a(x_1, t)\}$, which is the $\varphi(a)$ -level

set of $v(\cdot, t)$. Then $\Gamma_a(t)$ has mean curvature in the direction of increasing r given by

$$H = \frac{r_a''}{[1 + (r_a')^2]^{3/2}} - \frac{n-2}{r_a \sqrt{1 + (r_a')^2}}$$

in the viscosity sense, and normal velocity

$$V = \frac{\frac{\partial r_a}{\partial t}}{\sqrt{1 + (r_a')^2}}.$$

Thus, the hypothesis that r_a is a subsolution of (2.1) is equivalent to the statement that $V \leq H + g(y, r_a, t)$, in the viscosity sense.

Observe that v is nondecreasing (resp. nonincreasing) in the direction of increasing $r = |x'|$, if φ is nondecreasing (resp. nonincreasing). Namely, for any unit vector $\omega = \frac{x'}{|x'|} \in S^{n-2}$, we have

$$v(y, r_a(y, t)\omega, t) = \varphi(a).$$

But $\frac{\partial}{\partial a} r_a(y, t) > 0$, and the composition of two monotone functions of one real variable is monotone in the sense consistent with the chain rule.

Consider first the case $\varphi(a) \equiv a$; note that $\frac{\partial v}{\partial r}$ exists and is positive in this case.

In order to verify that v is a viscosity supersolution to equation (1.1), let ψ be a C^2 test function, with $\psi(x_0, t_0) = v(x_0, t_0)$ for an arbitrary point (x_0, t_0) , and with $\psi \leq v$ in a neighborhood of (x_0, t_0) . Then $\nabla\psi(x_0, t_0)$ is a subdifferential for v at (x_0, t_0) , so $\frac{\partial\psi}{\partial r}(x_0, t_0) = \frac{\partial v}{\partial r}(x_0, t_0) > 0$. Let $\tilde{\Gamma}_a(t)$ be the a -level hypersurface of ψ . A straightforward application of the chain rule shows that $\tilde{\Gamma}_a(t)$ has normal velocity (in the direction of increasing r)

$$(2.4) \quad \tilde{V} = -\frac{\frac{\partial\psi}{\partial t}}{|\nabla\psi|}$$

and mean curvature (in the direction of increasing r)

$$(2.5) \quad \tilde{H} = -\operatorname{div} \frac{\nabla\psi}{|\nabla\psi|}.$$

Write $a_0 = \varphi(a_0) = v(x_0, t_0) = \psi(x_0, t_0)$. We have $x_0 \in \Gamma_{a_0}(t_0) \cap \tilde{\Gamma}_{a_0}(t_0)$. For each a near a_0 and for each x in the smooth hypersurface $\tilde{\Gamma}_a(t)$, with (x, t) near (x_0, t_0) , since $\psi \leq v$, we see that x lies above or on $\Gamma_a(t)$, where ‘‘above’’ means in the direction of increasing r . Since $\Gamma_a(t)$ is a viscosity subsolution of $V = H + g(y, r_a, t)$, treated as a PDE for $r = |x'|$ as a function of t and the remaining x variables, $\tilde{\Gamma}_{a_0}(t_0)$ satisfies $\tilde{V} \leq \tilde{H} + g(x_1, |x'|, t)$ at $x = x_0$. Equivalently, at (x_0, t_0) ,

$$\frac{\partial\psi}{\partial t} \geq |\nabla\psi| \left(\operatorname{div} \frac{\nabla\psi}{|\nabla\psi|} - g(x_1, |x'|, t) \right).$$

This shows that v is a viscosity supersolution of the equation (2.2) in this case.

The case when $\varphi(a) \equiv -a$ is similar, although $\frac{\partial v}{\partial r} < 0$, which forces a change of sign in equations (2.4) and (2.5) above. The resulting change of sign of g is seen in

the comparison of equations (2.2) and (2.3). Other cases follow by replacing v with $-v$.

The general case, with a continuous and locally monotone function φ , follows from the case $\varphi(a) \equiv a$ since (1.1) is a *geometric* PDE (compare Theorem 5.2, p. 772 of [CGG].) **Q.E.D.**

Remark 1. As may be seen from the proof above, a sufficient hypothesis on the family $\Gamma_a(t)$ is that the family is a transversely C^1 foliation and that each leaf is a viscosity subsolution (resp. supersolution) of $V = H + g(x_1, |x'|, t)$. It is convenient, but not necessary, to assume that the hypersurfaces $\Gamma_a(t)$ are obtained by rotation about an axis.

3. LOWER BOUND ON GROWTH OF THE LEVEL SET

In this section, we will demonstrate a lower bound for the size of the fattened level set at time t , of the form: if $|x| < ct^{1/3}$, $0 < t < \delta$ and x lies outside $\Sigma^+(t)$ and outside $\Sigma^-(t)$, then $u(x, t) = 0$ (Theorem 3.5.) Note that Lemmas 3.1 and 3.4 and Proposition 3.2 do not require $\Sigma^\pm(t)$ to evolve by a geometric flow, but only to be smooth.

Throughout this section, a positive number δ will be required repeatedly to be small enough, and will still be denoted δ by abuse of notation.

Lemma 3.1. *Let $\Sigma^\pm(t)$ be two smooth, oriented hypersurfaces of \mathbb{R}^n which evolve smoothly in time $t \in (-T, T)$. Suppose that $\Sigma^+(t) \cap \Sigma^-(t) = \emptyset$ for $t \neq 0$, that $\Sigma^+(0)$ and $\Sigma^-(0)$ meet externally at the origin O of \mathbb{R}^n (and possibly elsewhere), and that the coordinate hyperplane $x_1 = 0$ is the common tangent hyperplane to $\Sigma^+(0)$ and $\Sigma^-(0)$ at the origin. Then there are positive numbers b , δ and Δ , and a real number B , such that for all $-\delta \leq t \leq \delta$, the graphs*

$$x_1 = \pm b|x'|^2 \pm bt^2 + Bt, \quad |x'| \leq \Delta$$

lie inside or on $\Sigma^\pm(t)$, respectively.

Proof. Choose Δ and δ small enough that $\Sigma^\pm(t) \cap \{|x'| \leq \Delta\}$ is a graph over the hyperplane $x_1 = 0$ for all $-\delta < t < \delta$. Write $\Sigma^\pm(t)$ locally as $\pm x_1 = \varphi_\pm(x', t)$ for some smooth function φ_\pm on $B_\Delta^{n-1}(O) \times (-\delta, \delta)$.

Let B be the common velocity of $\Sigma^\pm(t)$ at $t = 0$, $x = O$ in the positive x_1 -direction, and let $2b$ be an upper bound on second directional derivatives of φ_\pm in the (x', t) variables on $B_\Delta^{n-1}(O) \times (-\delta, \delta)$. Then the only nonvanishing first derivative of φ_\pm at $(O, 0)$ is $\frac{\partial \varphi_\pm}{\partial t}(O, 0) = \pm B$. It follows from Taylor's theorem that

$$|\varphi_\pm(x', t) \mp Bt| \leq b|x'|^2 + bt^2$$

for $(x', t) \in B_\Delta^{n-1}(O) \times (-\delta, \delta)$.

Q.E.D.

As in Koo's paper [K], we shall construct a family of hypersurfaces of revolution which expand by mean curvature. In [K], this family was self-similar, and was constructed in [ACI] via the solution of an ordinary differential equation. In this paper, we will need to solve directly a parabolic partial differential equation in one space variable. This will be done in the following proposition, by constructing sub- and supersolutions satisfying the given boundary conditions and by deriving an *a-priori* gradient bound, with reference to well-known existence and regularity results.

Choose a positive constant a , and define two positive increasing functions of $t > 0$ by $\beta(t) := (t/a)^{1/3}$ and $\alpha(t) := 2a(\beta(t)^2 + t^2)$. Note that $0 < \frac{d\alpha}{dt} < \frac{d\beta}{dt}$ for all $0 < t < t_0(a)$, for some $t_0(a) > 0$.

Proposition 3.2. *Let $\Sigma^\pm(t)$ and the numbers b , B , and Δ be as in the statement of Lemma 3.1. Then for each $a \geq b$, there is $\delta > 0$ and a subsolution $r = r_a(y, t)$ of the initial-boundary value problem*

$$(3.1) \quad \frac{\partial r}{\partial t} = \frac{r''}{1 + (r')^2} - \frac{n-2}{r}, \quad |y - Bt| < \frac{\alpha(t)}{\sqrt{2}}, \quad 0 < t < \delta;$$

$$(3.2) \quad r\left(Bt \pm \frac{\alpha(t)}{\sqrt{2}}, t\right) = \beta(t) - \frac{\alpha(t)}{\sqrt{2}}, \quad 0 < t < \delta; \quad \text{and}$$

$$(3.3) \quad r(y, t) \rightarrow 0 \quad \text{uniformly for} \quad |y - Bt| < \frac{\alpha(t)}{\sqrt{2}} \quad \text{as} \quad t \rightarrow 0^+.$$

r_a is smooth on the closure of its domain, except at $(0, 0)$, where it is continuous. The hypersurfaces of revolution $|x'| = r_a(x_1, t)$ in \mathbb{R}^n generated by the graphs of the solutions r_a define a foliation by hypersurfaces $\Gamma_a(t)$ moving by mean curvature. The boundary of $\Gamma_a(t)$ has two components, one inside or on $\Sigma^+(t)$ and the other inside or on $\Sigma^-(t)$. Moreover, the distance $r_a(y, t)$ from any point of $\Gamma_a(t)$ to the x_1 -axis satisfies the uniform estimate

$$(3.4) \quad r_a(y, t) = (t/a)^{1/3} + O(t^{2/3})$$

as $t \rightarrow 0^+$.

Proof. We shall first construct a subsolution $\tilde{r}(y, t)$ and a supersolution $\hat{r}(y, t)$ to (3.1) on the moving domain $|y - Bt| < \alpha(t)/\sqrt{2}$, both satisfying the boundary conditions (3.2). We shall construct the graph of $\tilde{r}(\cdot, t)$ as the lower quarter-circle of increasing radius $\alpha(t)$ and center $(y, r) = (Bt, \beta(t))$; $\hat{r}(\cdot, t)$ will describe the chord joining its endpoints.

For convenience, we may introduce the system of moving coordinates (x, t) , where $x := y - Bt$. Then equation (3.1) is equivalent to

$$(3.5) \quad \frac{\partial r}{\partial t} = Br' + \frac{r''}{1 + (r')^2} - \frac{n-2}{r}, \quad |x| < \frac{\alpha(t)}{\sqrt{2}}, \quad 0 < t < \delta,$$

since $r' = \frac{\partial r}{\partial y} = \frac{\partial r}{\partial x}$.

Let $\delta > 0$ be as in the conclusion of Lemma 3.1. By abuse of notation, we shall replace δ by smaller positive values as needed, which will still be called δ . In particular, we may assume that $\delta \leq t_0(a)$. Further, since the leading term of $\alpha(t)$ is $2a^{1/3}t^{2/3} = 2a\beta^2$, we may choose δ small enough that

$$(H1) \quad \frac{1 - 4a\beta}{3a\beta^2} - 4at + |B| \leq \frac{1}{\alpha} - \frac{n-2}{\beta-\alpha},$$

for all $0 < t < \delta$. Hypothesis (H1) and the computation $\frac{d\beta}{dt} = \frac{1}{3a\beta^2}$ imply that

$$(3.6) \quad \frac{d\beta}{dt} - \frac{d\alpha}{dt} + |B| \leq \frac{1}{\alpha} - \frac{n-2}{\beta-\alpha}.$$

This last inequality shows that the moving quarter-circle

$$(3.7) \quad \tilde{r}(x, t) := \beta(t) - \sqrt{\alpha(t)^2 - x^2}, \quad |x| \leq \frac{\alpha(t)}{\sqrt{2}},$$

satisfies

$$(3.8) \quad \frac{\partial \tilde{r}}{\partial t} \leq \frac{\tilde{r}''}{1 + (\tilde{r}')^2} - \frac{n-2}{\tilde{r}},$$

since $\sqrt{1 + (\tilde{r}')^2} \geq 1$ and $\tilde{r} \geq \beta(t) - \alpha(t)$. That is, $\tilde{r}(x, t)$ is a subsolution of equation (3.5) on the domain $\left\{ (x, t) \mid 0 < t < \delta, |x| < \frac{\alpha(t)}{\sqrt{2}} \right\}$.

Let us now require δ to be small enough that

$$(H2) \quad 3\alpha(t)\sqrt{2} \leq \beta(t)$$

for all $0 < t < \delta$. Then we may also construct a supersolution $\hat{r}(x, t) = \hat{r}(t)$ of (3.5) on the interval $|x| \leq \frac{\alpha(t)}{\sqrt{2}}$, satisfying the same boundary conditions (3.2) as $r_a(x, t)$, by defining

$$\hat{r}(x, t) := \beta(t) - \frac{\alpha(t)}{\sqrt{2}}$$

for all x in the interval $\left[-\frac{\alpha(t)}{\sqrt{2}}, \frac{\alpha(t)}{\sqrt{2}}\right]$. In fact, since $\hat{r}'' = 0$, we only need to show that $\frac{\partial \hat{r}}{\partial t} \geq 0$. But $\frac{d\beta}{dt} = \frac{1}{3a\beta^2}$, while $\frac{d\alpha}{dt} = 4a \left(\beta \frac{\partial \beta}{\partial t} + t \right)$, so \hat{r} is nondecreasing if and only if

$$2\sqrt{2}a\beta^2 + 6\sqrt{2}a^2\beta^3t \leq \beta,$$

which is a consequence of (H2), since the left-hand side is $< 3\sqrt{2}\alpha(t)$.

We shall need one last hypothesis regarding δ : for a given $0 \leq \eta < \pi/12$, we will assume that

$$(H\eta) \quad (n-2) \frac{3\alpha^2}{2\pi(\beta-\alpha)^2} c_\eta + (1 + c_\eta^2) |B| \frac{\alpha}{\sqrt{2}} < 1,$$

for all $0 < t < \delta$. Here we define $c_\eta := \tan(5\pi/12 + \eta) < \infty$. Note that hypothesis (H0) is the special case of (H η) with $\eta = 0$. Also, note that hypothesis (H0) implies (H η) for sufficiently small $\eta > 0$.

We shall return to the proof of Proposition 3.2 after establishing the following existence statement for a modified equation:

Lemma 3.3. *Assume that hypotheses (H1), (H2) and (H0) hold. Fix $\eta > 0$ such that (H η) is valid. Choose $\varepsilon > 0$, $p_1 > c_\eta$ and $0 < z_0 < \beta(\varepsilon) - \alpha(\varepsilon)$. Then there is a solution $r(x, t)$ to the initial-boundary value problem*

$$(3.9) \quad \frac{\partial r}{\partial t} = A^{11}(r')r'' + A(r, r'), \quad \varepsilon < t < \delta, \quad |x| < \frac{\alpha(t)}{\sqrt{2}}$$

with boundary conditions

$$(3.10) \quad r\left(\pm \frac{\alpha(t)}{\sqrt{2}}, t\right) = \hat{r}(t), \quad \varepsilon \leq t \leq \delta$$

and initial condition

$$(3.11) \quad r(x, \varepsilon) = \hat{r}(\varepsilon), \quad |x| \leq \frac{\alpha(\varepsilon)}{\sqrt{2}}.$$

Here, the coefficients of the modified equation (3.9) are defined by

$$A^{11}(p) := (1 + p^2)^{-1}, \quad p \leq c_\eta, \quad A^{11}(p) := (1 + p_1^2)^{-1}, \quad p \geq p_1,$$

and smooth for $c_\eta \leq p \leq p_1$; and by

$$A(z, p) := -\frac{n-2}{z \vee z_0} + Bp.$$

Moreover, $r(x, t)$ satisfies the a-priori gradient bound

$$(3.12) \quad |r'(x, t)| \leq c_0,$$

and the upper and lower bounds

$$(3.13) \quad \tilde{r}(x, t) \leq r(x, t) \leq \hat{r}(x, t)$$

on its domain.

Remark 2. Note that if the estimates (3.12) and (3.13) hold, then the modified equation (3.9) is equivalent to (3.5). Namely, since $\delta < t_0(a)$, the minimum value of $\tilde{r}(x, t)$ for $\varepsilon < t < \delta$ is $\tilde{r}(0, \varepsilon) = \beta(\varepsilon) - \alpha(\varepsilon)$.

Proof. The modified equation (3.9) is uniformly parabolic, and the existence theory for such equations, in a domain such as $\varepsilon < t < \delta$, $|x| < \alpha(t)/\sqrt{2}$ is well known ([F], [LSU], [L]). Specifically, the existence of a unique solution, for which r'' and $\frac{\partial r}{\partial t}$ are locally Hölder continuous in the interior follows from Theorem 12.16 of [L]. The Hölder continuity of r' at boundary points $x = \pm\alpha(t)/\sqrt{2}$ may be seen from Theorem 12.5 of [L]. Smoothness of $r(x, t)$ near the initial line $t = \varepsilon$ follows from a standard reflection technique and Theorem 12.16 of [L].

It remains to prove the estimates (3.12) and (3.13). For this purpose, consider the weaker inequalities

$$(3.14) \quad |r'(x, t)| \leq c_\eta,$$

and

$$(3.15) \quad z_0 \leq r(x, t).$$

Observe that the estimates (3.12) and (3.13) will be valid for $r(x, t)$ on a short time interval $\varepsilon < t < t_1$. Namely, the initial condition (3.11) implies $r' = 0$ at the initial line $t = \varepsilon$, hence (3.12) holds on a one-sided neighborhood. Since $r'' = 0$ on the initial line, and since \hat{r} is a strict supersolution, we have $\frac{\partial r}{\partial t} < \frac{\partial \hat{r}}{\partial t}$ on the initial line and hence $r < \hat{r}$ for a short time after ε . This shows that (3.13) holds for a short time after ε . Let $t_1 \in (\varepsilon, \delta]$ be the largest number so that (3.12) and (3.13) are valid for $\varepsilon < t \leq t_1$; and let $t_2 \in [t_1, \delta]$ be the largest number so that (3.14) and (3.15) are valid for $\varepsilon < t \leq t_2$. We need to show that $t_1 = \delta$; we shall show that otherwise, t_2 must be both greater than and equal to t_1 .

Since $z_0 < \beta(t) - \alpha(t) \leq \tilde{r}(x, t)$ and $|\tilde{r}'(x, t)| \leq 1 < c_\eta$ for $\varepsilon \leq t \leq t_2$, \tilde{r} is also a subsolution of the modified equation (3.9). Therefore, by the comparison principle, $r(x, t) \geq \tilde{r}(x, t) \geq \beta(t) - \alpha(t)$ for all $\varepsilon < t \leq t_2$, $|x| \leq \alpha(t)/\sqrt{2}$. Similarly, $r(x, t) \leq \hat{r}(t)$. That is, inequality (3.13) holds for all $\varepsilon < t \leq t_2$.

The gradient bound (3.12) requires more work. Let $\theta(x, t) \in (-\pi/2, \pi/2)$ be defined by $\tan \theta(x, t) := r'(x, t)$. Then θ satisfies the equation

$$(3.16) \quad \sec^2 \theta \left[\frac{\partial \theta}{\partial t} - B\theta' \right] - \theta'' = \frac{n-2}{r^2} \tan \theta,$$

wherever (3.14) and (3.15) are satisfied, in particular for $\varepsilon \leq t \leq t_2$. (When $r < z_0$ or when $|r'| > c_\eta$, the equation is more complicated, and will not be needed.) As we have seen, r' and therefore θ are Hölder continuous up to the boundary $x = \pm\alpha(t)/\sqrt{2}$. It may be seen from inequality (3.13) that at the boundary, $|r'(\pm\alpha(t)/\sqrt{2}, t)| \leq 1$, that is, $|\theta(\pm\alpha(t)/\sqrt{2}, t)| \leq \pi/4$.

We introduce a corrector function

$$\phi(x, t) := \frac{\pi}{3\alpha(t)^2} \left(\frac{\alpha(t)^2}{2} - x^2 \right).$$

Then $0 \leq \phi(x, t) \leq \pi/6$ on the domain of $r(x, t)$, while $\phi(\pm\alpha(t)/\sqrt{2}, t) = 0$. We shall show that $Q(x, t) := |\theta(x, t)| - \phi(x, t)$ satisfies a maximum principle, from which it will follow that $|\theta| \leq \phi + \pi/4 \leq 5\pi/12$, since $|\theta| \leq \pi/4$ and $\phi = 0$ on the boundary $|x| = \alpha(t)/\sqrt{2}$, while $\theta = 0$ and $0 \leq \phi \leq \pi/6$ on the initial line $t = \varepsilon$, $|x| \leq \alpha(t)/\sqrt{2}$. In particular, it will follow that (3.12) holds for $\varepsilon < t < t_2$.

Consider the first time when $Q = |\theta| - \phi$ reaches the value $\pi/4 + \eta$, which could only happen at an interior point. Then we have $|\theta| \leq 5\pi/12 + \eta$ there. We may

therefore compute that, at that point,

$$\begin{aligned} \frac{3\alpha^2}{2\pi} \left[\sec^2 \theta \left(\frac{\partial \phi}{\partial t} - B\phi' \right) - \phi'' \right] &= \sec^2 \theta \left(\frac{x^2}{\alpha} \frac{d\alpha}{dt} + Bx \right) + 1 \\ &\geq -(1 + c_\eta^2) |B| \frac{\alpha}{\sqrt{2}} + 1 \\ &> (n-2) \frac{3\alpha^2}{2\pi(\beta-\alpha)^2} c_\eta \end{aligned}$$

according to hypothesis (H η). Meanwhile, we have seen that $r(x, t) \geq \beta(t) - \alpha(t)$, which implies that the right-hand side of (3.16) is in absolute value at most $\frac{n-2}{(\beta-\alpha)^2} c_\eta$. Therefore there can be no interior point where a local first maximum value $\pi/4 + \eta$ occurs for $Q = \pm\theta - \phi$. This shows that $|\theta| < \pi/6 + \pi/4 + \eta$, and hence $|r'| < c_\eta$; therefore the gradient estimate (3.14) is valid on $\varepsilon < t < t_2$, $|x| \leq \alpha(t)/\sqrt{2}$. But the same argument holds for all smaller values of $\eta > 0$; this implies that, in fact, inequality (3.12) holds for $\varepsilon < t < t_2$.

We have just shown that in fact, $t_2 = t_1$. Now if $t_1 < \delta$, since t_1 is defined by the inequalities (3.12) and (3.13), then by continuity the weaker inequalities (3.14) and (3.15) (with our original, fixed, $\eta > 0$) would continue to hold for a short time after t_1 , that is to say $t_2 > t_1$. We conclude that $t_1 = \delta$, which means that the estimates (3.12) and (3.13) hold for all $\varepsilon \leq t \leq \delta$, $|x| \leq \alpha(t)/\sqrt{2}$. **Q.E.D.**

Proof of Proposition 3.2, cont. Write $r^{(\varepsilon)}(x, t)$ for the solution $r(x, t)$ of the modified equation (3.9) satisfying initial conditions (3.11) on the line $t = \varepsilon$, as given in the conclusion of Lemma 3.3. Then $r^{(\varepsilon)}$ is also a solution of equation (3.5), because of the estimates (3.12) and (3.13). For $0 < \varepsilon_0 < \varepsilon_1$, inequality (3.13) implies that $r^{(\varepsilon_0)}(x, \varepsilon_1) < \hat{r}(\varepsilon_1)$, which are the initial data for $r^{(\varepsilon_1)}$; meanwhile, $r^{(\varepsilon_0)}$ and $r^{(\varepsilon_1)}$ share the same boundary data. Therefore, by the strong maximum principle, $r^{(\varepsilon_0)} < r^{(\varepsilon_1)}$ on the domain of $r^{(\varepsilon_1)}$. That is, the solutions $r^{(\varepsilon)}(x, t)$ are increasing as a function of ε . As $\varepsilon \rightarrow 0$, we therefore have pointwise monotone convergence of $r^{(\varepsilon)}$ to some function $r^{(0)}$ on the domain $0 \leq t \leq \delta$, $|x| \leq \alpha(t)/\sqrt{2}$. The convergence is smooth, implying that $r^{(0)}$ satisfies the gradient bound (3.12), except at $(0, 0)$. Similarly, $r^{(0)}$ satisfies the inequality (3.13), and it follows that $r^{(0)}$ is continuous at $(0, 0)$, since both \tilde{r} and \hat{r} converge to zero there.

We now write $r_a(x, t)$ in place of $r^{(0)}(x, t)$, and $r_a^{(\varepsilon)}(x, t)$ in place of $r^{(\varepsilon)}(x, t)$, where $a \geq b$ is the parameter which was used to define $\beta(t)$ and $\alpha(t)$.

Returning to the original (y, t) coordinates, the foliation property of the family $r_a(y, t)$ of solutions to equation (3.1) may be seen by showing that $q(y, t) := \frac{\partial r_a}{\partial a}(y, t)$ is negative everywhere in its domain $0 < t \leq \delta$, $|y - Bt| \leq \alpha(t)/\sqrt{2}$. In fact, q satisfies

the homogeneous, uniformly parabolic partial differential equation

$$(3.17) \quad \frac{\partial q}{\partial t} = \frac{q''}{1 + (r')^2} - \frac{2r'r''q'}{[1 + (r')^2]^2} + \frac{n-2}{r^2} q$$

on its domain. Its boundary values are given by $\hat{q}(t) := \frac{\partial \hat{f}}{\partial a}(t) = -(\beta\sqrt{2} + 2a\beta^2 + 6at^2)/3a\sqrt{2} < 0$, as follows from the definitions of $\beta(t)$ and $\alpha(t)$. But $\hat{q}(t)$ are also the initial and boundary data for $q^{(\varepsilon)}(y, t) := \frac{\partial r_a^{(\varepsilon)}}{\partial a}(y, t)$, which also satisfies (3.17). It follows from the maximum principle that $q^{(\varepsilon)}(y, t) < 0$ for all $\varepsilon \leq t \leq \delta$, $|y - Bt| \leq \alpha(t)/\sqrt{2}$. Therefore $q(y, t) \leq 0$; moreover by the strong maximum principle $q(y, t) < 0$ for $t > 0$, since it has negative boundary values $\hat{q}(t)$.

Finally, at boundary points of the surface of revolution $\Gamma_a(t)$ generated by the graph of $r_a(y, t)$, we have $x_1 = Bt \pm \frac{\alpha(t)}{\sqrt{2}}$ and $|x'| = \beta(t) - \frac{\alpha(t)}{\sqrt{2}} < \beta(t)$. Since $a \geq b$, it follows that $\pm(x_1 - Bt) \geq b(|x'|^2 + t^2)$ and thus from Lemma 3.1 that these points lie inside or on $\Sigma^\pm(t)$, respectively. **Q.E.D.**

We are now ready to construct a solution v of the homogeneous equation (3.18) below, whose level sets will be formed by the family, just established, of hypersurfaces $\Gamma_a(t)$ moving by mean curvature.

In the remainder of this paper, we shall write $\Omega(t)$ for the open set in \mathbb{R}^n lying outside of $\Sigma^+(t)$ and of $\Sigma^-(t)$.

Lemma 3.4. *Let $\Sigma^\pm(t)$ and $b > 0$ be as in Lemma 3.1. Let $\delta > 0$ and the foliation $\{\Gamma_a(t) \mid a > b, 0 < t < \delta\}$ be as in Proposition 3.2. Choose $K > 0$ and $a_1 > a_0 > b$. Then there is a smooth real-valued function $v(x, t)$, defined for $x \in \Omega(t)$ and $t \in (0, \delta)$ satisfying*

$$(3.18) \quad \frac{\partial v}{\partial t} = |\nabla v| \operatorname{div} \left(\frac{\nabla v}{|\nabla v|} \right),$$

such that $v \geq 0$; $v(x, t) = K$ unless $x \in \Gamma_a(t)$ for some $a \in (b, a_1)$; and so that $v(x, t) = 0$ for all $x \in \Gamma_{a_0}(t)$.

Proof. According to Proposition 3.2, each hypersurface of revolution $\Gamma_a(t)$ moves by mean curvature and has two boundary components, one inside or on $\Sigma^+(t)$ and the other inside or on $\Sigma^-(t)$. Choose a smooth, locally monotone function $\varphi : \mathbb{R} \rightarrow [0, K]$ such that $\varphi(a) = K$ for all $a \leq b$ and all $a \geq a_1$; and such that $\varphi(a) = 0$ for all a in a neighborhood of a_0 . Define $v(x, t) := \varphi(a)$ if $x \in \Gamma_a(t)$ for some $a \in (b, a_1)$, and otherwise $v(x, t) := K$. Then v vanishes identically on the evolving hypersurface $\Gamma_{a_0}(t)$. Moreover, according to Proposition 2.1, v satisfies the partial differential equation (3.18) on the set

$$\bigcup_{0 < t < \delta} \Omega(t) \times \{t\} \subset \mathbb{R}^n \times \mathbb{R}.$$

Q.E.D.

Theorem 3.5. *Let $\Sigma^\pm(t)$ be two smooth, oriented hypersurfaces in \mathbb{R}^n which evolve according to*

$$(3.19) \quad V = H + g(x, t)$$

for some nonnegative continuous forcing term $g(x, t)$. Suppose that $\Sigma^+(t) \cap \Sigma^-(t) = \emptyset$ for $t \neq 0$, $-T \leq t < T$, and that there is a point $x_0 \in \Sigma^+(0) \cap \Sigma^-(0)$. Let $u(\cdot, t)^{-1}(0)$ be the generalized solution to (3.19) with initial condition $u(\cdot, -T)^{-1}(0) = \Sigma^+(-T) \cup \Sigma^-(-T)$. That is, $u(x, t)$ satisfies the equation

$$(3.20) \quad \frac{\partial u}{\partial t} = |\nabla u| \left(\operatorname{div} \left(\frac{\nabla u}{|\nabla u|} \right) - g(x, t) \right)$$

and $u(x, -T)$ vanishes iff $x \in \Sigma^\pm(-T)$. Then there exists $\delta > 0$ such that for all $0 < t < \delta$, the generalized solution $u(\cdot, t)^{-1}(0)$ has nonempty interior. Moreover, there is $c > 0$ so that for all $0 < t < \delta$, $u(x, t)$ vanishes whenever

$$(3.21) \quad x \in \Omega(t) \cap B_{ct^{1/3}}(x_0).$$

Remark 3. It was shown by Koo [K] that under the hypotheses of Theorem 3.5, fattening of the zero level set occurs immediately after contact. An examination of Koo's proof, for example, formula (3.16) of [K], shows that the size of the level set after time t is at least $\operatorname{const} \cdot t^{1/2}$. Thus, the main interest in Theorem 3.5 is the more rapid rate of growth (3.21).

Remark 4. It follows from Theorem 4.4 below that the exponent $\frac{1}{3}$ is sharp. However, the upper bound $\kappa_0 t^{1/3}$ of that theorem appears to have a constant κ_0 which is much larger than the best constant. We expect that the sharp constant might be $\kappa_0 = b^{-1/3}$, where $2b$ is an lower bound for the inward principal curvatures of $\Sigma^\pm(0)$. (We do not expect the convexity hypothesis of Theorem 4.4 to be necessary.) Any constant $c < b^{-1/3}$, where $2b$ is an upper bound for the principal curvatures of $\Sigma^\pm(0)$, is valid for Theorem 3.5.

Proof. As in [K], our proof will be based on the function $v(x, t)$ given in Lemma 3.4, whose level sets are generalized solutions for flow by mean curvature. Since we have assumed $g(x, t) \geq 0$, v is a supersolution of (3.20). Assume for simplicity $x_0 = O \in \mathbb{R}^n$. Since the PDE (3.20) is geometric, we may assume without loss of generality that u is uniformly bounded: $|u(x, t)| \leq K$ for all $x \in \mathbb{R}^n$, $t \in [-T, T]$. In fact, the conclusion refers only to the zero level set of u , which is unchanged if $u(x, t)$ is replaced by the bounded function $\tanh u(x, t)$. For similar reasons, we may assume $u(x, 0) > 0$ on $\Omega(0)$ and $u(x, 0) < 0$ for x inside $\Sigma^\pm(0)$. This function remains a viscosity solution of (3.20); see Theorem 5.6 of [CGG]. Let $\delta > 0$ be as in the conclusion of Lemma 3.4. Write Ω for the open set $\{(x, t) \mid x \in \Omega(t), 0 < t < \delta\}$ in $\mathbb{R}^n \times \mathbb{R}$. Then v is continuous on $\bar{\Omega}$ except at time $t = 0$; when $t = 0$, we have $v(x, 0) = K$ for $x \neq O$ and the lower semi-continuous envelope $v_*(O, 0) = 0$.

In particular, $v_*(x, 0) \geq u(x, 0)$ on $\overline{\Omega(0)}$. Further, $u(x, t) = 0$ for all $x \in \partial\Omega(t)$, so $v_* \geq u$ on the parabolic boundary $\{(x, t) \in \partial\Omega \mid 0 \leq t < \delta\}$. It follows from the comparison principle that $v \geq u$ everywhere on $\overline{\Omega}$ (see [GGIS], p. 463). In particular, $u(x, t) = 0$ for all $x \in \Gamma_{a_0}(t)$, $0 < t < \delta$.

Let $D(t)$ be the bounded open set in \mathbb{R}^n whose boundary consists of portions of $\Sigma^+(t)$, $\Sigma^-(t)$ and the surface of revolution $\Gamma_{a_0}(t)$, for each $0 < t < \delta$. Write $D = \{(x, t) \mid x \in D(t), 0 < t < \delta\}$. Then u vanishes identically on the parabolic boundary $\{(x, t) \in \partial D \mid 0 \leq t < \delta\}$. Applying the comparison principle on D , we see that $u \equiv 0$ on D . Finally, estimate (3.4) implies that $D(t)$ contains all $x \in \Omega(t)$ with $|x| < (t/a_0)^{1/3} + O(t^{2/3})$, and conclusion (3.21) follows. **Q.E.D.**

4. UPPER BOUND ON GROWTH OF THE LEVEL SET

In this section, we will demonstrate an upper bound for the size of the fattened level set at time t , of the form: if $|x'| > \kappa_0 t^{1/3}$ then $u(0, x', t) > 0$. (Theorem 4.4 below.) We would like to point out some differences between this section and section 3 above, in addition to the obvious change in direction of the inequality we wish to prove. In section 3, it was necessary to find both a subsolution and a supersolution, as barriers; these were required to have their boundaries inside $\Sigma^\pm(t)$, in order to sweep out the region $\Omega(t)$ outside. In this section, a two-parameter family of supersolutions (only) will be needed. However, the supersolution must lie entirely outside $\Sigma^\pm(t)$, in such a way that every nonzero point of the intersection of $\Omega(t)$ with the hyperplane $x_1 = 0$ is in one of the supersolutions of the two-parameter family. For this purpose, the simple geometric constructions (quarter-circle and horizontal line segment) which were sufficient for section 3 must be replaced by the well-known Grim Reaper, extended by two of its tangent lines. Extending the Grim Reaper by its tangent lines serves to overcome the effects of the forcing term.

Note that Lemmas 4.1–4.3 do not require $\Sigma^\pm(t)$ to evolve by a geometric flow, but only to be smooth.

Throughout this section, as in Section 3 above, a positive number δ will be required repeatedly to be small enough, and will still be denoted δ by abuse of notation. By further abuse of notation, the proof of Theorem 1.1 requires δ to be smaller than the last version of the number δ of this section, and also smaller than the last version of the number δ of section 3. Recall that for $x \in \mathbb{R}^n$, we write $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$. We also assume that $g(x, t)$ is a continuous function defined on $\mathbb{R}^n \times \mathbb{R}$ throughout this section.

We shall use methods analogous to the proof of Lemma 3.1 to show

Lemma 4.1. *Let $\Sigma^\pm(t)$ be two smooth, oriented hypersurfaces of \mathbb{R}^n which evolve smoothly in time $t \in (-T, T)$. Suppose that $\Sigma^+(t) \cap \Sigma^-(t) = \emptyset$ for $t \neq 0$, that $\Sigma^+(0)$ and $\Sigma^-(0)$ meet externally at the origin O of \mathbb{R}^n (and possibly elsewhere),*

and that the coordinate hyperplane $x_1 = 0$ is the common tangent hyperplane to $\Sigma^+(0)$ and $\Sigma^-(0)$ at the origin. Moreover, assume that $\Sigma^\pm(t)$ are strictly convex at $x = O$, $t = 0$. Then there are positive numbers b , b' , δ and Δ , and a real number B , such that for all $-\delta \leq t \leq \delta$, the graphs

$$x_1 = \pm b|x'|^2 \mp b't^2 + Bt, \quad |x'| \leq \Delta$$

lie outside or on $\Sigma^\pm(t)$, respectively.

Note that the constant B which appears in the conclusion of Lemma 4.1 is the same constant which appears in Lemma 3.1. However, the constants written as b in the two lemmas are different: one is roughly an upper bound, the other a lower bound, for principal curvatures of $\Sigma^\pm(0)$.

Proof. Choose Δ and δ small enough that $\Sigma^\pm(t) \cap \{|x'| \leq \Delta\}$ is strictly convex, and is a graph over the hyperplane $x_1 = 0$ for all $-\delta < t < \delta$. Write $\Sigma^\pm(t)$ locally as $\pm x_1 = \varphi_\pm(x', t)$ for some smooth function φ_\pm on $B_\Delta^{n-1}(O) \times (-\delta, \delta)$.

Let B be the common velocity of $\Sigma^\pm(t)$ at $t = 0$, $x = O$ in the positive x_1 -direction; let $4b$ be a positive lower bound on second directional derivatives in the x' -variables; and let $2\tilde{b}$ be an upper bound on the absolute value of its second directional derivatives in the (x', t) -variables on $B_\Delta^{n-1}(O) \times (-\delta, \delta)$. Then the only nonvanishing first derivative of φ_\pm at $(O, 0)$ is $\frac{\partial \varphi_\pm}{\partial t}(O, 0) = \pm B$. It follows from Taylor's theorem that

$$|\varphi_\pm(x', t) \mp Bt| \geq b|x'|^2 - b't^2$$

for $(x', t) \in B_\Delta^{n-1}(O) \times (-\delta, \delta)$. Here we may choose $b' := \tilde{b} + 4\tilde{b}^2/b$. The computation is based on Schwartz' inequality with appropriate weights, with respect to the positive semi-definite matrix $D^2\varphi_\pm + 2\tilde{b}I$. **Q.E.D.**

We are now ready to construct the two-parameter family of supersolutions $\Gamma_{a,k}(t)$ which comprise the main tool for the results of this section. Note that the parameter $k \geq 0$ will play the role of the parameter a in Proposition 2.1.

Lemma 4.2. *Let $\Sigma^\pm(t)$ and the numbers b , B , δ , and Δ be as in the statement of Lemma 4.1. Let us also define a continuous function*

$$g_{\max}(t) = \max \left\{ 0, \max \{ g(x, t) \mid |x| \leq \Delta \} \right\}.$$

Then for each $a \in (0, \min\{2(\pi^3 b)^{-1}, 1\})$, there is a positive continuous viscosity supersolution $r = h_a(y, t)$ of the equation

$$(4.1) \quad \frac{\partial h_a}{\partial t} = \frac{h_a''}{1 + (h_a')^2} - \frac{n-2}{h_a} + g_{\max}(t) \sqrt{1 + (h_a')^2}, \quad y \in \mathbb{R}^n, \quad 0 < t < \delta$$

with initial condition

$$h_a(y, 0) = \begin{cases} a \log \sec \frac{y}{a} - a \log \sec \frac{y_a}{a} + \sqrt{\frac{y_a}{b}} & \text{if } |y| \leq y_a, \\ (|y| - y_a) \tan \frac{y_a}{a} + \sqrt{\frac{y_a}{b}} & \text{if } |y| \geq y_a \end{cases}$$

where y_a is defined in (4.3) below. For each $a > 0$, the hypersurfaces of revolution $|x'| = h_a(x_1, t) + k$ in \mathbb{R}^n generated by the graph of $h_a + k$, with $k \geq 0$, defines a foliation by hypersurfaces $\Gamma_{a,k}(t)$ moving by the normal velocity $V \geq H + g_{\max}(t)$. Moreover,

$$(4.2) \quad \Gamma_{a,k}(t) \cap \Sigma^\pm(t) = \emptyset \quad \text{for } |x'| < \Delta, k > 0, \text{ and } 0 \leq t < \delta.$$

Proof. We shall first prove $\Gamma_{a,k}(t) \cap \Sigma^+(t) = \emptyset$. The case $\Sigma^-(t)$ is equivalent, after changing the sign of B .

Let $f(y, t)$ be defined for $y \geq 0$, $0 \leq t < \delta$ by :

$$r = f(y, t) = f_t(y) := \sqrt{(y + bt^2)/b + B't}, \quad \text{where } B' = \max\{-B/b, 0\}.$$

According to Lemma 4.1, the hypersurface of revolution $|x'| = f(x_1, t)$ in \mathbb{R}^n generated by the graph of $f(x_1, t)$, for $f(x_1, t) < \Delta$, $0 \leq t < \delta$, lies outside or on $\Sigma^+(t)$. Also for any $a > 0$, define $G : \mathbb{R} \rightarrow [0, \infty)$ by

$$y = G(y) := a \log \sec \frac{y}{a},$$

which is known as the Grim Reaper, and denote by y_a the unique value in $(0, a\pi/2)$ satisfying

$$(4.3) \quad f'_0(y_a) = G'(y_a) \iff \frac{1}{2\sqrt{by_a}} = \tan \frac{y_a}{a}.$$

Then define a continuously differentiable, positive function $h_a : \mathbb{R} \times [0, \infty) \rightarrow (0, \infty)$ for each $a > 0$ by extending the Grim Reaper linearly, and moving upward with (large) constant velocity:

$$(4.4) \quad h_a(y, t) = \begin{cases} a \log \sec \frac{y}{a} - a \log \sec \frac{y_a}{a} + \sqrt{\frac{y_a}{b}} + C \frac{t}{a} & \text{if } |y| \leq y_a, \\ (|y| - y_a) \tan \frac{y_a}{a} + \sqrt{\frac{y_a}{b}} + C \frac{t}{a} & \text{if } |y| \geq y_a \end{cases}$$

where C is defined in (4.6) and (4.10) below. Write $y_a =: a(\pi/2 - \theta)$ for some $0 < \theta < \pi/2$. Then we have

$$\frac{2}{\pi} \theta \leq \sin \theta = 2 \cos \theta \sqrt{b \cdot a \left(\frac{\pi}{2} - \theta\right)} \leq 2 \sqrt{b \cdot a \left(\frac{\pi}{2} - \theta\right)}$$

where the second equality follows from (4.3). It follows that

$$0 < \theta \leq \sqrt{b \cdot a \frac{\pi^3}{2}} \leq 1 \quad \text{for } a < \frac{2}{\pi^3 b}.$$

Consequently

$$(4.5) \quad \frac{1}{\sin \theta} = \frac{1}{2\sqrt{a} \cos \theta \sqrt{b(\pi/2 - \theta)}} \leq \frac{1}{2\sqrt{a} \cos 1 \sqrt{b(\pi/2 - 1)}} \quad \text{for } a < \frac{2}{\pi^3 b}.$$

Thus by choosing C satisfying

$$(4.6) \quad C \geq 1 + g_0 \left[\frac{1}{2(\cos 1)\sqrt{b(\pi/2 - 1)}} \right],$$

where g_0 is chosen as an upper bound of $g_{\max}(t)$ for $0 \leq t \leq \delta$, one has

$$1 + g_0 \frac{a}{\sin \theta} \leq C \quad \text{for } a < \min \left\{ \frac{2}{\pi^3 b}, 1 \right\}.$$

Then for $|y| \leq y_a$ and $a < \min\{2(\pi^3 b)^{-1}, 1\}$

$$\begin{aligned} \frac{h_a''}{1 + (h_a')^2} + g_{\max}(t)\sqrt{1 + (h_a')^2} &= \frac{1}{a} + g_{\max}(t) \sec \frac{y}{a} \\ &\leq \frac{1}{a} + g_0 \sec \frac{y_a}{a} = \frac{1}{a} + g_0 \frac{1}{\sin \theta} = \frac{1}{a} \left(1 + g_0 \frac{a}{\sin \theta} \right) \leq \frac{C}{a} = \frac{\partial h_a}{\partial t}. \end{aligned}$$

On the other hand, for $|y| \geq y_a$,

$$g_0 \sqrt{1 + (h_a')^2} = g_0 \sqrt{1 + \tan^2 \frac{y_a}{a}} = g_0 \sec \frac{y_a}{a} = g_0 \frac{1}{\sin \theta} \leq \frac{C}{a} = \frac{\partial h_a}{\partial t}.$$

Therefore, since $h_a'' = 0$, h_a is a viscosity supersolution of (4.1) for $|y| \geq y_a$, as well as for $|y| \leq y_a$. Note that h_a is continuously differentiable.

We claim that any C^1 function $h(y, t)$ which is a smooth supersolution except along a smooth curve (the line $y = y_a$ in our case) and C^2 up to the curve from either side, is a viscosity supersolution. To see this, let (y_0, t_0) be a point of the curve, and suppose a smooth test function $\psi(y_0, t_0) = h(y_0, t_0)$ and $\psi \leq h$ in a neighborhood. We need to show that ψ is a supersolution at (y_0, t_0) . But the first partial derivatives of ψ at (y_0, t_0) agree with those of h . Moreover by the one-sided second-derivative test, the second directional derivatives of ψ at (y_0, t_0) are less than or equal to those of h , where the second derivatives of h are computed on either side of the curve. It follows that ψ is a supersolution of the PDE at (y_0, t_0) , and hence that h is a viscosity supersolution.

To prove (4.2), let us first denote by y_t the value at which the function $h_a(y, t) - f(y, t)$ attains its minimum as a function of y for each time $t \in (0, \delta)$. Then y_t satisfies

$$(4.7) \quad \frac{\partial}{\partial y} h_a(y_t, t) = \frac{\partial}{\partial y} f(y_t, t) \iff \tan \frac{y_t}{a} = \frac{1}{2b\sqrt{(y + b't^2)/b + B't}}.$$

Further, y_t decreases as t increases since $\tan(y/a)$ is monotonically increasing in y and $\left(2b\sqrt{(y + b't^2)/b + B't}\right)^{-1}$ is monotonically decreasing in time t . Thus we have

$$(4.8) \quad y_t \leq y_a \quad \text{for } t \in (0, \delta).$$

Then one has

$$(4.9) \quad \frac{\partial}{\partial t} \left[\min_y \{h_a(y, t) - f(y, t)\} \right] = \frac{\partial}{\partial t} [h_a(y_t, t) - f(y_t, t)] = \\ \left[\frac{\partial}{\partial y} h_a(y_t, t) - \frac{\partial}{\partial y} f(y_t, t) \right] \cdot \frac{\partial y_t}{\partial t} + \frac{C}{a} - \frac{b't/b + B'/2}{\sqrt{(y + b't^2)/b + B't}} = \\ \frac{C}{a} - 2b \left(\frac{b't}{b} + \frac{B'}{2} \right) \tan \frac{y_t}{a},$$

where the last equality follows from (4.7). Now by choosing C satisfying

$$(4.10) \quad \frac{C}{2b(b'\delta/b + B'/2)} > \frac{1}{2(\cos 1)\sqrt{b(\pi/2 - 1)}},$$

one has for $a < \min\{2(\pi^3 b)^{-1}, 1\}$

$$(4.11) \quad \frac{C}{2b(b'\delta/b + B'/2)} > a \frac{1}{\sin \theta} \geq a \tan \frac{y_a}{a} \geq a \tan \frac{y_t}{a}$$

where the first inequality is from (4.5) and the third inequality follows from (4.8). Then by (4.9) and (4.11), one finds, whenever $f(y, t) < \Delta$ and $0 \leq t < \delta$, that

$$\frac{\partial}{\partial t} \left[\min_y \{h_a(y, t) - f(y, t)\} \right] \geq 0 \Rightarrow \min_y \{h_a(y, t) - f(y, t)\} \geq h_a(y_a, 0) - f(y_a, 0) = 0,$$

which implies (4.2). **Q.E.D.**

Having constructed the hypersurfaces $\Gamma_{a,k}(t)$ which are supersolutions of $V \geq H + g_{max}(t)$, we may now define a family of subsolutions v_a of equation (1.1), whose level sets are given by $\Gamma_{a,k}(t)$ for various values of $k \geq 0$.

Recall that, for given hypersurfaces $\Sigma^\pm(t)$, we write $\Omega(t)$ for the open set in \mathbb{R}^n lying outside of both $\Sigma^+(t)$ and of $\Sigma^-(t)$.

Lemma 4.3. *Let $\Sigma^\pm(t)$, b , and $\Delta > 0$ be as in the statement of Lemma 4.1. Let u_0 be a continuous function, which is positive on $\Omega(0) \subset \mathbb{R}^n$, and is equal to 0 on $\Sigma^\pm(0)$. Let u be the corresponding viscosity solution of (4.12) below with initial condition u_0 . Fix $a \in (0, \min\{\frac{2}{\pi^3 b}, 1, \frac{\Delta^2 b}{\pi}\})$ and let the foliation $\{\Gamma_{a,k}(t) \mid k \geq 0, 0 \leq t < \delta\}$ be as in Lemma 4.2. Then there is a positive number $\tilde{\delta} > 0$ and a continuous real-valued function $v_a(x, t)$ defined for*

$$(x, t) \in \bigcup_{0 \leq t < \tilde{\delta}} [B_\Delta^n \cap \Omega(t)] \times \{t\} \subset \mathbb{R}^n \times \mathbb{R},$$

where $B_\Delta^n := \{x \in \mathbb{R}^n \mid |x| < \Delta\}$, which is a viscosity subsolution of

$$(4.12) \quad \frac{\partial v}{\partial t} = |\nabla v| \left(\operatorname{div} \frac{\nabla v}{|\nabla v|} - g(x, t) \right);$$

such that for all $(x, t) \in \bigcup_{0 \leq t < \delta} [B_\Delta \cap \Omega(t)] \times \{t\}$

$$\begin{cases} 0 \leq v_a(x, t) \leq u(x, t), \\ v_a\left(0, \left(\frac{Ct}{a} + \sqrt{\frac{\pi a}{2b}}\right) \hat{e}, t\right) > 0 \quad \forall \hat{e} \in \mathbb{R}^{n-1}, |\hat{e}| = 1, \text{ where } C \text{ is as in (4.4)}, \\ v_a \text{ is nondecreasing in the } r = |x'| \text{ direction,} \\ v_a \equiv \text{const on each } \Gamma_{a,k}(t). \end{cases}$$

Proof. For a fixed value $a > 0$, we first define a set S_Δ^a by

$$S_\Delta^a = \{x \in \Omega(0) \mid |x| = \Delta \quad \text{and} \quad |x'| \geq h_a(x_1, 0)\}.$$

(See (4.4) for the definition of h_a .) By the continuity of u and the fact that u_0 is positive on $\Omega(0)$, we can find $\delta' > 0$ such that

$$(4.13) \quad \delta' < \delta \text{ of Lemma 4.1 and } m_a := \min_{\substack{x \in S_\Delta^a \\ 0 \leq t < \delta'}} u(x, t) > 0.$$

We shall show next that m_a is nonincreasing as a function of a , from which it follows that δ' can be chosen independently of a .

We first claim that $(y_a, h_a(y_a, 0))$ is inside the circle of radius Δ , or equivalently, that

$$(4.14) \quad y_a^2 + h_a(y_a, 0)^2 = y_a^2 + \left(\sqrt{\frac{y_a}{b}}\right)^2 < \Delta^2.$$

Since $0 < y_a < \frac{\pi a}{2}$, it is enough to show that

$$\left(\frac{\pi a}{2}\right)^2 + \frac{\pi a}{2b} < \Delta^2.$$

But the second term of the left-hand side is less than $\frac{1}{2}\Delta^2$, since $a < \frac{\Delta^2 b}{\pi}$; multiplying this last inequality for a with the hypothesis $a < \frac{2}{\pi^3 b}$ makes the first term of the left-hand side less than $\frac{\Delta^2}{2\pi^2}$, and the claim follows.

We next observe that y_a is increasing as a function of a ; this follows from the fact that $y = y_a$ solves $1/(2\sqrt{by}) = \tan(y/a)$ (see equation (4.3)), where $1/(2\sqrt{by})$ is a decreasing function of y , while $\tan y/a$ is increasing in y and decreasing in a .

Finally, given $0 < a < a' < \min\left\{\frac{2}{\pi^3 b}, 1, \frac{\Delta^2 b}{\pi}\right\}$, we have

$$h_a(y_{a'}, 0) > f_0(y_{a'}) = \sqrt{\frac{y_{a'}}{b}} = h_{a'}(y_{a'}, 0),$$

where the first inequality holds since the straight-line part of $h_a(\cdot, 0)$ is tangent to the concave function f_0 at y_a , which is less than $y_{a'}$. Moreover,

$$\tan \frac{y_a}{a} = \frac{1}{2\sqrt{by_a}} > \frac{1}{2\sqrt{by_{a'}}} = \tan \frac{y_{a'}}{a'},$$

that is, the slope of the straight-line part of $h_a(\cdot, 0)$ is greater than the slope of the straight-line part of $h_{a'}(\cdot, 0)$. Hence, we conclude that, for $y \geq y_{a'}$, $h_a(y, 0) > h_{a'}(y, 0)$. Then, by this conclusion and the fact that the graph of $h_a(\cdot, 0)$ crosses

the circle of radius Δ on the straight-line part, which follows from (4.14) above, we conclude that S_a^Δ is larger than $S_{a'}^\Delta$ as sets. It now follows that $m_a \geq m_{a'}$, and in particular that δ' can be chosen independently of a .

For any given $a > 0$ and $x \in B_\Delta^n \cap \Omega(0)$, if $|x'| \geq h_a(x_1, 0)$, define $k(x)$ to be the unique value such that $x \in \Gamma_{a,k(x)}(0)$, by Lemma 4.2. Then define

$$\begin{aligned} v_a(x, 0) &= \min \left\{ m_a, \min_{k \geq k(x)} \min_{z \in \Gamma_{a,k}(0) \cap B_\Delta^n} u_0(z) \right\} & \text{if } |x'| \geq h_a(x_1, 0), \\ v_a(x, 0) &= 0 & \text{if } |x'| < h_a(x_1, 0). \end{aligned}$$

Finally, define

$$(4.15) \quad v_a(x, t) = v_a(x_1, x', t) = \begin{cases} v_a(x_1, x' - \frac{C}{a}t \frac{x'}{|x'|}, 0) & \text{if } |x'| \geq h_a(x_1, t), \\ 0 & \text{if } |x'| < h_a(x_1, t). \end{cases}$$

It follows directly from the construction that $v_a \equiv \text{const}$ on each $\Gamma_{a,k}(t)$, and $v_a(x, t) = \varphi(k)$ for some nondecreasing function φ whenever $(x, t) \in \Gamma_{a,k}(t)$, and $v_a(x, t) = 0$ otherwise, which means that v_a is nondecreasing in the r direction. Therefore v_a is a viscosity subsolution of

$$\frac{\partial v}{\partial t} = |\nabla v| \left(\text{div} \frac{\nabla v}{|\nabla v|} - g_{\max}(t) \right)$$

by Proposition 2.1 and Lemma 4.2, where $g_{\max}(t)$ is as in the statement of Lemma 4.2. Since $g(x, t) \leq g_{\max}(t)$ for all $|x| < \Delta$, it follows that v_a is a viscosity subsolution of (4.12). Moreover, by the construction of $v_a(x, t)$,

$$0 \leq v_a(x, t) \leq u(x, t) \quad \text{on} \quad ([B_\Delta^n \cap \Omega(0)] \times \{0\}) \cup \left[\bigcup_{0 \leq t < \delta'} \partial [B_\Delta^n \cap \Omega(t)] \times \{t\} \right].$$

It follows from the comparison principle that $0 \leq v_a(x, t) \leq u(x, t)$ for all $(x, t) \in \bigcup_{0 \leq t < \delta'} [B_\Delta^n \cap \Omega(t)] \times \{t\}$ (see [GGIS], p. 463). Moreover, since $a < \frac{\Delta^2 b}{\pi}$ by the assumption on a , which implies $\sqrt{\frac{\pi a}{2b}} < \Delta$, we can define $\tilde{\delta} > 0$ by

$$(4.16) \quad \tilde{\delta} := \min \left\{ \frac{a}{C} \left(\Delta - \sqrt{\frac{\pi a}{2b}} \right), \delta' \right\}$$

where δ' is defined in (4.13). Then

$$(4.17) \quad \frac{Ct}{a} + \sqrt{\frac{\pi a}{2b}} < \Delta \quad \text{whenever} \quad 0 \leq t < \tilde{\delta}.$$

Moreover, with the same function f_0 as in Lemma 4.2,

$$(4.18) \quad \sqrt{\frac{\pi a}{2b}} = f_0 \left(\frac{\pi}{2} a \right) > f_0(y_a) = h_a(y_a, 0) > h_a(0, 0).$$

It follows from (4.17), (4.18), and Lemma 4.2 that

$$(4.19) \quad \left(0, \left(\frac{Ct}{a} + \sqrt{\frac{\pi a}{2b}}\right) \hat{e}\right) \in B_{\Delta}^n \cap \Omega(t),$$

and, thus, $v_a(0, (\frac{Ct}{a} + \sqrt{\frac{\pi a}{2b}}) \hat{e}, t)$ is well defined for any $|\hat{e}| = 1$ and for $0 \leq t < \tilde{\delta}$. Then, since $(0, \sqrt{\frac{\pi a}{2b}} \hat{e}, 0) \in \Gamma_{a,k}(0)$ for some $k > 0$ by (4.18) and Lemma 4.2, $v_a(0, \sqrt{\frac{\pi a}{2b}} \hat{e}, 0) > 0$ by the definition of v_a and the fact that u_0 is positive on $\Omega(0)$. Further, since $|(\frac{Ct}{a} + \sqrt{\frac{\pi a}{2b}}) \hat{e}| = f_0(\frac{\pi a}{2}) + \frac{Ct}{a} > h_a(0, 0) + \frac{Ct}{a} = h_a(0, t)$, $v_a(0, (\frac{Ct}{a} + \sqrt{\frac{\pi a}{2b}}) \hat{e}, t) = v_a(0, \sqrt{\frac{\pi a}{2b}} \hat{e}, 0) > 0$ for $0 \leq t < \tilde{\delta}$ by (4.15). **Q.E.D.**

Remark 5. Lemmas 3.4 and 4.3 show: if $\Sigma^{\pm}(t)$ are part of the moving boundary for a nonnegative solution of (2.1), with the boundary condition: $u = 0$ on $\Sigma^{\pm}(t)$, disjoint except at $t = 0$, $\Sigma^+(0) \cap \Sigma^-(0) = O$, $\Sigma^{\pm}(0)$ strictly convex at O , then for $0 < t < \delta$, the generalized solution $\{u = 0\}$ includes a piece of $\Omega(t)$ of size $\propto t^{1/3}$.

We are finally ready to prove an upper bound $|x| < \kappa_0 t^{1/3}$ on the size of the level set after fattening; this upper bound only applies to points $x = (0, x')$ in the hyperplane $x_1 = 0$. Of course, one expects that the fattening may appear instantaneously at great distances along the hypersurfaces $\Sigma^{\pm}(t)$, $t > 0$; therefore some restriction similar to $x_1 = 0$ is necessary in general.

Theorem 4.4. *Let $\Sigma^{\pm}(t)$ be two smooth, oriented hypersurfaces in \mathbb{R}^n which evolve according to*

$$(4.20) \quad V = H + g(x, t)$$

for some continuous forcing term $g(x, t)$. Suppose that $\Sigma^+(t) \cap \Sigma^-(t) = \emptyset$ for $t \neq 0$, $-T \leq t < T$, and that there is a point $x_0 \in \Sigma^+(0) \cap \Sigma^-(0)$. Moreover, assume that $\Sigma^{\pm}(t)$ are strictly convex at $x = x_0$, $t = 0$. Let $u(\cdot, t)^{-1}(0)$ be the generalized solution to (4.20) with initial condition $u(\cdot, -T)^{-1}(0) = \Sigma^+(-T) \cup \Sigma^-(-T)$. That is, $u(x, t)$ satisfies the equation

$$\frac{\partial u}{\partial t} = |\nabla u| \left(\operatorname{div} \left(\frac{\nabla u}{|\nabla u|} \right) - g(x, t) \right)$$

and $u(x, -T)$ vanishes iff $x \in \Sigma^{\pm}(-T)$. Then there exists $\delta > 0$ such that for all $0 < t < \delta$, the generalized solution $u(\cdot, t)^{-1}(0)$ has nonempty interior. Furthermore, recall that $\Omega(t)$ denotes the open set in \mathbb{R}^n lying outside of both $\Sigma^+(t)$ and of $\Sigma^-(t)$, and define $B_{\Delta}^n := \{x \in \mathbb{R}^n \mid |x - x_0| < \Delta\}$. Also, let b be as in the statement of Lemma 4.3. Then there is $\kappa_0 > 0$ and $\delta > 0$ so that for all $0 < t < \delta$ and $P_x := x_0 + (0, x') \in B_{\Delta}^n \cap \Omega(t)$, if $u(P_x, t)$ vanishes, then necessarily

$$P_x \in B_{\kappa_0 t^{1/3}}(x_0).$$

Furthermore, for $|\hat{e}| = 1$ and $0 < t < \delta$,

$$(4.21) \quad x_0 + (0, \kappa_0 t^{1/3} \hat{e}) \in B_{\Delta}^n \cap \Omega(t), \quad \text{and thus} \quad u(x_0 + (0, \kappa_0 t^{1/3} \hat{e}), t) > 0.$$

Proof. Our proof will be based on the family of subsolutions $v_a(x, t)$ given in Lemma 4.3. Assume for simplicity $x_0 = O \in \mathbb{R}^n$. We may assume $u(x, 0) > 0$ for $x \in \Omega(0)$ and $u(x, 0) < 0$ for x inside $\Sigma^\pm(0)$; see Theorem 5.6 of [CGG].

Let us pick any a satisfying

$$(A1) \quad 0 < a < \min \left\{ \frac{2}{\pi^3 b}, 1, \frac{\Delta^2 b}{\pi} \right\},$$

$$(A2) \quad t_a := a^{3/2} \frac{\sqrt{\pi}}{C\sqrt{8b}} < \tilde{\delta} = \min \left\{ \frac{a}{C} \left(\Delta - \sqrt{\frac{\pi a}{2b}} \right), \delta' \right\}$$

as in (4.16), that is, δ' is defined in (4.13) and C is defined in (4.4). (See (4.23) below to understand the meaning of t_a .) Let b be as in the statement of Lemma 4.1 and

$$f_{a,\kappa}(t) = \frac{Ct}{a} + \sqrt{\frac{\pi a}{2b}} - \kappa t^{1/3} \quad (\kappa > 0).$$

Since

$$f'_{a,\kappa}(t_a) = 0 \quad \text{at} \quad t_a = \left(\frac{a\kappa}{3C} \right)^{3/2},$$

we may compute

$$f_{a,\kappa}(t_a) = \frac{C}{a} t_a + \sqrt{\frac{\pi a}{2b}} - \kappa t_a^{1/3} = \sqrt{a} \left[-\frac{2}{3} \sqrt{\frac{1}{3}} \kappa^{3/2} \frac{1}{\sqrt{C}} + \sqrt{\frac{\pi}{2b}} \right].$$

Thus, we define κ_0 independent of a by

$$a^{-1/2} f_{a,\kappa_0}(t_a) = -\frac{2}{3} \sqrt{\frac{1}{3}} \kappa_0^{3/2} \frac{1}{\sqrt{C}} + \sqrt{\frac{\pi}{2b}} = 0 \Leftrightarrow \kappa_0 := \left(\frac{27\pi C}{8b} \right)^{1/3}.$$

Then since $f''_{a,\kappa}(t) > 0$,

$$(4.22) \quad f_{a,\kappa_0}(t_a) = 0 \quad \text{and} \quad f_{a,\kappa_0}(t) > 0 \quad \text{if} \quad t \neq t_a \quad \text{and} \quad t > 0.$$

For this choice of κ_0 , we have

$$(4.23) \quad t_a = \left(\frac{a\kappa_0}{3C} \right)^{3/2} = a^{3/2} \left(\frac{1}{3C} \right)^{3/2} \left(\frac{27\pi C}{8b} \right)^{1/2} = a^{3/2} \frac{\sqrt{\pi}}{C\sqrt{8b}}.$$

Note that $r = \kappa_0 t^{1/3}$ is tangent to the straight line $r = \frac{Ct}{a} + \sqrt{\frac{\pi a}{2b}}$ at $t = t_a$, and forms the envelope of this family of straight lines with parameter a . Then, by (A2), t_a satisfies $t_a < \tilde{\delta}$ for $\tilde{\delta}$ defined in (4.16), and we can apply Lemma 4.3 for this t_a whenever a satisfies (A1)–(A2), or equivalently, as long as

$$0 < t_a < \min \left\{ \tilde{\delta}, a_{\min}^{3/2} \frac{\sqrt{\pi}}{C\sqrt{8b}} \right\} =: \delta$$

where $a_{\min} := \min \left\{ \frac{2}{\pi^3 b}, 1, \frac{\Delta^2 b}{\pi} \right\}$. Furthermore, by the fact that v_a is nondecreasing in the r direction, we obtain

$$u(0, x', t_a) \geq v_a(0, x', t_a) \geq v_a \left(0, \left(\frac{Ct_a}{a} + \sqrt{\frac{\pi a}{2b}} \right) \frac{x'}{|x'|}, t_a \right) > 0$$

whenever $(0, x') \in B_{\Delta}^n \cap \Omega(t_a)$, $0 < t_a < \delta$ and $|x'| \geq \left| \frac{Ct_a}{a} + \sqrt{\frac{\pi a}{2b}} \right| = \kappa_0 t_a^{1/3}$ (the last equality follows from (4.22)). By replacing t_a by t , we conclude that $u(0, x', t) > 0$ whenever $(0, x') \in B_{\Delta}^n \cap \Omega(t)$, $0 < t < \delta$, and $|x'| \geq \kappa_0 t^{1/3}$. Therefore, for $0 < t < \delta$ and $(0, x') \in B_{\Delta}^n \cap \Omega(t)$,

$$u(0, x', t) = 0 \quad \text{may only occur if} \quad |x'| < \kappa_0 t^{1/3}.$$

Finally, (4.21) follows from (4.19) and (4.22).

Q.E.D.

Remark 6. Although Theorem 1.1 deals with two disjoint pieces of hypersurface $\Sigma^{\pm}(t)$ evolving by $V = H + g(x, t)$, the reader may note that this includes the case of a *connected* hypersurface $\Sigma(t)$ which touches itself at some time $t = 0$ and then pulls away. In this situation, $\Sigma^{\pm}(t)$ may be chosen as appropriate neighborhoods of the contact point.

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SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA; DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CONNECTICUT

E-mail address: `gulliver@math.umn.edu`, `koo@math.uconn.edu`