Research Statement

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I am interested in probability theory and its applications. Currently I am working on random walks in random environments (RWRE). In what follows I will present some of my past results and my plan about future research.

1 Current and completed work on RWRE

An *environment* $\omega : \mathbb{Z}^d \times \{e \in \mathbb{Z}^d : |e| = 1\} \to [0,1]$ is a function that satisfies

$$\sum_{e:|e|=1} \omega(x,e) = 1$$

for all x. In other words, it is a transition probability on the lattice. We endow the set of all possible environments Ω with a probability measure P and assume that it is ergodic with respect to spatial shifts. A special case is when the transition probabilities on different sites are independent and identically distributed (iid).

In the model of RWRE, an environment ω is sampled randomly according to P. Then on this fixed ω, the random walks (X_n) is defined to be the Markov chain with transition probability P_ω given by

$$P_\omega(X_{n+1} = x + e | X_n = x) = \omega(x,e).$$

The measure P_ω of the paths under a fixed environment ω is called the *quenched* measure. The average over all quenched measures, $\mathbb{P} := P \otimes P_\omega$, is called the *annealed* measure.

1.1 Random walks in a balanced environment

The first model we considered is the random walks in balanced random environment. We say that an environment is *balanced* if it has no drift, and *elliptic* if P-almost every ω is strictly positive. A special case is the simple random walk on $\mathbb{Z}^d$, where the walker moves to each direction with equal probability $1/2d$. In [11], together with my advisor Ofer Zeitouni, I proved a quenched invariance principle for random walks in balanced elliptic iid environments in $\mathbb{Z}^d, d \geq 2$. Namely, for almost every ω, the continuous rescaled trajectory converges weakly to a Brownian motion.

In a previous work, Lawler [14] proved the invariance principle under the uniform ellipticity assumption that $\omega > \epsilon$ for almost every ω and some constant $\epsilon > 0$. One goal of our paper [11] is to explore the extent to which the uniform ellipticity assumption can be dropped. Surprisingly, in the iid case, we can show that no assumptions of uniform ellipticity are needed at all.

Our approach uses percolation arguments. Unlike the uniform-elliptic case, one may worry that the walker will be trapped in certain "bad locations" for a long time, which can destroy the invariance principle. Our key observation is that the bad sites are percolation clusters of an iid percolation, which are very rare if they are "very bad". This allows us to use percolation argument to get good control of the bad sites.

Recently, Berger and Deuschel have generalized our ideas and extended the invariance principle to the general non-elliptic case where the environment is only required to be iid and "genuinely d-dimensional". See [3] for details.

In addition to the invariance principle, we also obtained results concerning the transience and recurrence of the walks. When $d = 2$, the recurrence of the walks follows from the invariance principle, by an argument of Kesten (see [21, p.281]). When $d \geq 3$, we proved that the walks in the iid balanced environment is transient. Our idea is to consider long range jumps which are essentially uniformly (up to a constant determined by the percolation radius) elliptic walks on a different d-dimensional mesh. The key ingredient in our proof is a new maximum principle for (not uniformly elliptic) difference operators on general meshes, see Theorem 15 in [11].

1.2 Limiting velocity of random walks in mixing random environment

In my second paper [10], I studied the limiting velocity $\lim_{n\to\infty} X_n/n$ of random walks in mixing random environment.

Much progress has been made in recent years in the study of the limiting velocity in iid environment. For one-dimensional RWRE, the law of large numbers (LLN) is well known [19]. For $d \geq 2$, a conditional law of large numbers (CLLN) is proved in [20, 22] (see [21] for the full version), which states that the limiting velocity exists almost surely and its support has at most two elements. Moreover, for $d = 2$, the LLN follows from combining the CLLN and Zerner and Merkl's 0-1 law [23] for two-dimensional RWRE: for any direction ℓ,

$$P(\text{the walk is transient in the direction } \ell) \in \{0, 1\}. \qquad \text{(0-1 Law)}$$

In high dimensions ($d \geq 5$), Berger [2] showed that the limiting velocity can take at most one non-zero value.

It is interesting to consider the case when the environments on different sites are allowed to be dependent. Of special interest is the environment that is produced by a Gibbsian particle system and satisfies Dobrushin-Shlosman's strong mixing condition IIIc in [7], see [16, 17, 18, 5, 6] for related works. An important feature of this model is that the influence of environments in remote locations decays exponentially as the distance grows. However, the lack of finite-range dependence destroys the regeneration properties of the random path.

The first main result of my paper [10] is the CLLN for random walks in the strong-mixing Gibbsian environment. This result is a minor extension of Rassoul-Agha's CLLN in [18], in which the assumption is slightly stronger than strong-mixing. Yet, my proof is very different from the proof in [18], which is based on a large deviation principle [17]. The novel contribution of my proof is a new definition of the regeneration structure, which enables us to divide a random path in the mixing environment into "almost iid" parts (the so-called *regeneration slabs*).

My second main result in [10] states that in dimensions $d \geq 5$, there is at most one non-zero velocity in the strong-mixing Gibbsian environment. Inspired by [2], I consider two walks that are ballistic in opposite directions: one starts at the origin, the other starts near the n-th regeneration position of the first path, with $n \to \infty$. As [2] shows, to prove the uniqueness of the non-zero velocity, one only needs to show that the two paths are "almost independent". For the iid environment considered in [2], it suffices to prove that they do not intersect, since environments in different locations are independent. But, we need more in the mixing case, namely, we need the two paths to become further and further away from each other. The main tool is a heat kernel estimate for walks in the mixing environment, which I obtained using coupling arguments.

1.3 Einstein relation for RWRE

The following is a current project I am working on.

Einstein [8, pp. 1-18] investigated the movement of suspended particles in a liquid under the influence of an external force. He established the following linear relation between the diffusion constant D and the *mobility* (i.e. the ratio between the drift velocity and the force) μ:

$$D = k_B T \mu,$$

where k_B is Boltzmann's constant and T is the absolute temperature.

Recently, there has been much interest in studying the Einstein relation for reversible motions in random media, see [15, 12, 9, 1]. So far, little work has been done in the non-reversible set-ups, e.g. RWRE. Together with my advisor Ofer Zeitouni, I am working on the proof of the Einstein relation for RWRE, which is formulated as follows.

Assume that we have a uniformly elliptic balanced random environment. As was shown in [14], the scaled path of the random walks in the balanced environment converges to a Brownian motion with a diffusion matrix $\mathbf{D}$. For a given unit vector $\ell \in \mathbb{R}^d$ and $\lambda \in (0, 1)$, define the perturbed environment $\omega^{\lambda,\ell}$ of a balanced environment ω by

$$\omega^{\lambda,\ell}(x, e) = (1 + \lambda\ell \cdot e)\omega(x, e).$$

Then the drift caused by the perturbation will impart a velocity $v_\lambda \in \mathbb{R}^d$ to the random walks on $\omega^{\lambda,\ell}$. The Einstein relation we want to prove is

$$\lim_{\lambda \to 0} \frac{v_\lambda}{\lambda} = \mathbf{D}\ell.$$

Following the argument of Lebowitz and Rost [15], one can show that

$$\lim_{\alpha \to \infty} \lim_{\lambda \to 0} E_{\lambda,\ell} \frac{X_{\alpha/\lambda^2}}{\alpha/\lambda} = \mathbf{D}\ell,$$

where $E_{\lambda,\ell}$ is the annealed expectation with respect to the perturbed measure. To identify the left-hand side with the mobility of the RWRE, one needs some moment estimates for the regeneration times, which can be reduced to estimates on regeneration times for biased simple random walk. I have studied the latter and concluded that they do not possess the required tail. To fix this, we adapt the approach of Gantert, Mathieu and Piatnitski [9] and use modified regeneration times. However, due to the lack of reversibility, we don't have good heat kernel estimates, which were crucially employed in [9] in the construction of the modified regeneration times. We hope to use a modification of the Harnack inequality for difference operators in [13] to bypass this point and complete the argument.

2 Future work

There are several problems that I am interested in working on in the near-term future.

1. Einstein relation for RWRE with the presence of cut points. In [4], a law of large numbers and a central limit theorem are proved for RWRE where cut points (see the definition in [4, pp.530]) exist. The Einstein relation in this case is not yet explored and is of interest to me.

2. Is the random walk in iid random environment transient when $d \geq 3$?

3. The 0-1 law for RWRE when $d \geq 3$. As mentioned before, for two-dimensional RWRE the 0-1 law is proved by Zerner and Merkl [23]. For $d \geq 3$ this problem is still wide open.

I am also interested in other probability models, e.g. random polymers, SLE (I participated in a student-run seminar on SLE in Minnesota in 2009), KPZ equation, optimal transport theory and etc. I enjoy learning new things and hope to expand my scope of research in the future.

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