# HOMEWORK SOLUTIONS FOR MATH 5651

## HOMEWORK FOR WEEK 1 OF SPRING 2008

Text: de Groot and Schervish, 3rd edition
Assignment: §1.4: 4,6,7; §1.5:4,5,6,8,9; §1.6: 2,3; §1.7: 3,5,7,8,9;

### Problem 4 of §1.4

We have

$$AB \cup AB^c = A(B \cup B^c) = AS = A$$

where the first equality holds by the result of problem 2 above, the second equality holds by a result on p. 9 of the text (right near the middle) and the third holds by a result on p. 8 of the text (right near the bottom). Similarly we have

$$\begin{aligned}
AB \cap AB^c &= (A \cap B) \cap (A \cap B^c) \\
&= A \cap (B \cap (A \cap B^c)) \\
&= A \cap ((A \cap B^c) \cap B) \\
&= A \cap (A \cap (B^c \cap B)) \\
&= A \cap (A \cap (B \cap B^c)) \\
&= A \cap (A \cap \emptyset) \\
&= A \cap \emptyset \\
&= \emptyset
\end{aligned}$$

where each step is justified by some fact you can find on pages 8 and 9 of the text.

### Problem 6 of §1.4

We have a sample space

$$S = \{\text{red}, \text{blue}\} \times \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}.$$

for this experiment. We are given that

$$\begin{aligned}
A &= \text{“an even-numbered card is selected”} \\
&= \{\text{red}, \text{blue}\} \times \{2, 4, 6, 8, 10\}, \\
\\
B &= \text{“a blue card is selected”} \\
&= \{\text{blue}\} \times \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}, \\
\\
C &= \text{“a card with a number} < 5 \text{ is selected”} \\
&= \{\text{red}, \text{blue}\} \times \{1, 2, 3, 4\}.
\end{aligned}$$

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We then have

$$\begin{aligned} ABC &= \text{“an even-numbered blue card numbered} < 5 \text{ is selected”} \\ &= \{\text{blue}\} \times \{2, 4\}, \end{aligned}$$

$$\begin{aligned} BC^c &= \text{“a blue card numbered} \geq 5 \text{ is selected”} \\ &= \{\text{blue}\} \times \{5, 6, 7, 8, 9, 10\}, \end{aligned}$$

$$\begin{aligned} A \cup B \cup C &= \text{“a card is selected which is even-numbered, or blue,} \\ &\quad \text{or numbered} < 5\text{”} \\ &= (\{\text{blue}\} \times \{1, 2, 3, 4, 6, 7, 8, 9, 10\}) \\ &\quad \cup (\{\text{red}\} \times \{1, 2, 3, 4, 6, 8, 10\}), \end{aligned}$$

$$\begin{aligned} A(B \cup C) &= AB \cup AC \\ &= \text{“a card is selected which is even-numbered-and-blue} \\ &\quad \text{or even-numbered-and-numbered-}<5\text{”} \\ &= (\{\text{blue}\} \times \{2, 4, 6, 8, 10\}) \cup (\{\text{red}\} \times \{2, 4\}), \end{aligned}$$

$$\begin{aligned} A^c B^c C^c &= \text{“an odd-numbered red card numbered} \geq 5 \text{ is selected”} \\ &= \{\text{red}\} \times \{5, 7, 9\}. \end{aligned}$$

## Problem 7 of §1.4

[I corrected a typo on Tuesday, March 4, 2008.] We are given events

$$\begin{aligned} A &= \{x : 1 \leq x \leq 5\}, \\ B &= \{x : 3 < x \leq 7\}, \\ C &= \{x : x \leq 0\}. \end{aligned}$$

all of which are subsets of the real line. Then we have

$$\begin{aligned} A^c &= \{x : x < 1\} \cup \{x : x > 5\} \ \text{(clear)}, \\ A \cup B &= \{x : 1 \leq x \leq 7\} \ \text{(note overlap)}, \\ BC^c &= \{x : 3 < x \leq 7\} \cap \{x : x > 0\} = B, \\ A^c B^c C^c &= (A \cup B \cup C)^c = (\{x : 1 \leq x \leq 7\} \cup \{x : x \leq 0\})^c \\ &= \{x : x > 7\} \cup \{x : 0 < x < 1\}, \\ (A \cup B)C &= \{x : 1 \leq x \leq 7\} \cap \{x : x \leq 0\} = \emptyset. \end{aligned}$$

This kind of manipulation will come up a lot in our later work.

## Problem 4 of §1.5

Let $F_A$ be the event “student A fails the exam”. Let $F_B$ be the event “student B fails the exam”. Note that $F_A F_B$ is the event “students A and B both fail the exam” and that $F_A \cup F_B$ is the event “at least one of the students A and B fail the exam”. We are given that

$$\Pr(F_A) = 0.5, \ \ \Pr(F_B) = 0.2, \ \ \Pr(F_A F_B) = 0.1.$$

We therefore have

$$\Pr(F_A \cup F_B) = \Pr(F_A) + \Pr(F_B) - \Pr(F_A F_B) = 0.5 + 0.2 - 0.1 = 0.6.$$

## Problem 5 of §1.5

Keep the notation of the previous problem. Note that $F_A^c F_B^c$ is the event that neither A nor B fail the exam. We have

$$\Pr(F_A^c F_B^c) = \Pr((F_A \cup F_B)^c) = 1 - \Pr(F_A \cup F_B) = 1 - 0.6 = 0.4$$

by using the result of the previous problem.

## Problem 6 of §1.5

Keep the notation of the previous two problems. Note that $F_A F_B^c \cup F_A^c F_B$ is the event that exactly one of A and B fail the exam. Notice that

$$F_A \cup F_B = F_A F_B^c \cup F_A^c F_B \cup F_A F_B$$

and that the events on the right side of the equation are disjoint. Therefore

$$\Pr(F_A F_B^c \cup F_A^c F_B) = \Pr(F_A \cup F_B) - \Pr(F_A F_B) = 0.6 - 0.1 = 0.5$$

by using the result of problem 4 above.

## Problem 8 of §1.5

To turn this into a probability problem, consider the experiment of randomly selecting a family from the city. Let $A$ be the event that the selected family subscribes to the morning paper and let $B$ be the event that the selected family subscribes to the afternoon paper. Then $A \cup B$ is the event that the selected family subscribes to one or the other of the newspapers, and $AB$ is the event that the selected family subscribes to both newspapers. We are given

$$\Pr(A) = 0.5, \quad \Pr(B) = 0.65, \quad \Pr(A \cup B) = 0.85,$$

and this gives us an equation

$$0.85 = \Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(AB) = 1.15 - \Pr(AB)$$

we can solve for $\Pr(AB)$. We find that $\Pr(AB) = 1.15 - 0.85 = 0.30$.

## Problem 9 of §1.5

For any events $A$ and $B$, we have that

$$A^c B \cup AB^c = \text{“exactly one of the events } A \text{ and } B \text{ occur”}.$$

In general we have

$$A \cup B = A^c B \cup AB^c \cup AB$$

and the union on the right side is disjoint. So we have

$$\Pr(A^c B \cup AB^c) = \Pr(A \cup B) - \Pr(AB)$$

$$= \Pr(A) + \Pr(B) - \Pr(AB) - \Pr(AB) = \Pr(A) + \Pr(B) - 2\Pr(AB),$$

which finishes the proof.

## Problem 2 of §1.6

If the sum of the number of spots showing on a pair of dice is even, then either both dice show an even number of spots, or else both show an odd number of spots. The probability in question is therefore $(3^2 + 3^2)/6^2 = 18/36 = 1/2$.

## Problem 3 of §1.6

Consider the following diagram. Asterisks mark the possible outcomes of the experiment corresponding to the event in question.

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | * | * | * | | | |
| 2 | * | * | * | * | | |
| 3 | * | * | * | * | * | |
| 4 | | * | * | * | * | * |
| 5 | | | * | * | * | * |
| 6 | | | | * | * | * |

Since there are 24 asterisks, the probability in question is $24/36 = 2/3$.

## Problem 3 of §1.7

The five letters $a, b, c, d, e$ can be arranged in $P_{5,5} = 5!$ ways.

## Problem 5 of §1.7

To roll four dice and get four different numbers is the same as choosing a permutation of 6 things taken 4 at a time. There are $6^4$ possible outcomes for the experiment of rolling 4 dice. So the probability in question here is $P_{6,4}/6^4 = 6 \cdot 5 \cdot 4 \cdot 3/6^4$.

## Problem 7 of §1.7

The experiment of throwing 12 balls into 20 boxes has $20^{12}$ possible outcomes. The event that all the balls end up in different boxes is the same as forming a permutation of 20 things taken 12 at a time, so there are $P_{20,12}$ possible outcomes with balls all in different boxes. Thus the probability in question is $P_{20,12}/20^{12}$.

## Problem 8 of §1.7

The sample space $S$ for the experiment is all possible ways for five people each to choose one of seven floors of the building at which to exit the elevator. The event $A$ we are interested in is the one in which all the passengers choose distinct floors. In other words (cf. p. 24 of the text), $A$ is the set of permutations of the seven floors of the building taken five at a time. All the outcomes in the sample space $S$ are equally likely. We find that

$$\Pr(A) = \frac{\#A}{\#S} = \frac{P_{7,5}}{7^5} = \frac{7!}{2!7^5}.$$

## Problem 9 of §1.7

The possible outcomes of the experiment are the permutations of the six runners taken six at a time. The possible outcomes of the experiment with team A taking first, second and third places and team B taking fourth, fifth and sixth places are (in effect) ordered pairs consisting of permutations of 3 things taken 3 at a time. The probability in question is therefore $(3!)^2/6^!$.