KRONECKER-WEBER PLUS EPSILON

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Abstract. We say that a group is *almost abelian* if every commutator is central and squares to the identity. Now let G be the Galois group of the algebraic closure of the field $\mathbb{Q}$ of rational numbers in the field of complex numbers. Let $G^{\mathrm{ab}+\epsilon}$ be the quotient of G universal for continuous homomorphisms to almost abelian profinite groups and let $\mathbb{Q}^{\mathrm{ab}+\epsilon}/\mathbb{Q}$ be the corresponding Galois extension. We prove that $\mathbb{Q}^{\mathrm{ab}+\epsilon}$ is generated by the roots of unity, the fourth roots of the rational primes and the square roots of certain algebraic sine-monomials. The inspiration for the paper came from recent studies of algebraic Γ-monomials by P. Das and by S. Seo.

1. Introduction

We say that a group is *almost abelian* if every commutator is central and squares to the identity. Let G be the Galois group of the algebraic closure of the field of rational numbers $\mathbb{Q}$ in the field $\mathbb{C}$ of complex numbers. Let $G^{\mathrm{ab}+\epsilon}$ be the quotient of G universal for continuous homomorphisms to almost abelian profinite groups. Let G^{ϵ} be the kernel of the natural map of $G^{\mathrm{ab}+\epsilon}$ to the abelianization G^{ab} of G. By construction the group G^{ϵ} is central in $G^{\mathrm{ab}+\epsilon}$ and killed by 2. Let $\mathbb{Q}^{\mathrm{ab}}$ (resp. $\mathbb{Q}^{\mathrm{ab}+\epsilon}$) be the Galois extension of $\mathbb{Q}$ in $\mathbb{C}$ with Galois group G^{ab} (resp. $G^{\mathrm{ab}+\epsilon}$). The Kronecker-Weber theorem determines the structure of the group G^{ab} and provides an explicit description of the field $\mathbb{Q}^{\mathrm{ab}}$. The theory of [Fröhlich 1983] in principle determines the structure of the group $G^{\mathrm{ab}+\epsilon}$ but does not provide an explicit description of the field $\mathbb{Q}^{\mathrm{ab}+\epsilon}$. Kummer theory identifies the Pontryagin dual of G^{ϵ} with $H^0(G^{\mathrm{ab}}, \mathbb{Q}^{\mathrm{ab}\times}/\mathbb{Q}^{\mathrm{ab}\times 2})$. Our purpose in this paper is to exhibit for the latter group an explicit $\mathbb{Z}/2\mathbb{Z}$-basis, thereby obtaining a description of the field $\mathbb{Q}^{\mathrm{ab}+\epsilon}$ as explicit as that provided for the field $\mathbb{Q}^{\mathrm{ab}}$ by the Kronecker-Weber theorem. Our method is more or less elementary and independent of the theory of [Fröhlich 1983]. The inspiration for our work came from the recent studies [Das 2000] and [Seo 2001] of algebraic Γ-monomials.

Our main results are as follows. Let $\mathcal{A}$ be the free abelian group on symbols of the form

$$[a] \quad (a \in \mathbb{Q}),$$

modulo the identifications

$$[a] = [b] \Leftrightarrow a - b \in \mathbb{Z}.$$

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For all prime numbers $p<q$, if $2<p$ put

$$\mathbf{a}_{pq} := \sum_{i=1}^{\frac{p-1}{2}} \left(\left[\frac{i}{p}\right] - \sum_{k=0}^{\frac{q-1}{2}} \left[\frac{i}{pq}+\frac{k}{q}\right] \right) - \sum_{j=1}^{\frac{q-1}{2}} \left(\left[\frac{j}{q}\right] - \sum_{\ell=0}^{\frac{p-1}{2}} \left[\frac{j}{pq}+\frac{\ell}{p}\right] \right),$$

e. g.,

$$\mathbf{a}_{3\cdot 5} = \left[\frac{1}{3}\right] + \left[\frac{2}{15}\right] - \left[\frac{4}{15}\right] - \left[\frac{1}{5}\right],$$

and if $2=p$ put

$$\mathbf{a}_{pq} := \left(\left[\frac{1}{4}\right] - \sum_{k=0}^{\frac{q-1}{2}} \left[\frac{1}{4q}+\frac{k}{q}\right] \right) - \sum_{j=1}^{\frac{q-1}{2}} \left(\left[\frac{j}{q}\right] + \left[-\frac{1}{2q}+\frac{j}{q}\right] - \left[\frac{j}{2q}\right] - \left[-\frac{1}{4q}+\frac{j}{2q}\right] \right),$$

e. g.,

$$\mathbf{a}_{2\cdot 3} := \left[\frac{1}{4}\right] - \left[\frac{5}{12}\right] - \left[\frac{1}{3}\right].$$

Let

$$\sin : \mathcal{A} \to \mathbb{Q}^{\mathrm{ab}\,\times}$$

be the unique homomorphism such that

$$\sin[a] = \begin{cases} 2\sin \pi a \left(= \left|1-e^{2\pi i a}\right|\right) & \text{if } 0<a<1 \\ 1 & \text{if } a=0 \end{cases} \qquad (a \in \mathbb{Q}\cap[0,1)).$$

We prove that the family of real numbers

$$\left\{\sqrt{\ell}\right\}_{\ell:\mathrm{prime}} \bigcup \{\sin \mathbf{a}_{pq}\}_{\substack{p,q:\mathrm{prime}\\ p<q}}$$

projects to a $\mathbb{Z}/2\mathbb{Z}$-basis of the group $H^0\left(G^{\mathrm{ab}}, \mathbb{Q}^{\mathrm{ab}\,\times}/\mathbb{Q}^{\mathrm{ab}\,\times 2}\right)$ and hence that

$$\mathbb{Q}^{\mathrm{ab}\,+\epsilon} = \mathbb{Q}^{\mathrm{ab}} \left(\left\{\sqrt[4]{\ell}\right\}_{\ell:\mathrm{prime}} \bigcup \left\{\sqrt{\sin \mathbf{a}_{pq}}\right\}_{\substack{p,q:\mathrm{prime}\\ p<q}} \right).$$

We actually prove more. We define a canonical injective homomorphism

$$\mathbf{D} : H^0\left(G^{\mathrm{ab}}, \mathbb{Q}^{\mathrm{ab}\,\times}/\mathbb{Q}^{\mathrm{ab}\,\times 2}\right) \to \bigwedge^2 H^1(G^{\mathrm{ab}}, \mathbb{Z}/2\mathbb{Z}) \left(= \bigwedge^2 \left(\mathbb{Q}^\times/\mathbb{Q}^{\times 2}\right) \right)$$

and exhibit a preimage for each element of a natural $\mathbb{Z}/2\mathbb{Z}$-basis of the target. (In particular, it turns out that $\mathbf{D}$ is an isomorphism.) Our main results are the "Auxiliary Formula" (§3.4) and the "Main Formula" (§4.4.1). Our main technical tools are the "Log Wedge Formula" (§3.3) and a family of combinatorial identities of "Das type" (§4.2).

Now the fact that the homomorphism $\mathbf{D}$ is an isomorphism is not really new: one can easily deduce it from [Fröhlich 1983, Theorem 4.10, p. 56]. (See Remark 4.4.3 below for a sketch of the deduction.) Rather, it is the explicit procedure developed here for inverting the map $\mathbf{D}$ that is really new.

A key role is played in this paper by the *universal odd ordinary distribution* U^-, namely the quotient of $\mathcal{A}$ by the subgroup generated by all expressions of the form

$$[a] - \sum_{i=0}^{N-1} \left[\frac{a+i}{N}\right], \quad [a]+[1-a] \quad (a\in\mathbb{Q},\ N\text{: positive integer}).$$

A study of U^- and related objects was made in [Kubert 1978a] and [Kubert 1978b], building on [Sinnott 1978]. In particular, it was proved that the torsion subgroup of U^- is killed by 2. Kubert's results combined with the idea behind the algebraicity criterion of [Koblitz-Ogus 1979] yield a more or less mechanical procedure for determining whether a given element of $\mathcal{A}$ represents a torsion element of U^-. The double complex method of [Anderson 1999] yields a canonical $\mathbb{Z}/2\mathbb{Z}$-basis for the torsion subgroup of U^- indexed by finite sets of prime numbers of even cardinality. In [Das 2000, Sec. 3 and Sec. 9] it is proved that the family $\{\mathbf{a}_{pq}\}_{2<p<q}$ represents the "two-odd-prime" part of the canonical basis. The method of Das can easily be modified to prove that the family $\{\mathbf{a}_{pq}\}_{p<q}$ represents the "two-prime" part of the canonical basis.

The torsion subgroup of U^- plays an important role in the theory of algebraic Γ-monomials. Let

$$\Gamma : \mathcal{A} \to \mathbb{R}^\times$$

be the unique homomorphism such that

$$\Gamma([a]) = \begin{cases} \sqrt{2\pi}/\Gamma(a) & \text{if } 0 < a < 1 \\ 1 & \text{if } a = 0 \end{cases} \qquad (a \in \mathbb{Q} \cap [0,1)).$$

Now fix $\mathbf{a} \in \mathcal{A}$ representing a torsion element of U^-. By straightforward manipulation of standard functional equations satisfied by $\Gamma(s)$ one verifies that $\Gamma(\mathbf{a})$ is an algebraic number. Numbers of the form $\Gamma(\mathbf{a})$ are the so called *algebraic Γ-monomials.* Under mild hypotheses, a reciprocity law [Deligne-Milne-Ogus-Shih 1982, Thm. 7.15, p. 91] due to Deligne links $\Gamma(\mathbf{a})$ to an analogously defined Jacobi sum Hecke character. Deligne reciprocity is the main motivation for studying algebraic Γ-monomials.

In [Das 2000] the algebraic Γ-monomials were related to algebraic sine-monomials and properties of the latter were investigated. Again fix $\mathbf{a} \in \mathcal{A}$ representing a torsion element of U^-. By [Das 2000, Thm. 6], under mild hypotheses, $\Gamma(\mathbf{a})$ factors as the root of a rational number times $\sqrt{\sin \mathbf{a}}$, and the factorization can in principle be worked out explicitly. For example, one has

$$\Gamma(\mathbf{a}_{3\cdot 5}) = \frac{\Gamma\left(\frac{4}{15}\right)\Gamma\left(\frac{1}{5}\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{15}\right)} = 3^{-\frac{1}{5}}5^{\frac{1}{12}}\sqrt{\frac{\sin\frac{\pi}{3}\cdot\sin\frac{2\pi}{15}}{\sin\frac{4\pi}{15}\cdot\sin\frac{\pi}{5}}} = 3^{-\frac{1}{5}}5^{\frac{1}{12}}\sqrt{\sin \mathbf{a}_{3\cdot 5}}$$

and

$$\Gamma(\mathbf{a}_{2\cdot 3}) = \frac{\Gamma\left(\frac{5}{12}\right)\Gamma\left(\frac{1}{3}\right)}{\sqrt{2\pi}\,\Gamma\left(\frac{1}{4}\right)} = 2^{-\frac{1}{4}}3^{\frac{1}{8}}\sqrt{\frac{\sin\frac{\pi}{4}}{2\cdot\sin\frac{5\pi}{12}\cdot\sin\frac{\pi}{3}}} = 2^{-\frac{1}{4}}3^{\frac{1}{8}}\sqrt{\sin \mathbf{a}_{2\cdot 3}}.$$

Indeed, for all primes $p < q$ one has

$$\frac{\Gamma(\mathbf{a}_{pq})}{\sqrt{\sin \mathbf{a}_{pq}}} = \begin{cases} p^{\frac{-(q-1)^2}{16q}}q^{\frac{(p-1)^2}{16p}} & \text{if } 2 < p, \\ 2^{-\frac{q-1}{8}}q^{\frac{1}{8}} & \text{if } 2 = p, \end{cases}$$

as is explained below in Remark 4.3.3. By [Das 2000, Thm. 11], under mild hypotheses, the extension $\mathbb{Q}^{\text{ab}}\left(\sqrt{\sin \mathbf{a}}\right)/\mathbb{Q}$ is Galois, and in fact the latter is true in general; see Proposition 4.1.8 below. By [Das 2000, Thm. 22], if $\mathbf{a}$ represents an element of the canonical basis for the torsion subgroup of U^- indexed by a finite set of odd primes of cardinality at least four, then $\sqrt{\sin \mathbf{a}} \in \mathbb{Q}^{\text{ab}}$.

The theory of [Das 2000] begs the question as to the nature of the Galois extensions of the form $\mathbb{Q}^{\mathrm{ab}}\left(\sqrt{\sin \mathbf{a}_{pq}}\right)/\mathbb{Q}$. In order to begin answering that question, Das calculated (see [Das 2000, end of Sec. 16]) the structure of the central extension

$$1 \to \{\pm 1\} \to \mathrm{Gal}\left(\mathbb{Q}\left(e^{\frac{2\pi i}{60}}, \sqrt{\sin \mathbf{a}_{3\cdot 5}}\right)/\mathbb{Q}\right) \to (\mathbb{Z}/60\mathbb{Z})^{\times} \to 1$$

explicitly and in particular proved that the middle group is nonabelian. That calculation of Das was the primary inspiration for this paper.

Another important inspiration was the striking result [Seo 2001, Prop. 2.3] according to which, for all odd primes $p < q$, one has

$$(-1)^{v_p(\sin \mathbf{a}_{pq})} = \left(\begin{array}{c} q \\ p \end{array}\right), \quad (-1)^{v_q(\sin \mathbf{a}_{pq})} = \left(\begin{array}{c} p \\ q \end{array}\right),$$

where $\left(\begin{array}{c} \cdot \\ \cdot \end{array}\right)$ is the Legendre symbol, v_p is any additive valuation of $\mathbb{Q}^{\mathrm{ab}}$ above p such that $v_p\left(1 - e^{\frac{2\pi i}{p}}\right) = 1$, and v_q is analogously chosen above q. It follows that if $\left(\begin{array}{c} p \\ q \end{array}\right) = -1$ or $\left(\begin{array}{c} q \\ p \end{array}\right) = -1$, then $\sqrt{\sin \mathbf{a}_{pq}}$ cannot belong to $\mathbb{Q}^{\mathrm{ab}}$. Seo's result greatly encouraged the author to attempt the calculations detailed in this paper. In Remark 5.2.7 we briefly sketch another proof of Seo's formula.

2. Group-theoretical background

2.1. **The category of $\mathbb{N}$-profinite groups.** We say that a topological group G is $\mathbb{N}$-*profinite* if G is compact and there exists a chain $U_1 \supseteq U_2 \supseteq U_3 \supseteq \dots$ of open normal subgroups of G forming a neighborhood base at the origin. A *morphism* of $\mathbb{N}$-profinite groups is by definition a continuous homomorphism of topological groups; thus the $\mathbb{N}$-profinite groups form a category. The Galois group of any Galois extension of countable fields is $\mathbb{N}$-profinite, and so the category of $\mathbb{N}$-profinite groups is large enough for our number-theoretic purposes. (But the Galois group of an algebraic closure of the field of rational functions in one variable over the field of complex numbers is *not* $\mathbb{N}$-profinite.) One can show that a compact topological group G is $\mathbb{N}$-profinite if and only if G has a countable neighborhood base consisting of open compact sets.

Proposition 2.2. *Let G be an $\mathbb{N}$-profinite group. Let H be a closed normal subgroup of G. There exists a function $\vartheta : G \to G$ with the following properties:*

- *ϑ is continuous.*
- *ϑ is constant on cosets of H in G.*
- *$\vartheta(\sigma) \in \sigma H$ for all $\sigma \in G$.*

Proof. Fix a descending chain $U_1 \supseteq U_2 \supseteq U_3 \supseteq \dots$ of open normal subgroups of G forming a neighborhood base at the origin. Choose a set S_1 of representatives for the cosets of HU_1 in G. For each index $i > 1$ choose a set $S_i \subseteq U_{i-1}$ of representatives for the cosets of HU_i in HU_{i-1}. For each index i let $\vartheta_i : G \to G$ be the unique map collapsing each coset of HU_i in G to its unique representative in the finite set $S_1 \cdots S_i$. The uniform limit ϑ of the maps ϑ_i exists and has all the desired properties. □

2.3. **Cochains, cocycles, coboundaries and cohomology classes.** Let G be a topological group. Let A be a topological abelian group equipped with a continuous left G-action. For each nonnegative integer n, let $C^n(G,A)$ denote the group of continuous functions

$$a = ((\sigma_1,\dots,\sigma_n) \mapsto a_{\sigma_1,\dots,\sigma_n}) : G^n \to A,$$

and equip the graded group

$$C^*(G,A) := \bigoplus C^n(G,A)$$

with a differential δ of degree 1 by the standard rule

$$(\delta a)_{\sigma_1,\dots,\sigma_{n+1}} = \sigma_1 a_{\sigma_2,\dots,\sigma_{n+1}} + \cdots + (-1)^i a_{\sigma_1,\dots,\sigma_i\sigma_{i+1},\dots,\sigma_{n+1}} + \cdots + (-1)^{n+1} a_{\sigma_1,\dots,\sigma_n}.$$

Elements of $C^n(G,A)$ will be called *cochains* of degree n. In low degree the differential δ takes the form

$$\begin{array}{rcll} (\delta a)_{\sigma_1} & = & \sigma_1 a - a & \text{if } n=0,\\ (\delta a)_{\sigma_1,\sigma_2} & = & \sigma_1 a_{\sigma_2} - a_{\sigma_1\sigma_2} + a_{\sigma_1}, & \text{if } n=1,\\ (\delta a)_{\sigma_1,\sigma_2,\sigma_3} & = & \sigma_1 a_{\sigma_2,\sigma_3} - a_{\sigma_1\sigma_2,\sigma_3} + a_{\sigma_1,\sigma_2\sigma_3} - a_{\sigma_1,\sigma_2} & \text{if } n=2. \end{array}$$

Elements of the groups

$$\begin{array}{rcl} Z^n(G,A) & := & \ker\left(\delta : C^n(G,A) \to C^{n+1}(G,A)\right),\\ B^n(G,A) & := & \mathrm{image}\left(\delta : C^{n-1}(G,A) \to C^n(G,A)\right),\\ H^n(G,A) & := & Z^n(G,A)/B^n(G,A) \end{array}$$

will be called *n-cocycles*, *n-coboundaries* and *cohomology classes of degree n*, respectively. If G is profinite and A is discretely topologized, then $H^n(G,A)$ is the usual Galois cohomology group of degree n.

2.4. **Cup product.** Let G be a topological group and let A be a commutative topological ring equipped with a continuous G-action. Given $a \in C^p(G,A)$ and $b \in C^q(G,A)$, put

$$(a \cup b)_{\sigma_1,\dots,\sigma_{p+q}} = a_{\sigma_1,\dots,\sigma_p} \cdot \sigma_1 \cdots \sigma_p b_{\sigma_{p+1},\dots,\sigma_{p+q}},$$

thereby defining the *cup product* $a \cup b \in C^{p+q}(G,A)$ of the cochains a and b. For $p = q = 1$ one has

$$(a \cup b)_{\sigma_1,\sigma_2} = a_{\sigma_1} \cdot \sigma_1 b_{\sigma_2}.$$

In general one has

$$\delta(a \cup b) = (\delta a) \cup b + (-1)^p a \cup (\delta b)$$

and thus the cup product construction induces a product in $H^*(G,A)$.

2.5. **Extensions in the category of $\mathbb{N}$-profinite groups.** Let G be an $\mathbb{N}$-profinite group and let A be an $\mathbb{N}$-profinite abelian group equipped with a continuous left G-action. An exact sequence

$$\Sigma : \quad 1 \to A \xrightarrow{i} U \xrightarrow{p} G \to 1$$

in the category of $\mathbb{N}$-profinite groups such that

$$ui(a)u^{-1} = i(p(u)a) \quad (u \in U,\ a \in A)$$

is called an *extension* of G by A in the category of $\mathbb{N}$-profinite groups. If A is a trivial G-module, the homomorphism $i : A \to U$ takes values in the center of U; in such a case one says that Σ is a *central* extension. Extensions Σ and Σ' of G by A are said to be *isomorphic* if there exists an isomorphism $\psi : U \xrightarrow{\sim} U'$ of $\mathbb{N}$-profinite groups such that $\psi \circ i = i'$ and $p' \circ \psi = p$. A *set-theoretic splitting* $\vartheta : G \to U$ of Σ is a continuous map such that $p \circ \vartheta = \mathrm{id}_G$. Set-theoretic splittings exist by Proposition 2.2. We say that Σ is *trivial* if there exists a set-theoretic splitting that is also a group homomorphism. Any two trivial extensions of G by A are isomorphic. Each set-theoretic splitting ϑ of Σ gives rise to a cochain $a \in C^2(G, A)$ by the rule

$$\vartheta(\sigma)\vartheta(\tau) = i(a_{\sigma,\tau})\vartheta(\sigma\tau) \quad (\sigma, \tau \in G).$$

Associativity of the group law in U forces a to be a 2-cocycle. The cohomology class of the 2-cocycle a depends only the isomorphism class of the extension Σ and accordingly is denoted $[\Sigma]$. Every 2-cocycle representing the cohomology class $[\Sigma]$ arises from some set-theoretic splitting of Σ. The construction $\Sigma \mapsto [\Sigma]$ puts isomorphism classes of extensions of G by A in the category of $\mathbb{N}$-profinite groups into bijective correspondence with $H^2(G, A)$ and sends every trivial extension to 0.

2.6. **Almost abelian $\mathbb{N}$-profinite groups.** Let G be an $\mathbb{N}$-profinite group and put

$$[\sigma, \tau] := \sigma\tau\sigma^{-1}\tau^{-1} \quad (\sigma, \tau \in G).$$

The *abelianization* G^{ab} of G is the abelian quotient of G universal for morphisms to abelian $\mathbb{N}$-profinite groups. One has

$$G^{\mathrm{ab}} = G/[G, G],$$

where $[G, G]$ is the closed subgroup of G topologically generated by the set

$$\{[\sigma, \tau] \mid \sigma, \tau \in G\}.$$

We say that G is *almost abelian* if $[\sigma, \tau]$ is central and 2-torsion for all $\sigma, \tau \in G$. We define $G^{\mathrm{ab}+\epsilon}$ to be the almost abelian quotient of G universal for morphisms to almost abelian $\mathbb{N}$-profinite groups. One has

$$G^{\mathrm{ab}+\epsilon} = G/[G, G]^\epsilon,$$

where $[G, G]^\epsilon$ is the closed subgroup of $[G, G]$ topologically generated by the set

$$\{[\sigma, \tau]^2 \mid \sigma, \tau \in G\} \cup \{[\sigma, [\tau, \eta]] \mid \sigma, \tau, \eta \in G\}.$$

Put

$$G^\epsilon := [G, G]/[G, G]^\epsilon.$$

The preceding constructions fit together to form a canonical central extension

$$\Sigma_G^\epsilon : \quad 1 \to G^\epsilon \to G^{\mathrm{ab}+\epsilon} \to G^{\mathrm{ab}} \to 1$$

of $\mathbb{N}$-profinite groups. We remark that the Pontryagin dual of G^ϵ is canonically isomorphic to $H^1(G^\epsilon, \mathbb{Z}/2\mathbb{Z})$ and that the latter is a vector space over $\mathbb{Z}/2\mathbb{Z}$ of at most countably infinite dimension.

Proposition 2.7. *Let G be an $\mathbb{N}$-profinite group. Let $H^2(G^{\mathrm{ab}}, \mathbb{Z}/2\mathbb{Z})^-$ be the quotient of $H^2(G^{\mathrm{ab}}, \mathbb{Z}/2\mathbb{Z})$ by the subgroup consisting of all cohomology classes of the form $[\Sigma]$ for some extension Σ of G^{ab} by $\mathbb{Z}/2\mathbb{Z}$ the middle group of which is abelian. For each $c \in H^1(G^\epsilon, \mathbb{Z}/2\mathbb{Z})$, view c as a continuous homomorphism $G^\epsilon \to \mathbb{Z}/2\mathbb{Z}$, and let*

$$(a \mapsto c \circ a) : H^2(G^{\mathrm{ab}}, G^\epsilon) \to H^2(G^{\mathrm{ab}}, \mathbb{Z}/2\mathbb{Z})$$

be the map induced by c in degree 2 cohomology. The homomorphism

$$(c \mapsto c \circ [\Sigma_G^\epsilon]) : H^1(G^\epsilon, \mathbb{Z}/2\mathbb{Z}) \to H^2(G^{\mathrm{ab}}, \mathbb{Z}/2\mathbb{Z})$$

followed by the projection

$$H^2(G^{\mathrm{ab}}, \mathbb{Z}/2\mathbb{Z}) \to H^2(G^{\mathrm{ab}}, \mathbb{Z}/2\mathbb{Z})^-$$

is injective.

Proof. Fix $0 \neq c \in H^1(G^\epsilon, \mathbb{Z}/2\mathbb{Z})$ and consider the extension

$$\Sigma : \ 0 \to \mathbb{Z}/2\mathbb{Z} \to G^{\mathrm{ab}+\epsilon}/\ker c \to G^{\mathrm{ab}} \to 1.$$

Then

$$[\Sigma] = c \circ [\Sigma_G^\epsilon]$$

and the middle group of Σ is nonabelian by definition of G^{ab}. □

Lemma 2.8. *Let $G = \prod_{i=1}^\infty G_i$ be an abelian $\mathbb{N}$-profinite group decomposed as a countable product of cyclic profinite groups G_i. Let an extension*

$$\Sigma : \ \ 0 \to \mathbb{Z}/2\mathbb{Z} \to U \to G \to 1$$

of $\mathbb{N}$-profinite groups be given. Let

$$a \in Z^2(G, \mathbb{Z}/2\mathbb{Z})$$

be any 2-cocycle representing the cohomology class

$$[\Sigma] \in H^2(G, \mathbb{Z}/2\mathbb{Z}).$$

For each index i fix $\sigma_i \in G$ with the following properties:

- *σ_i projects to a topological generator of G_i.*
- *σ_i projects to $1 \in G_j$ for all indices $j \neq i$.*

Put

$$\alpha_{ij} :\equiv a_{\sigma_i,\sigma_j} + a_{\sigma_j,\sigma_i} \bmod 2$$

for all indices $i < j$. Then the following hold:

(1) *α_{ij} vanishes for all but finitely many pairs of indices.*
(2) *α_{ij} vanishes if and only if σ_i and σ_j have liftings to U that commute.*
(3) *α_{ij} depends only on the cohomology class $[\Sigma]$.*
(4) *If σ_i or σ_j is the square of an element of G, then α_{ij} vanishes.*
(5) *The middle group U of Σ is abelian if and only if all the α_{ij} vanish.*

Proof. Fix a set-theoretic splitting ϑ of Σ giving rise to the 2-cocycle a.

1. By hypothesis one has

$$(\delta a)_{1,1,\sigma} - (\delta a)_{\sigma,1,1} \equiv a_{1,\sigma} + a_{\sigma,1} \equiv 0 \bmod 2 \quad (\sigma \in G)$$

and there exists an open normal subgroup $V \subseteq G$ such that $a_{\sigma,\tau}$ depends only on the pair of cosets $(\sigma V, \tau V)$. It follows that if either $\sigma_i \in V$ or $\sigma_j \in V$, then α_{ij} vanishes. But one has $\sigma_i \in V$ for all but finitely many indices i.

2. The following statements are equivalent:

- The coefficient α_{ij} vanishes.
- $\vartheta(\sigma_i)$ and $\vartheta(\sigma_j)$ commute.
- σ_i and σ_j have liftings to U that commute.
- Any liftings of σ_i and σ_j to U commute.

3. By what we have already proved, the coefficients α_{ij} depend only on the isomorphism class of the extension Σ and hence only on the cohomology class $[\Sigma]$.

4. Suppose, say, that $\sigma_i = \tau^2$ for some $\tau \in G$. Then

$$\vartheta(\tau)^2 c = \vartheta(\sigma_i), \quad \vartheta(\tau)\vartheta(\sigma_j) = \vartheta(\sigma_j)\vartheta(\tau)c'$$

where c and c' belong to the center of U, and hence

$$\vartheta(\sigma_i)\vartheta(\sigma_j) = \vartheta(\tau)\vartheta(\sigma_j)\vartheta(\tau)c'c = \vartheta(\sigma_j)\vartheta(\tau)c'\vartheta(\tau)c'c = \vartheta(\sigma_j)\vartheta(\sigma_i),$$

i. e., $\vartheta(\sigma_i)$ and $\vartheta(\sigma_j)$ commute, and hence the coefficient α_{ij} vanishes.

5. All the coefficients α_{ij} vanish if and only if the elements of the family $\{\vartheta(\sigma_i)\}$ commute among themselves. But the union of the latter family with $\ker(U \to G)$ generates U topologically. □

Proposition 2.9. *Let G be an abelian $\mathbb{N}$-profinite group admitting decomposition as a countably infinite product of cyclic profinite groups. The cup product induces an isomorphism*

$$\bigwedge^2 H^1(G, \mathbb{Z}/2\mathbb{Z}) \tilde{\to} H^2(G, \mathbb{Z}/2\mathbb{Z})^-$$

of vector spaces over $\mathbb{Z}/2\mathbb{Z}$, where $H^2(G, \mathbb{Z}/2\mathbb{Z})^-$ is as defined in Proposition 2.7.

Proof. For each $e \in Z^1(G, \mathbb{Z}/2\mathbb{Z}) = H^1(G, \mathbb{Z}/2\mathbb{Z})$, the cohomology class of the 2-cocycle $e \cup e$ vanishes in the quotient $H^2(G, \mathbb{Z}/2\mathbb{Z})^-$ by Lemma 2.8. Therefore the cup product induces a well defined map

$$\bigwedge^2 H^1(G, \mathbb{Z}/2\mathbb{Z}) \to H^2(G, \mathbb{Z}/2\mathbb{Z})^-.$$

Fix a decomposition $G = \prod_{i=1}^\infty G_i$ of G as a countably infinite product of cyclic profinite groups G_i and for each index i fix $\sigma_i \in G$ with the properties specified in Lemma 2.8. Let I be the set of indices i such that $G_i \not\subseteq G^2$. For each index $i \in I$ there exists a unique 1-cocycle

$$e_i \in Z^1(G, \mathbb{Z}/2\mathbb{Z}) = H^1(G, \mathbb{Z}/2\mathbb{Z})$$

such that

$$(e_i)_{\sigma_j} \equiv \delta_{ij} \bmod 2$$

for all indices j. Then the family

$$\{e_i\}_{i \in I}$$

is a $\mathbb{Z}/2\mathbb{Z}$-basis for $H^1(G, \mathbb{Z}/2\mathbb{Z})$ and hence the family

$$\{e_i \wedge e_j\}_{\substack{i,j \in I \\ i<j}}$$

is a $\mathbb{Z}/2\mathbb{Z}$-basis for $\bigwedge^2 H^1(G, \mathbb{Z}/2\mathbb{Z})$. By Lemma 2.8 the cohomology classes of the 2-cocycles belonging to the family

$$\{e_i \cup e_j\}_{\substack{i,j \in I \\ i<j}}$$

remain independent over $\mathbb{Z}/2\mathbb{Z}$ in $H^2(G, \mathbb{Z}/2\mathbb{Z})^-$. Therefore the map in question is injective. For any $a \in Z^2(G, \mathbb{Z}/2\mathbb{Z})$ the cohomology class of the two-cocycle

$$a + \sum_{\substack{i,j \in I \\ i<j}} (a_{\sigma_i, \sigma_j} + a_{\sigma_j, \sigma_i}) e_i \cup e_j$$

vanishes in the quotient $H^2(G,\mathbb{Z}/2\mathbb{Z})^-$ by Lemma 2.8. Therefore the map in question is surjective. □

Proposition 2.10. *For any* $\mathbb{N}$*-profinite group* G*, the group* $[G,G]^\epsilon$ *is the intersection of all relatively open subgroups of* $[G,G]$ *that are of index* 2 *in* $[G,G]$ *and normal in* G*.*

Proof. Let H be the intersection of all relatively open subgroups of $[G,G]$ that are of index 2 in $[G,G]$ and normal in G. Since the quotient G/H is almost abelian, one has $[G,G]^\epsilon \subseteq H$. Since the continuous linear functionals $G^\epsilon \to \mathbb{Z}/2\mathbb{Z}$ separate points, one has $H \subseteq [G,G]^\epsilon$. □

3. The homomorphism D

3.1. The setting for the rest of the paper.

3.1.1. Let $\overline{\mathbb{Q}}$ be the algebraic closure of the field $\mathbb{Q}$ of rational numbers in the field $\mathbb{C}$ of complex numbers. Let $\mathbb{Q}^{\mathrm{ab}}$ be the compositum of all subfields $K \subseteq \overline{\mathbb{Q}}$ abelian over $\mathbb{Q}$. Let $\mathbb{Q}^{\mathrm{ab}+\epsilon}$ be the compositum of all subfields $K \subseteq \overline{\mathbb{Q}}$ that are quadratic over $\mathbb{Q}^{\mathrm{ab}}$ and Galois over $\mathbb{Q}$. (In the terminology of [Das 2000] such extensions $K/\mathbb{Q}$ would be called *double coverings* of $\mathbb{Q}^{\mathrm{ab}}$.) Let G denote the Galois group of $\overline{\mathbb{Q}}/\mathbb{Q}$. One has

$$G^{\mathrm{ab}} = \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}), \quad G^{\mathrm{ab}+\epsilon} = \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}+\epsilon}/\mathbb{Q}), \quad G^{\epsilon} = \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}+\epsilon}/\mathbb{Q}^{\mathrm{ab}})$$

by Proposition 2.10 and the definitions.

3.1.2. Put

$$S := \{-1\} \cup \{\text{rational prime numbers}\}.$$

The family

$$\{p \bmod \mathbb{Q}^{\times 2}\}_{p\in S}$$

is a $\mathbb{Z}/2\mathbb{Z}$-basis for $\mathbb{Q}^\times/\mathbb{Q}^{\times 2}$. For each $p \in S$ we define a 1-cocycle

$$e_p \in Z^1(G^{\mathrm{ab}},\mathbb{Z}/2\mathbb{Z}) = H^1(G^{\mathrm{ab}},\mathbb{Z}/2\mathbb{Z})$$

by the rule

$$\sigma\sqrt{p} = (-1)^{(e_p)_\sigma}\sqrt{p} \quad (\sigma \in G^{\mathrm{ab}}).$$

By Kummer theory the family

$$\{e_p\}_{p\in S}$$

is a $\mathbb{Z}/2\mathbb{Z}$-basis for $H^1(G^{\mathrm{ab}},\mathbb{Z}/2\mathbb{Z})$.

3.1.3. Let $G_{-1} \subset G^{\mathrm{ab}}$ be the subgroup generated by the restriction of complex conjugation to $\mathbb{Q}^{\mathrm{ab}}$. For each odd prime p, let $G_p \subset G^{\mathrm{ab}}$ be the inertia subgroup at p. Let $G_2 \subset G^{\mathrm{ab}}$ be the subgroup of the inertia subgroup at 2 fixing $\sqrt{-1}$. For all $p \in S$ the profinite group G_p is cyclic and has nonzero 2-rank. By the Kronecker-Weber theorem the family $\{G_p\}_{p\in S}$ is the family of "coordinate axes" for a decomposition of G^{ab} into a product of cyclic profinite groups indexed by S. It follows by Proposition 2.9 that we can identify $H^2(G^{\mathrm{ab}},\mathbb{Z}/2\mathbb{Z})^-$ with $\bigwedge^2 H^1(G^{\mathrm{ab}},\mathbb{Z}/2\mathbb{Z})$. Note that the family

$$\{e_p \wedge e_q\}_{\substack{p,q\in S\\ p<q}}$$

is a $\mathbb{Z}/2\mathbb{Z}$-basis for $\bigwedge^2 H^1(G^{\mathrm{ab}},\mathbb{Z}/2\mathbb{Z})$.

3.1.4. For each $p \in S$ we arbitrarily fix a topological generator σ_p of G_p. Note that σ_{-1} is the restriction to $\mathbb{Q}^{\mathrm{ab}}$ of complex conjugation. Note that

$$(e_p)_{\sigma_q} \equiv \delta_{pq} \bmod 2 \quad (p, q \in S),$$

i. e., the families $\{\sigma_p\}_{p\in S}$ and $\{e_p\}_{p\in S}$ are in a convenient way dual.

3.2. **The definition of D.** We define

$$\mathbf{D} : H^0\left(G^{\mathrm{ab}}, \mathbb{Q}^{\mathrm{ab}\,\times}/\mathbb{Q}^{\mathrm{ab}\times 2}\right) \to \bigwedge^2 H^1(G^{\mathrm{ab}}, \mathbb{Z}/2\mathbb{Z})$$

to be the injective homomorphism obtained as the composite of the following three homomorphisms:

- The canonical isomorphism
$$H^0(G^{\mathrm{ab}}, \mathbb{Q}^{\mathrm{ab}\,\times}/\mathbb{Q}^{\mathrm{ab}\,\times 2}) \xrightarrow{\sim} H^1(G^\epsilon, \mathbb{Z}/2\mathbb{Z})$$
provided by Kummer theory combined with Proposition 2.10.

- The canonical injective homomorphism
$$H^1(G^\epsilon, \mathbb{Z}/2\mathbb{Z}) \to H^2(G^{\mathrm{ab}}, \mathbb{Z}/2\mathbb{Z})^-$$
provided by Proposition 2.7.

- The inverse of the canonical isomorphism
$$\bigwedge^2 H^1(G^{\mathrm{ab}}, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\sim} H^2(G^{\mathrm{ab}}, \mathbb{Z}/2\mathbb{Z})^-$$
provided by Proposition 2.9.

The map $\mathbf{D}$ is the focus of our investigation.

3.3. **The Log Wedge Formula.** We carry out the diagram-chase necessary to make the map $\mathbf{D}$ usably explicit.

3.3.1. Fix $u \in \mathbb{Q}^{\mathrm{ab}\,\times}$ such that

$$(u \bmod \mathbb{Q}^{\mathrm{ab}\,\times 2}) \in H^0(G^{\mathrm{ab}}, \mathbb{Q}^{\mathrm{ab}\,\times}/\mathbb{Q}^{\mathrm{ab}\,\times 2}).$$

The 1-cocycle

$$c \in Z^1(G^\epsilon, \mathbb{Z}/2\mathbb{Z}) = H^1(G^\epsilon, \mathbb{Z}/2\mathbb{Z})$$

defined by the rule

$$\sigma\sqrt{u} = (-1)^{c_\sigma}\sqrt{u} \quad (\sigma \in G^\epsilon)$$

is the one corresponding via Kummer theory to the class $u \bmod \mathbb{Q}^{\mathrm{ab}\,\times 2}$.

3.3.2. Let

$$\vartheta : G^{\mathrm{ab}} \to G^{\mathrm{ab}\,+\epsilon}$$

be any set-theoretic splitting of the canonical central extension Σ_G^ϵ. The 2-cocycle

$$a \in Z^2(G^{\mathrm{ab}}, \mathbb{Z}/2\mathbb{Z})$$

defined by the formula

$$a_{\sigma,\tau} = c_{\vartheta(\sigma)\vartheta(\tau)\vartheta(\sigma\tau)^{-1}} \quad (\sigma, \tau \in G^{\mathrm{ab}})$$

represents the cohomology class $c \circ [\Sigma_G^\epsilon] \in H^2(G^{\mathrm{ab}}, \mathbb{Z}/2\mathbb{Z})$.

3.3.3. Let

$$\{v_\sigma\}_{\sigma\in G^{\mathrm{ab}}}$$

be a locally constant family of elements of $\mathbb{Q}^{\mathrm{ab}\,\times}$ such that

$$\sigma u = v_\sigma^2 u \quad (\sigma \in G^{\mathrm{ab}}).$$

Define a 1-cochain

$$f \in C^1(G^{\mathrm{ab}}, \mathbb{Z}/2\mathbb{Z})$$

by the rule

$$\vartheta(\sigma)\sqrt{u} = (-1)^{f_\sigma} v_\sigma \sqrt{u} \quad (\sigma \in G^{\mathrm{ab}}).$$

A straightforward if tedious calculation gives the formula

$$(-1)^{a_{\sigma,\tau}}\sqrt{u} = \vartheta(\sigma)\vartheta(\tau)\vartheta(\sigma\tau)^{-1}\sqrt{u} = (-1)^{(\delta f)_{\sigma,\tau}}(\delta v)_{\sigma,\tau}\sqrt{u} \quad (\sigma,\tau \in G^{\mathrm{ab}}).$$

The latter says that the 2-cocycle

$$\delta \log_{-1} v \in Z^2(G^{\mathrm{ab}}, \mathbb{Z}/2\mathbb{Z})$$

also represents the cohomology class $c \circ [\Sigma_G^\epsilon]$, where

$$\log_{-1} : \{\pm 1\} \xrightarrow{\sim} \mathbb{Z}/2\mathbb{Z}$$

is the evident (and only) isomorphism.

3.3.4. Put

$$\alpha_{pq} := \frac{(\delta v)_{\sigma_p,\sigma_q}}{(\delta v)_{\sigma_q,\sigma_p}} = \frac{\sigma_p v_{\sigma_q}/v_{\sigma_q}}{\sigma_q v_{\sigma_p}/v_{\sigma_p}} \in \{\pm 1\} \quad \left(\begin{array}{c} p,q \in S \\ p<q \end{array}\right).$$

By Lemma 2.8 one has $\alpha_{pq} = 1$ for all but finitely many pairs $\{p < q\}$ and the image in $H^2(G^{\mathrm{ab}}, \mathbb{Z}/2\mathbb{Z})^-$ of the cohomology class of the 2-cocycle

$$\delta \log_{-1} v + \sum_{\substack{p,q\in S\\ p<q}} \log_{-1}\alpha_{pq} \cdot e_p \cup e_q$$

vanishes. We arrive finally at the result

$$\mathbf{D}(u \bmod \mathbb{Q}^{\mathrm{ab}\,\times 2}) = \sum_{\substack{p,q\in S\\ p<q}} \log_{-1}\alpha_{pq} \cdot e_p \wedge e_q$$

to which in the sequel we refer as the *Log Wedge Formula*.

3.4. **The Auxiliary Formula.** Here is an easy first application of the Log Wedge Formula. Let ℓ be any prime number. One has $\sqrt{\ell} \in \mathbb{Q}(\zeta_{4\ell})^\times$ and hence there exists a unique function

$$(\sigma \mapsto v_\sigma) : G^{\mathrm{ab}} \to \{1, i\}$$

factoring through $\mathrm{Gal}(\mathbb{Q}(\zeta_{4\ell})/\mathbb{Q})$ such that

$$\sigma\sqrt{\ell} = v_\sigma^2 \sqrt{\ell} \quad \left(\text{equivalently: } v_\sigma^2 = (-1)^{(e_\ell)_\sigma}\right) \qquad (\sigma \in G^{\mathrm{ab}}).$$

For all $p, q \in S$ one has

$$\sigma_p v_{\sigma_q}/v_{\sigma_q} = \begin{cases} -1 & \text{if } p = -1 \text{ and } q = \ell, \\ 1 & \text{otherwise,} \end{cases}$$

and hence

$$\mathbf{D}\left(\sqrt{\ell} \bmod \mathbb{Q}^{\mathrm{ab}\,\times 2}\right) = e_{-1} \wedge e_\ell$$

by the Log Wedge Formula. We refer to the result above as the *Auxiliary Formula*.

3.5. **Motivating example.**

3.5.1. The main work of the paper, which commences formally in §4, is the construction of sine-monomial identities shedding light on the structure of the homomorphism $\mathbf{D}$ via the Log Wedge Formula. To motivate that work we present a calculation of $\mathbf{D}\left(\sin \mathbf{a}_{3\cdot 5} \bmod \mathbb{Q}^{\mathrm{ab}\,\times 2}\right)$ that illustrates in a relatively nontechnical way the process of extracting group-extension information from sine-monomial identities. The sine-monomial notation and the symbol

$$\mathbf{a}_{3\cdot 5} = \left[\frac{1}{3}\right] + \left[\frac{2}{15}\right] - \left[\frac{4}{15}\right] - \left[\frac{1}{5}\right]$$

are as defined in the introduction to the paper. The calculation to be presented is essentially just a reworking in our set up of the analysis of the central extension

$$1 \to \{\pm 1\} \to \mathrm{Gal}\left(\mathbb{Q}\left(e^{\frac{2\pi i}{60}}, \sqrt{\sin \mathbf{a}_{3\cdot 5}}\right)/\mathbb{Q}\right) \to (\mathbb{Z}/60\mathbb{Z})^\times \to 1$$

carried out in [Das 2000, end of Sec. 16].

3.5.2. Put

$$\zeta_{60} := e^{\frac{2\pi i}{60}}.$$

We may assume that the generators $\{\sigma_p\}_{p\in S}$ of G^{ab} are chosen in such a way that

$$\sigma_p \zeta_{60} = \begin{cases} \zeta_{60}^{41} & \text{if } p = 3, \\ \zeta_{60}^{37} & \text{if } p = 5, \\ \zeta_{60}^{-1} & \text{if } p = -1, \\ \zeta_{60} & \text{if } p = 2 \text{ or } p > 5 \end{cases}$$

for all $p \in S$. It turns out that one has

$$\sigma_p \sin \mathbf{a}_{3\cdot 5} = (\sin \mathbf{a}_{3\cdot 5})/\left(\sin \mathbf{c}_p\right)^2$$

for all $p \in S$ where

$$\mathbf{c}_p := \begin{cases} \left[\frac{1}{15}\right] + \left[\frac{2}{15}\right] & \text{if } p = 3, \\ -\left[\frac{1}{15}\right] + \left[\frac{2}{15}\right] - \left[\frac{4}{15}\right] & \text{if } p = 5, \\ 0 & \text{if } p = -1,\ p = 2, \text{ or } p > 5. \end{cases}$$

For p distinct from 3 and 5 the identity above is of course trivial. See Remark 4.3.4 for a proof in the nontrivial cases $p = 3$ and $p = 5$. By writing down the family of identities above we beg the question as to how one goes about systematically constructing such identities. Answering that question is the business of the next section of the paper. Note that the family of sine-monomial identities above confirms the observation of Das that the extension $\mathbb{Q}\left(\zeta_{60}, \sqrt{\sin \mathbf{a}_{3\cdot 5}}\right)/\mathbb{Q}$ is Galois.

3.5.3. If we take

$$v_{\sigma_p} = 1/\sin \mathbf{c}_p$$

for all $p \in S$ and correspondingly take

$$\alpha_{pq} = \frac{\sigma_q \sin \mathbf{c}_p / \sin \mathbf{c}_p}{\sigma_p \sin \mathbf{c}_q / \sin \mathbf{c}_q} = \operatorname{sign} \frac{\sigma_q \sin \mathbf{c}_p}{\sigma_p \sin \mathbf{c}_q}$$

for all $p, q \in S$ such that $p < q$, then the formula

$$\mathbf{D}\left(\sin \mathbf{a}_{3\cdot 5} \bmod \mathbb{Q}^{\mathrm{ab}\times 2}\right) = \sum_{\substack{p,q\in S\\ p<q}} \log_{-1} \operatorname{sign} \frac{\sigma_q \sin \mathbf{c}_p}{\sigma_p \sin \mathbf{c}_q} \cdot e_p \wedge e_q$$

is what we get by correspondingly specializing the Log Wedge Formula. Note that every term in the sum above save the one indexed by $(p, q) = (3, 5)$ vanishes for more or less trivial reasons. To finish off the calculation, observe that

$$\begin{aligned}
\sigma_5 \sin \mathbf{c}_3 &= \frac{\zeta_{60}^{2\cdot 37} - \zeta_{60}^{-2\cdot 37}}{i} \cdot \frac{\zeta_{60}^{4\cdot 37} - \zeta_{60}^{-4\cdot 37}}{i} \\
&= \frac{\zeta_{60}^{14} - \zeta_{60}^{-14}}{i} \cdot \frac{\zeta_{60}^{28} - \zeta_{60}^{-28}}{i} > 0 \\
\sigma_3 \sin \mathbf{c}_5 &= \frac{i}{\zeta_{60}^{2\cdot 41} - \zeta_{60}^{-2\cdot 41}} \cdot \frac{\zeta_{60}^{4\cdot 41} - \zeta_{60}^{-4\cdot 41}}{i} \cdot \frac{i}{\zeta_{60}^{8\cdot 41} - \zeta_{60}^{-8\cdot 41}} \\
&= \frac{i}{\zeta_{60}^{22} - \zeta_{60}^{-22}} \cdot \frac{\zeta_{60}^{44} - \zeta_{60}^{-44}}{i} \cdot \frac{i}{\zeta_{60}^{28} - \zeta_{60}^{-28}} < 0
\end{aligned}$$

and hence

$$\mathbf{D}\left(\sin \mathbf{a}_{3\cdot 5} \bmod \mathbb{Q}^{\mathrm{ab}\times 2}\right) = e_3 \wedge e_5.$$

This last confirms the observation of Das that the Galois extension $\mathbb{Q}\left(\zeta_{60}, \sqrt{\sin \mathbf{a}_{3\cdot 5}}\right)/\mathbb{Q}$ is nonabelian and suggests (as is in fact the case) that one has $\mathbf{D}\left(\sin \mathbf{a}_{pq} \bmod \mathbb{Q}^{\mathrm{ab}\times 2}\right) = e_p \wedge e_q$ in general.

4. The Main Formula

4.1. **The universal ordinary distribution and related apparatus.**

4.1.1. We denote by $\mathcal{A}$ the free abelian group on symbols of the form

$$[a] \quad (a \in \mathbb{Q})$$

modulo the identifications

$$[a] = [b] \Leftrightarrow a - b \in \mathbb{Z}.$$

Put

$$\mathbf{e}(a) := e^{2\pi i a} \quad (a \in \mathbb{Q}).$$

We equip $\mathcal{A}$ with the unique action of G^{ab} such that

$$\sigma[a] = [b] \Leftrightarrow \sigma\mathbf{e}(a) = \mathbf{e}(b) \quad (a, b \in \mathbb{Q},\ \sigma \in G^{\mathrm{ab}}).$$

4.1.2. For each prime number p we define

$$Y_p : \mathcal{A} \to \mathcal{A}$$

to be the unique endomorphism such that

$$Y_p[a] := [a] - \sum_{i=0}^{p-1} \left[\frac{a+i}{p}\right] \quad (a \in \mathbb{Q}).$$

The operator Y_p is G^{ab}-equivariant. The operators Y_p commute among themselves. Put

$$U := \frac{\mathcal{A}}{\sum_p Y_p \mathcal{A}}, \quad U^- := \frac{\mathcal{A}}{(1+\sigma_{-1})\mathcal{A} + \sum_p Y_p \mathcal{A}},$$

thereby defining the *universal ordinary distribution* and the *universal odd ordinary distribution*, respectively. The action of G^{ab} on $\mathcal{A}$ descends to U and U^-. By [Kubert 1978a], the group U is free abelian. It follows that the torsion subgroup of U^- can be identified with the cohomology group $H^2(\langle \sigma_{-1} \rangle, U)$, and in particular that the torsion subgroup of U^- is killed by 2. By [Kubert 1978b], the group G^{ab} operates trivially on the torsion subgroup of U^-. We mention that Kubert's work built upon that of [Sinnott 1978].

4.1.3. We denote by $\mathcal{A}'$ the subgroup of $\mathcal{A}$ generated by all symbols of the form

$$[a] \quad \left(a \in \mathbb{Q} \setminus \frac{1}{2}\mathbb{Z}\right).$$

Notice that $\mathcal{A}'$ is stable under the action of G^{ab} and stable also under the action of all the operators Y_p. We call a homomorphism

$$H : \mathcal{A}' \to \mathcal{A}'$$

a *lifting operator* if the identities

$$H^2 = H, \quad H + \sigma_{-1} H \sigma_{-1} = 1$$

hold, in which case

$$\mathcal{A}' = H\mathcal{A}' \oplus \sigma_{-1} H \mathcal{A}'$$

and one has implications

$$\begin{array}{lcl} (1+\sigma_{-1})\mathbf{x} = 0 & \Rightarrow & \mathbf{x} = (1-\sigma_{-1})H\mathbf{x} \\ (1-\sigma_{-1})\mathbf{x} = 0 & \Rightarrow & \mathbf{x} = (1+\sigma_{-1})H\mathbf{x} \end{array} \qquad (\mathbf{x} \in \mathcal{A}').$$

4.1.4. To each partition

$$(0,1) \cap \left(\mathbb{Q} \setminus \frac{1}{2}\mathbb{Z}\right) = T \coprod \{1 - a \mid a \in T\}$$

is attached a unique lifting operator

$$\mathbf{H}_T : \mathcal{A}' \to \mathcal{A}'$$

such that

$$\mathbf{H}_T[a] = \begin{cases} [a] & \text{if } a \in T \\ 0 & \text{if } a \notin T \end{cases} \qquad \left(a \in (0,1) \cap \left(\mathbb{Q} \setminus \frac{1}{2}\mathbb{Z}\right)\right).$$

Following [Das 2000], we call

$$\mathbf{H} := \mathbf{H}_{(0,1/2)\cap\mathbb{Q}}$$

the *canonical lifting operator*.

4.1.5. We define

$$\xi : \mathcal{A} \to \mathbb{Q}^{\mathrm{ab}\,\times}$$

to be the unique homomorphism such that

$$\xi[a] = \begin{cases} 1 - \mathbf{e}(a) & \text{if } a \neq 0 \\ 1 & \text{if } a = 0 \end{cases} \qquad (a \in [0,1) \cap \mathbb{Q}).$$

The homomorphism ξ is Galois equivariant. One has

$$\xi\left(Y_p\,[a]\right) = \begin{cases} 1 & \text{if } a \neq 0 \\ p^{-1} = (p^{-1/2})^2 & \text{if } a = 0 \end{cases} \qquad (p\text{: prime, } a \in [0,1) \cap \mathbb{Q})$$

and

$$\xi\left((1+\sigma_{-1})[a]\right) = \begin{cases} 1 & \text{if } a = 0 \\ |1 - \mathbf{e}(a)|^2 = 4\sin^2 \pi a & \text{if } a \neq 0 \end{cases} \qquad (a \in [0,1) \cap \mathbb{Q}).$$

It follows that ξ induces a G^{ab}-equivariant homomorphism

$$U^- \to \mathbb{Q}^{\mathrm{ab}\,\times}/\mathbb{Q}^{\mathrm{ab}\,\times 2}$$

and hence a homomorphism

$$(\text{torsion subgroup of } U^-) \to H^0\left(G^{\mathrm{ab}}, \mathbb{Q}^{\mathrm{ab}\,\times}/\mathbb{Q}^{\mathrm{ab}\,\times 2}\right).$$

Note also that ξ kills $\sum_p Y_p \mathcal{A}'$.

4.1.6. We define a homomorphism

$$\sin : \mathcal{A} \to \mathbb{Q}^{\mathrm{ab}\,\times}$$

by the rule

$$\sin[a] := \begin{cases} 2\sin \pi a & \text{if } a \in \mathbb{Q} \cap (0,1), \\ 1 & \text{if } a = 0. \end{cases}$$

One then has

$$\sin \mathbf{a} = |\xi(\mathbf{a})| \quad (\mathbf{a} \in \mathcal{A}).$$

Let $\mathcal{A}''$ denote the subgroup of $\mathcal{A}'$ generated by elements of the form

$$[a] \quad (a \in (0,1) \cap \mathbb{Q} \cap \mathbb{Z}_2),$$

where $\mathbb{Z}_2$ denotes the 2-adic completion of $\mathbb{Z}$. Let

$$\mathbf{P} : \mathcal{A}'' \to \mathcal{A}''$$

be the unique idempotent endomorphism such that

$$\mathbf{P}[a] = \begin{cases} [a] & \text{if } a \text{ is a 2-adic unit} \\ 0 & \text{otherwise} \end{cases} \qquad (a \in (0,1) \cap \mathbb{Q} \cap \mathbb{Z}_2).$$

Now $a \in (0,1) \cap \mathbb{Q} \cap \mathbb{Z}_2$ is a 2-adic unit if and only if $1 - a$ is not a 2-adic unit; it follows that one has an identity

$$\mathbf{Pa} + \sigma_{-1}\mathbf{P}\sigma_{-1}\mathbf{a} = \mathbf{a} \quad (\mathbf{a} \in \mathcal{A}'').$$

Let

$$\deg : \mathcal{A} \to \mathbb{Z}$$

be the unique homomorphism such that

$$\deg[a] = 1 \quad (a \in \mathbb{Q}).$$

Proposition 4.1.7. *One has*

$$\operatorname{sign}\sigma \sin \mathbf{a} = \frac{\sigma \sin \mathbf{a}}{\sin \sigma\mathbf{a}} = (-1)^{\deg \mathbf{P}(1-\sigma)\mathbf{a}} \left(\frac{\sigma\sqrt{-1}}{\sqrt{-1}}\right)^{\deg \mathbf{a}} \quad (\mathbf{a} \in \mathcal{A}'',\ \sigma \in G^{\mathrm{ab}}).$$

Proof. For each $a \in \mathbb{Q} \cap \mathbb{Z}_2$ there exists a solution $\tilde{a} \in \mathbb{Q} \cap \mathbb{Z}_2$ of the congruence

$$2\tilde{a} \equiv a \bmod \mathbb{Z}$$

unique modulo $\mathbb{Z}$ because $(\mathbb{Q} \cap \mathbb{Z}_2)/\mathbb{Z}$ is the subgroup of $\mathbb{Q}/\mathbb{Z}$ consisting of elements of odd order. For all $a \in \mathbb{Q} \cap (0,1) \cap \mathbb{Z}_2$ one has

$$\frac{a}{2} \equiv \tilde{a} + \frac{1}{2} \bmod \mathbb{Z} \Leftrightarrow a \text{ is a 2-adic unit.}$$

It follows that one has an identity

$$|1 - \mathbf{e}(a)| = 2 \sin \pi a = (-1)^{\deg \mathbf{P}[a]} \ \frac{\mathbf{e}(\tilde{a}) - \mathbf{e}(-\tilde{a})}{\mathbf{e}(1/4)} \quad (a \in (0,1) \cap \mathbb{Q} \cap \mathbb{Z}_2).$$

Let

$$\tilde{\xi} : \mathcal{A}'' \to \mathbb{Q}^{\mathrm{ab}\,\times}$$

be the unique homomorphism such that

$$\tilde{\xi}[a] = \mathbf{e}(\tilde{a}) - \mathbf{e}(-\tilde{a}) \quad (a \in (0,1) \cap \mathbb{Q} \cap \mathbb{Z}_2).$$

Then $\tilde{\xi}$ is G^{ab}-equivariant and one has an identity

$$\sin \mathbf{a} = (-1)^{\deg \mathbf{Pa}} \frac{\tilde{\xi}(\mathbf{a})}{\mathbf{e}(1/4)^{\deg \mathbf{a}}} \quad (\mathbf{a} \in \mathcal{A}''),$$

whence the result. □

Proposition 4.1.8. *For every* $\mathbf{a} \in \mathcal{A}$ *representing a torsion element of* U^-*, the extension* $\mathbb{Q}^{\mathrm{ab}}(\sqrt{\sin \mathbf{a}})/\mathbb{Q}$ *is Galois. (Cf.* [Das 2000, Thm. 11]*.)*

Proof. For every $\mathbf{a} \in \mathcal{A}$ the ratio $\xi(\mathbf{a})/\sin \mathbf{a}$ is a root of unity. Therefore the homomorphism

$$(\text{torsion subgroup of } U^-) \to H^0\left(G^{\mathrm{ab}}, \mathbb{Q}^{\mathrm{ab}\,\times}/\mathbb{Q}^{\mathrm{ab}\,\times 2}\right)$$

induced by $\xi : \mathcal{A} \to \mathbb{Q}^{\mathrm{ab}\,\times}$ is also induced by $\sin : \mathcal{A} \to \mathbb{Q}^{\mathrm{ab}\,\times}$. □

4.2. **Identities of Das type.** We carry out by *ad hoc* methods tailored to our needs what are in effect diagram-chases through the double complex employed by [Das 2000].

4.2.1. Fix a lifting operator

$$H : \mathcal{A}' \to \mathcal{A}',$$

distinct prime numbers p and q, and $\mathbf{x} \in \mathcal{A}$ such that

$$Y_p\mathbf{x}, Y_q\mathbf{x} \in \mathcal{A}', \quad (1 - \sigma_{-1})\mathbf{x} = 0.$$

One has

$$(1 - \sigma_{-1})Y_p\mathbf{x} = 0, \quad (1 - \sigma_{-1})Y_q\mathbf{x} = 0$$

and hence

$$Y_p\mathbf{x} = (1 + \sigma_{-1})HY_p\mathbf{x}, \quad Y_q\mathbf{x} = (1 + \sigma_{-1})HY_q\mathbf{x}.$$

One has

$$(1 + \sigma_{-1})(Y_qHY_p - Y_pHY_q)\mathbf{x} = (Y_q(1 + \sigma_{-1})HY_p - Y_p(1 + \sigma_{-1})HY_q)\mathbf{x} = 0,$$

hence

$$(1-\sigma_{-1})(HY_pHY_q - HY_qHY_p)\mathbf{x} = (Y_pHY_q - Y_qHY_p)\mathbf{x},$$

and hence

$$\begin{array}{rl} & 2(HY_pHY_q - HY_qHY_p)\mathbf{x} \\ = & (Y_pHY_q - Y_qHY_p)\mathbf{x} + (1+\sigma_{-1})(HY_pHY_q - HY_qHY_p)\mathbf{x}. \end{array}$$

We refer to the latter relation as the *first Das identity.*

4.2.2. Fix lifting operators

$$H, \bar{H} : \mathcal{A}' \to \mathcal{A}',$$

distinct prime numbers p and q, and $\mathbf{x} \in \mathcal{A}$ such that

$$Y_p\mathbf{x}, Y_q\mathbf{x} \in \mathcal{A}', \quad (1-\sigma_{-1})\mathbf{x} = 0.$$

Put

$$\mathbf{a} := (HY_pHY_q - HY_qHY_p)\mathbf{x}, \quad \bar{\mathbf{a}} := (\bar{H}Y_p\bar{H}Y_q - \bar{H}Y_q\bar{H}Y_p)\mathbf{x},$$

and

$$\mathbf{b} := (Y_pH(H-\bar{H})Y_q - Y_qH(H-\bar{H})Y_p)\mathbf{x}, \;\; \mathbf{c} := H(\mathbf{a}-\bar{\mathbf{a}}) - H\mathbf{b}.$$

One has

$$(1+\sigma_{-1})(H-\bar{H})Y_q\mathbf{x} = 0, \quad (1+\sigma_{-1})(H-\bar{H})Y_p\mathbf{x} = 0,$$

hence

$$\begin{array}{rcl} (H-\bar{H})Y_q\mathbf{x} & = & (1-\sigma_{-1})H(H-\bar{H})Y_q\mathbf{x}, \\ (H-\bar{H})Y_p\mathbf{x} & = & (1-\sigma_{-1})H(H-\bar{H})Y_p\mathbf{x}, \end{array}$$

hence

$$(1-\sigma_{-1})(\mathbf{a}-\bar{\mathbf{a}}) = (Y_p(H-\bar{H})Y_q - Y_q(H-\bar{H})Y_p)\mathbf{x} = (1-\sigma_{-1})\mathbf{b}$$

and hence

$$\mathbf{a} - \bar{\mathbf{a}} = \mathbf{b} + (1+\sigma_{-1})\mathbf{c}.$$

We refer to the latter relation as the *second Das identity.*

4.2.3. Fix a lifting operator

$$H : \mathcal{A}' \to \mathcal{A}',$$

distinct primes p and q, and $\mathbf{x} \in \mathcal{A}$ such that

$$Y_p\mathbf{x}, Y_q\mathbf{x} \in \mathcal{A}', \quad (1-\sigma)\mathbf{x} = 0 \quad (\sigma \in G^{\mathrm{ab}}).$$

Put

$$\mathbf{a} := (HY_pHY_q - HY_qHY_p)\mathbf{x}.$$

For each $\sigma \in G^{\mathrm{ab}}$ put

$$\begin{array}{rcll} \mathbf{b}_\sigma & := & Y_pH(1-\sigma)HY_q\mathbf{x} - Y_qH(1-\sigma)HY_p\mathbf{x} & \in Y_p\mathcal{A}' + Y_q\mathcal{A}', \\ \mathbf{c}_\sigma & := & H(1-\sigma)\mathbf{a} - H\mathbf{b}_\sigma & \in \mathcal{A}'. \end{array}$$

After making the substitution $\bar{H} = \sigma H \sigma^{-1}$ in the second Das identity and simplifying, one obtains the family of identities

$$(1-\sigma)\mathbf{a} = \mathbf{b}_\sigma + (1+\sigma_{-1})\mathbf{c}_\sigma \quad (\sigma \in G^{\mathrm{ab}})$$

to which we refer in the sequel as the *Das conjugation formula.* Note that all the expressions $\mathbf{b}_\sigma$ belong to the kernel of the homomorphism ξ. It follows that the Das conjugation formula specializes under the homomorphism ξ to a family of numerical identities

$$\sigma\xi(\mathbf{a}) = \xi(\mathbf{a})/\sin^2\mathbf{c}_\sigma \quad (\sigma \in G^{\mathrm{ab}})$$

in which the expressions $\mathbf{b}_\sigma$ figure not at all.

4.3. **Das classes.**

4.3.1. Let a lifting operator

$$H : \mathcal{A}' \to \mathcal{A}'$$

and primes $p < q$ be given. Put

$$\mathbf{a} := (HY_pHY_q - HY_qHY_p) \cdot \begin{cases} [0] & \text{if } 2 < p < q, \\ [0] + [1/2] & \text{otherwise.} \end{cases}$$

In the present situation the identities of Das type have the following implications:

- $\mathbf{a}$ represents a 2-torsion G^{ab}-invariant element of U^- independent of H.
- $\xi(\mathbf{a})$ is a real number and hence $\xi(\mathbf{a}) = \pm\sin\mathbf{a}$.
- $\sin\mathbf{a}$ represents a class in $H^0\left(G^{\mathrm{ab}}, \mathbb{Q}^{\mathrm{ab}\,\times}/\mathbb{Q}^{\mathrm{ab}\,\times 2}\right)$ independent of H.
- One has a family of numerical identities

$$\sigma\sin\mathbf{a} = \sin\mathbf{a}/\sin^2\mathbf{c}_\sigma \quad (\sigma \in G^{\mathrm{ab}})$$

  where $\{\mathbf{c}_\sigma\}$ is the cochain figuring in the Das conjugation formula.

We call the torsion element of U^- represented by $\mathbf{a}$ the *Das class* associated to the pair $\{p < q\}$ of prime numbers.

4.3.2. For all primes $p < q$, put

$$\mathbf{a}_{pq} := (\mathbf{H}Y_p\mathbf{H}Y_q - \mathbf{H}Y_q\mathbf{H}Y_p) \cdot \begin{cases} [0] & \text{if } 2 < p < q, \\ [0] + [1/2] & \text{otherwise,} \end{cases}$$

thereby defining the *canonical representative* of the Das class associated to the pair $\{p < q\}$ of primes. For $2 < p < q$ one has

$$\mathbf{a}_{pq} = \left(\sum_{i=1}^{\frac{p-1}{2}}\left(\left[\frac{i}{p}\right] - \sum_{k=0}^{\frac{q-1}{2}}\left[\frac{i}{pq} + \frac{k}{q}\right]\right)\right) - \left(\sum_{j=1}^{\frac{q-1}{2}}\left(\left[\frac{j}{q}\right] - \sum_{\ell=0}^{\frac{p-1}{2}}\left[\frac{j}{pq} + \frac{\ell}{p}\right]\right)\right),$$

e. g.,

$$\mathbf{a}_{3\cdot 5} = \left[\frac{1}{3}\right] + \left[\frac{2}{15}\right] - \left[\frac{4}{15}\right] - \left[\frac{1}{5}\right].$$

For $2 = p < q$ one has

$$\mathbf{a}_{pq} = \left(\left[\frac{1}{4}\right] - \sum_{k=0}^{\frac{q-1}{2}}\left[\frac{1}{4q} + \frac{k}{q}\right]\right) - \sum_{j=1}^{\frac{q-1}{2}}\left(\left[\frac{j}{q}\right] + \left[-\frac{1}{2q} + \frac{j}{q}\right] - \left[\frac{j}{2q}\right] - \left[-\frac{1}{4q} + \frac{j}{2q}\right]\right),$$

e. g.,

$$\mathbf{a}_{2\cdot 3} := \left[\frac{1}{4}\right] - \left[\frac{5}{12}\right] - \left[\frac{1}{3}\right].$$

The discovery that $\mathbf{a}_{pq}$ for $2 < p < q$ represents a torsion element of U^- is due to Das; see [Das 2000, Sec. 9].

Remark 4.3.3. Let $\Gamma : \mathcal{A} \to \mathbb{R}^\times$ be the unique homomorphism such that

$$\Gamma([a]) = \begin{cases} \sqrt{2\pi}/\Gamma(a) & \text{if } 0 < a < 1 \\ 1 & \text{if } a = 0 \end{cases} \qquad (a \in \mathbb{Q} \cap [0,1)).$$

The well known functional equations satisfied by the Γ-function become in the present context the identities

$$\Gamma([a] + [-a]) = 2 \sin \pi a, \quad \Gamma(Y_p[a]) = p^{1/2-a} \quad (a \in \mathbb{Q} \cap (0,1)).$$

For all primes $p < q$, we then have

$$\begin{aligned} \frac{\Gamma(\mathbf{a}_{pq})}{\sqrt{\sin \mathbf{a}_{pq}}} &= \sqrt{\Gamma\left((Y_p\mathbf{H}Y_q - Y_q\mathbf{H}Y_p) \cdot \begin{cases} [0] & \text{if } 2 < p \\ [0] + [1/2] & \text{if } 2 = p \end{cases}\right)} \\ &= \begin{cases} p^{\frac{-(q-1)^2}{16q}} q^{\frac{(p-1)^2}{16p}} & \text{if } 2 < p \\ 2^{-\frac{q-1}{8}} q^{\frac{1}{8}} & \text{if } 2 = p \end{cases} \end{aligned}$$

by the first Das identity. The formula above (in the case $2 < p$) was discovered by Das (see [Das 2000, Sec. 9]) and the proof of that formula is where we got the idea for the technical tool that we are calling the first Das identity.

Remark 4.3.4. Consider the Das conjugation formula in the special case

$$H = \mathbf{H}, \quad p = 3, \quad q = 5, \quad \mathbf{x} = [0].$$

We have

$$(1 - \sigma)\mathbf{a} = \mathbf{b}_\sigma + (1 + \sigma_{-1})\mathbf{c}_\sigma \quad (\sigma \in G^{\mathrm{ab}})$$

where

$$\mathbf{a} = (\mathbf{H}Y_p\mathbf{H}Y_q - \mathbf{H}Y_q\mathbf{H}Y_p)[0] = \left[\frac{1}{3}\right] + \left[\frac{2}{15}\right] - \left[\frac{4}{15}\right] - \left[\frac{1}{5}\right] = \mathbf{a}_{3\cdot 5}$$

and

$$\begin{aligned} \mathbf{b}_\sigma &= Y_3\mathbf{H}(1-\sigma)\mathbf{H}Y_5[0] - Y_5\mathbf{H}(1-\sigma)\mathbf{H}Y_3[0] && \in Y_3\mathcal{A}' + Y_5\mathcal{A}', \\ \mathbf{c}_\sigma &= \mathbf{H}(1-\sigma)\mathbf{a} - \mathbf{H}\mathbf{b}_\sigma && \in \mathcal{A}'. \end{aligned}$$

For $\sigma_3, \sigma_5 \in G^{\mathrm{ab}}$ such that

$$\sigma_3\left[\frac{1}{60}\right] = \left[\frac{41}{60}\right], \quad \sigma_5\left[\frac{1}{60}\right] = \left[\frac{37}{60}\right],$$

we find after a tedious but completely straightforward calculation that

$$\mathbf{c}_{\sigma_3} = \left[\frac{1}{15}\right] + \left[\frac{2}{15}\right], \quad \mathbf{c}_{\sigma_5} = -\left[\frac{1}{15}\right] + \left[\frac{2}{15}\right] - \left[\frac{4}{15}\right],$$

which is enough to prove the sine-monomial identities of §3.5.2 in the nontrivial cases $p = 3$ and $p = 5$. It should be clear at this point that the idea for the technical tool that we are calling the Das conjugation formula came from the intriguing example [Das 2000, end of Sec. 16].

Remark 4.3.5. The double complex method of [Anderson 1999] gives rise to a canonical $\mathbb{Z}/2\mathbb{Z}$-basis for the torsion subgroup of U^- indexed by finite sets of prime numbers of even cardinality. In [Das 2000, §3 and §9] it is proved that the family $\{\mathbf{a}_{pq}\}_{2<p<q}$ represents the "two-odd-prime" part of the canonical basis. The method of Das can easily be modified to show that the family $\{\mathbf{a}_{pq}\}_{p<q}$ represents the "two-prime" part of the canonical basis.

4.4. **Statement of the Main Formula.**

4.4.1. The main result of this paper is the relation

$$\mathbf{D}\left(\sin \mathbf{a}_{pq} \bmod \mathbb{Q}^{\mathrm{ab}\,\times 2}\right) = e_p \wedge e_q \quad (p < q : \text{ prime})$$

to which we refer in the sequel as the *Main Formula*. The Main and Auxiliary Formulas together immediately imply that

$$\mathbb{Q}^{\mathrm{ab}\,+\epsilon} = \mathbb{Q}^{\mathrm{ab}}\left(\left\{\sqrt[4]{\ell}\right\}_{\ell:\mathrm{prime}} \bigcup \left\{\sqrt{\sin \mathbf{a}_{pq}}\right\}_{\substack{p,q:\mathrm{prime}\\ p<q}}\right).$$

Thus we get a quite explicit description of the field $\mathbb{Q}^{\mathrm{ab}\,+\epsilon}$.

Remark 4.4.2. By [Das 2000, Thm. 22] one has $\sqrt{\sin \mathbf{a}} \in \mathbb{Q}^{\mathrm{ab}}$ and hence $\mathbf{D}\left(\sin \mathbf{a} \bmod \mathbb{Q}^{\mathrm{ab}\,\times 2}\right) = 0$ for every $\mathbf{a} \in \mathcal{A}$ representing an element of the canonical basis of the torsion subgroup of U^- indexed by four or more odd primes. (In all likelihood the restriction to odd indexing primes can be dropped.) So in principle the value of $\mathbf{D}\left(\sin \mathbf{a} \bmod \mathbb{Q}^{\mathrm{ab}\,\times 2}\right)$ for (almost) all $\mathbf{a} \in \mathcal{A}$ representing a torsion element of U^- is determined by the Main and Auxiliary Formulas combined with the result of Das. Since the latter three results fit so naturally together, it is vexing in the extreme that the proofs for them we now possess are so different: whereas the proofs of the Main and Auxiliary Formulas given in this paper are essentially elementary, the proof of the result of Das given in [Das 2000] turns on a corollary to Deligne's decidedly nonelementary theory of absolute Hodge cycles on abelian varieties; the corollary in question is the Deligne reciprocity law [Deligne-Milne-Ogus-Shih 1982, Thm. 7.15, p. 91] linking algebraic Γ-monomials to Jacobi sum Hecke characters. It would be nice to have an elementary proof of the result of Das. It would be nicer still to have an elementary proof of Deligne reciprocity but that seems farther off. A proof of the Main and Auxiliary Formulas based on Deligne reciprocity would also be nice.

Perhaps an elementary proof of the result of Das could be constructed by generalizing the methods of this paper. However, we fear that a direct generalization avoiding the double complex would be distressingly complicated. We only get away with avoiding the double complex because our focus on "two-prime" torsion classes in U^- bounds the combinatorial complexity with which we have to contend; any attempt to give an elementary proof of the result of Das is going to involve far greater combinatorial complexity, and such complexity cannot be managed without some sort of machinery. An aesthetically appealing elementary proof of the result of Das would be one based on some fundamental enrichment of the family of structures with which the double complex is already known to be equipped, and presumably such a "good" proof would simultaneously prove the Main and Auxiliary Formulas.

Remark 4.4.3. The following is a corollary to the Main and Auxiliary Formulas:

($\star$) For $G = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, the canonical injective map

$$(c \mapsto c \circ [\Sigma_G^\epsilon]) : H^1(G^\epsilon, \mathbb{Z}/2\mathbb{Z}) \to H^2(G^{\mathrm{ab}}, \mathbb{Z}/2\mathbb{Z})^-$$

defined in Proposition 2.7 is an isomorphism.

But ($\star$) is not really new. One can get it very easily from the theory of [Fröhlich 1983], as we now explain. Let S be a finite set of primes of $\mathbb{Q}$ including 2 and the infinite prime. Let Γ be the maximal pro-2 quotient of G unramified outside the primes in S. To prove ($\star$) it is enough to prove that the canonical injective map

$$(e \mapsto e \circ [\Sigma_\Gamma^\epsilon]) : H^1(\Gamma^\epsilon, \mathbb{Z}/2\mathbb{Z}) \to H^2(\Gamma^{\mathrm{ab}}, \mathbb{Z}/2\mathbb{Z})^-$$

is an isomorphism. Let $(\tau \mapsto \tau^{\mathrm{ab}}) : \Gamma \to \Gamma^{\mathrm{ab}}$ be the canonical projection. For each odd finite prime $p \in S$, choose any $\tau_p \in \Gamma$ such that τ_p^{ab} topologically generates the inertia subgroup of Γ^{ab} at p. Choose any $\tau_2 \in \Gamma$ such that τ_2^{ab} topologically generates the subgroup of the inertia subgroup of Γ^{ab} at 2 fixing $\sqrt{-1}$. Choose any $\tau_\infty \in \Gamma$ so that $\tau_\infty^{\mathrm{ab}} \in \Gamma^{\mathrm{ab}}$ is the automorphism induced by complex conjugation. The family $\{\tau_p\}_{p \in S}$ is then a minimal set of generators for Γ; in particular, the 2-rank of Γ^{ab} is N, where N is the cardinality of S. By [Fröhlich 1983, Theorem 4.10, p. 56] a complete set of defining relations for Γ as a pro-2 group is lifted from the family of relations

$$\left(\tau_p^{\mathrm{ab}}\right)^{p-1} = 1 \quad (p : \text{odd finite prime in } S); \quad \left(\tau_\infty^{\mathrm{ab}}\right)^2 = 1.$$

It follows that the family of commutators

$$\{[\tau_p, \tau_q]\}_{\substack{p,q \in S \\ p<q}}$$

projects to a $\mathbb{Z}/2\mathbb{Z}$-basis for Γ^ϵ; in particular, the 2-rank of Γ^ϵ is $\frac{N(N-1)}{2}$. By Proposition 2.9, the 2-rank of $H^2(\Gamma^{\mathrm{ab}}, \mathbb{Z}/2\mathbb{Z})^-$ is $\frac{N(N-1)}{2}$. By dimension-counting it follows that the injective map $e \mapsto e \circ [\Sigma_\Gamma^\epsilon]$ is bijective. Thus ($\star$) is proved *à la* Fröhlich.

Remark 4.4.4. The papers [Thakur 1991], [Sinha 1997], [Bae-Gekeler-Yin 2000] and [Bae-Gekeler-Kang-Yin 2001] (just to mention the first several coming to mind plus one more brought to the author's attention by the referee) strongly suggest that there ought to be an analogue of the Main and Auxiliary Formulas over a global field of positive characteristic equipped with a distinguished place.

Remark 4.4.5. The relations standing between the Main Formula, the index formulas of [Sinnott 1978], Deligne reciprocity [Deligne-Milne-Ogus-Shih 1982, Thm. 7.15, p. 91], the theory of [Fröhlich 1983], the theory of [Das 2000], the theory of the group cohomology of the universal ordinary distribution (see [Ouyang 2001] and references therein) and Stark's conjecture and its variants (see [Tate 1984]) deserve to be thoroughly investigated. We have only scratched the surface here. Stark's conjecture is relevant in view of the well known expansion

$$\sum_{n=0}^{\infty} \frac{1}{(n+x)^s} = \frac{1}{2} - x + s \log\left(\frac{\Gamma(x)}{\sqrt{2\pi}}\right) + O(s^2)$$

of the Hurwitz zeta function at $s = 0$.

Remark 4.4.6. Perhaps there is an analogue of the Main Formula over an imaginary quadratic field involving elliptic units. This possibility seems especially intriguing.

5. Proof of the Main Formula

Fix prime numbers $p < q$.

5.1. Reductions.

5.1.1. Put

$$\mathbf{x} := \begin{cases} [0] & \text{if } 2 < p, \\ [0] + [1/2] & \text{if } 2 = p. \end{cases}$$

Fix a partition

$$(0,1) \cap \left(\mathbb{Q} \setminus \frac{1}{2}\mathbb{Z}\right) = T \coprod \{1 - a \mid a \in T\}$$

and put

$$H := \mathbf{H}_T.$$

Presently we are going to make an advantageous choice for the set T. Put

$$\mathbf{a} := (HY_p HY_q - HY_q HY_p)\,\mathbf{x}.$$

Then $\mathbf{a}$ represents the Das class associated to the pair $\{p < q\}$ of primes and

$$\sin \mathbf{a}_{pq} \equiv \sin \mathbf{a} \bmod \mathbb{Q}^{\mathrm{ab}\,\times 2}.$$

5.1.2. In the present situation the Das conjugation formula reads

$$(1 - \sigma)\mathbf{a} = \mathbf{b}_\sigma + (1 + \sigma_{-1})\mathbf{c}_\sigma \quad (\sigma \in G^{\mathrm{ab}})$$

where

$$\begin{aligned} \mathbf{b}_\sigma &:= (Y_p H(1-\sigma) HY_q - Y_q H(1-\sigma) HY_p)\,\mathbf{x}, \\ \mathbf{c}_\sigma &:= H(1-\sigma)\mathbf{a} - H\mathbf{b}_\sigma, \end{aligned}$$

and one has a family of numerical identities

$$\sigma \sin \mathbf{a} = \sin \mathbf{a} / \sin^2 \mathbf{c}_\sigma \quad (\sigma \in G^{\mathrm{ab}}).$$

By construction the function

$$(\sigma \mapsto \sin \mathbf{c}_\sigma) : G^{\mathrm{ab}} \to \mathbb{Q}^{\mathrm{ab}\,\times}$$

factors through $\mathrm{Gal}\left(\mathbb{Q}\left(\mathbf{e}\left(\frac{1}{2pq}\right)\right)/\mathbb{Q}\right)$ and takes values in $\mathbb{Q}\left(\mathbf{e}\left(\frac{1}{4pq}\right)\right)$.

5.1.3. The Log Wedge Formula says that

$$\mathbf{D}\left(\sin \mathbf{a}_{pq} \bmod \mathbb{Q}^{\mathrm{ab}\,\times 2}\right) = \mathbf{D}\left(\sin \mathbf{a} \bmod \mathbb{Q}^{\mathrm{ab}\,\times 2}\right) = \sum_{\substack{r,s \in S \\ r < s}} \log_{-1} \alpha_{rs} \cdot e_r \wedge e_s$$

where

$$\alpha_{rs} := \frac{\sigma_s \sin \mathbf{c}_{\sigma_r} / \sin \mathbf{c}_{\sigma_r}}{\sigma_r \sin \mathbf{c}_{\sigma_s} / \sin \mathbf{c}_{\sigma_s}} \in \{\pm 1\}.$$

(We employ r and s as dummy variables here since p and q have already been fixed.) Since all numbers of the form $\sin \mathbf{a}$ are positive real, we can calculate α_{rs} just by keeping track of signs, i. e., we have

$$\alpha_{rs} = \frac{\operatorname{sign} \sigma_s \sin \mathbf{c}_{\sigma_r}}{\operatorname{sign} \sigma_r \sin \mathbf{c}_{\sigma_s}}.$$

One has $\sin \mathbf{c}_{\sigma_{-1}} = 1$ and hence $\alpha_{-1,\ell} = 1$ for all primes ℓ. For all primes ℓ distinct from p and q, one has $\sin \mathbf{c}_{\sigma_\ell} = \sin \mathbf{c}_1 = 1$ and moreover σ_ℓ operates trivially on the field $\mathbb{Q}\left(\mathbf{e}\left(\frac{1}{4pq}\right)\right)$. (Recall that $\sigma_2\sqrt{-1}=\sqrt{-1}$.) It follows that $\alpha_{rs}=1$ for all primes $r<s$ such that $\{r,s\}\neq\{p,q\}$. It remains now only to prove that $\alpha_{pq}=-1$.

5.2. **Calculation of α_{pq} in the case $2<p$.** Assume that $2<p$, i. e., that both p and q are odd.

5.2.1. We have

$$\alpha_{pq}=(-1)^{\deg\left(\mathbf{P}(1-\sigma_q)\mathbf{c}_{\sigma_p}-\mathbf{P}(1-\sigma_p)\mathbf{c}_{\sigma_q}\right)}$$

by Proposition 4.1.7. In this way we reduce the calculation of α_{pq} to an essentially combinatorial problem.

5.2.2. Put

$$\Theta_p:=\sigma_p^0+\cdots+\sigma_p^{\frac{p-3}{2}},\qquad \Theta_q:=\sigma_q^0+\cdots+\sigma_q^{\frac{q-3}{2}},$$

thereby defining elements of the integral group ring of G^{ab}. The relations

$$\left.\begin{array}{l}(1-\sigma_p)\Theta_p[a+b]=\left(1-\sigma_p^{\frac{p-1}{2}}\right)[a+b]=[a+b]-[-a+b]\\ \\ (1-\sigma_q)\Theta_q[a+b]=\left(1-\sigma_q^{\frac{q-1}{2}}\right)[a+b]=[a+b]-[a-b]\end{array}\right\}\qquad\left(a\in\frac{1}{p}\mathbb{Z},\ \ b\in\frac{1}{q}\mathbb{Z}\right)$$

figure crucially in our calculations.

5.2.3. We make an advantageous choice for the set T now. We assume that T has been chosen in such a way that the associated lifting operator $H=\mathbf{H}_T$ has the following properties:

$$H\sigma_p^i\left[\tfrac{q}{p}\right]=\begin{cases}\sigma_p^i\left[\tfrac{q}{p}\right] & \text{for } 0\leq i<(p-1)/2,\\ 0 & \text{for } (p-1)/2\leq i<p-1,\end{cases}$$

$$H\sigma_q^i\left[\tfrac{p}{q}\right]=\begin{cases}\sigma_q^i\left[\tfrac{p}{q}\right] & \text{for } 0\leq i<(q-1)/2,\\ 0 & \text{for } (q-1)/2\leq i<q-1,\end{cases}$$

$$H\sigma_p^i\sigma_q^j\left[-\tfrac{1}{p}+\tfrac{1}{q}\right]=\begin{cases}\sigma_p^i\sigma_q^j\left[-\tfrac{1}{p}+\tfrac{1}{q}\right] & \text{for } 0\leq i<(p-1)/2 \text{ and } 0\leq j<q-1,\\ 0 & \text{for } (p-1)/2\leq i<p-1 \text{ and } 0\leq j<q-1.\end{cases}$$

The reason for making this choice is to force a lot of cancellation. It was in order to have the freedom to make this choice that we took the trouble to formulate the definition of the Das class in terms of an arbitrarily chosen lifting operator.

5.2.4. We make some preliminary calculations. One has

$$Y_q\,[0]=-\Theta_q\left(\left[\frac{p}{q}\right]+\left[-\frac{p}{q}\right]\right),\quad Y_p\,[0]=-\Theta_p\left(\left[\frac{q}{p}\right]+\left[-\frac{q}{p}\right]\right),$$

hence

$$HY_q\,[0]=-\Theta_q\left[\frac{p}{q}\right],\quad HY_p\,[0]=-\Theta_p\left[\frac{q}{p}\right],$$

and hence

$$H(1-\sigma_q)HY_q\,[0] = -\begin{bmatrix} p \\ q \end{bmatrix}, \quad H(1-\sigma_p)HY_p\,[0] = -\begin{bmatrix} q \\ p \end{bmatrix}.$$

One then has

$$\begin{aligned}
Y_pH(1-\sigma_q)HY_q\,[0] &= -\left(\left[\tfrac{p}{q}\right] - \left[\tfrac{1}{q}\right] - \Theta_p\left(\left[\tfrac{1}{p}+\tfrac{1}{q}\right] + \left[-\tfrac{1}{p}+\tfrac{1}{q}\right]\right)\right), \\
Y_qH(1-\sigma_p)HY_p\,[0] &= -\left(\left[\tfrac{q}{p}\right] - \left[\tfrac{1}{p}\right] - \Theta_q\left(\left[\tfrac{1}{p}+\tfrac{1}{q}\right] + \left[\tfrac{1}{p}-\tfrac{1}{q}\right]\right)\right), \\
Y_pHY_q\,[0] &= -\Theta_q\left(\left[\tfrac{p}{q}\right] - \left[\tfrac{1}{q}\right] - \Theta_p\left(\left[\tfrac{1}{p}+\tfrac{1}{q}\right] + \left[-\tfrac{1}{p}+\tfrac{1}{q}\right]\right)\right), \\
Y_qHY_p\,[0] &= -\Theta_p\left(\left[\tfrac{q}{p}\right] - \left[\tfrac{1}{p}\right] - \Theta_q\left(\left[\tfrac{1}{p}+\tfrac{1}{q}\right] + \left[\tfrac{1}{p}-\tfrac{1}{q}\right]\right)\right).
\end{aligned}$$

5.2.5. It follows that

$$\begin{aligned}
\mathbf{a} &= (HY_pHY_q - HY_qHY_p)[0] \\
&= \Theta_p\Theta_q\left[-\tfrac{1}{p}+\tfrac{1}{q}\right] - H\Theta_q\left(\left[\tfrac{p}{q}\right] - \left[\tfrac{1}{q}\right]\right) + H\Theta_p\left(\left[\tfrac{q}{p}\right] - \left[\tfrac{1}{p}\right]\right), \\
H(1-\sigma_p)\mathbf{a} &= \Theta_q\left[-\tfrac{1}{p}+\tfrac{1}{q}\right] + H(1-\sigma_p)H\Theta_p\left(\left[\tfrac{q}{p}\right] - \left[\tfrac{1}{p}\right]\right), \\
HY_qH(1-\sigma_p)HY_p\,[0] &= -H\left(\left[\tfrac{q}{p}\right] - \left[\tfrac{1}{p}\right]\right), \\
\mathbf{c}_{\sigma_p} &= H(1-\sigma_p)\mathbf{a} - HY_pH(1-\sigma_p)HY_q[0] + HY_qH(1-\sigma_p)HY_p[0] \\
&= \Theta_q\left[-\tfrac{1}{p}+\tfrac{1}{q}\right] + H(1-\sigma_p)H\Theta_p\left(\left[\tfrac{q}{p}\right] - \left[\tfrac{1}{p}\right]\right) - H\left(\left[\tfrac{q}{p}\right] - \left[\tfrac{1}{p}\right]\right), \\
H(1-\sigma_q)\mathbf{a} &= \Theta_p\left(\left[-\tfrac{1}{p}+\tfrac{1}{q}\right] - \left[-\tfrac{1}{p}-\tfrac{1}{q}\right]\right) - H(1-\sigma_q)H\Theta_q\left(\left[\tfrac{p}{q}\right] - \left[\tfrac{1}{q}\right]\right), \\
HY_pH(1-\sigma_q)HY_q\,[0] &= \Theta_p\left[-\tfrac{1}{p}+\tfrac{1}{q}\right] - H\left(\left[\tfrac{p}{q}\right] - \left[\tfrac{1}{q}\right]\right), \\
\mathbf{c}_{\sigma_q} &= H(1-\sigma_q)\mathbf{a} - HY_pH(1-\sigma_q)HY_q[0] + HY_qH(1-\sigma_q)HY_p[0] \\
&= -\Theta_p\left[-\tfrac{1}{p}-\tfrac{1}{q}\right] - H(1-\sigma_q)H\Theta_q\left(\left[\tfrac{p}{q}\right] - \left[\tfrac{1}{q}\right]\right) + H\left(\left[\tfrac{p}{q}\right] - \left[\tfrac{1}{q}\right]\right).
\end{aligned}$$

5.2.6. Finally, one has

$$\begin{aligned}
&\mathbf{P}(1-\sigma_q)\mathbf{c}_{\sigma_p} - \mathbf{P}(1-\sigma_p)\mathbf{c}_{\sigma_q} \\
= \;& \mathbf{P}\left(\left[-\frac{1}{p}+\frac{1}{q}\right] - \left[-\frac{1}{p}-\frac{1}{q}\right]\right) - \mathbf{P}\left(-\left[-\frac{1}{p}-\frac{1}{q}\right] + \left[\frac{1}{p}-\frac{1}{q}\right]\right) \\
= \;& \mathbf{P}\left[-\frac{1}{p}+\frac{1}{q}\right] - \mathbf{P}\left[\frac{1}{p}-\frac{1}{q}\right].
\end{aligned}$$

Now consider the rational numbers $1-\frac{1}{p}+\frac{1}{q}$ and $\frac{1}{p}-\frac{1}{q}$. Both numbers are 2-adically integral and belong to the open unit interval of the real line, but one and only one of them (namely the former) is a 2-adic unit. Therefore one has $\alpha_{pq} = -1$.

Remark 5.2.7. Let v_p be an additive valuation of $\mathbb{Q}^{\mathrm{ab}}$ above the odd prime p normalized by the requirement that $v_p\left(1-e^{\frac{2\pi i}{p}}\right) = 1$ and let v_q above q be analogously defined. In the course of the calculation of α_{pq} presented above we proved that

$$\mathbf{a} = \Theta_p\Theta_q\left[-\frac{1}{p}+\frac{1}{q}\right] - H\Theta_q\left(\left[\frac{p}{q}\right]-\left[\frac{1}{q}\right]\right) + H\Theta_p\left(\left[\frac{q}{p}\right]-\left[\frac{1}{p}\right]\right)$$

represents the Das class indexed by the pair $\{p<q\}$. Since $\sin\mathbf{a}_{pq}$ and $\sin\mathbf{a}$ agree up to a factor in $\mathbb{Q}^{\mathrm{ab}\,\times 2}$ we have a congruence

$$v_p(\sin\mathbf{a}_{pq}) \equiv v_p(\sin\mathbf{a}) \equiv \deg\left(H\Theta_p\left(\left[\frac{q}{p}\right]-\left[\frac{1}{p}\right]\right)\right) \bmod 2$$

in the ring $\mathbb{Z}\left[\frac{1}{p}\right]$ wherein the valuation v_p takes it values. Now recall the *Gauss Lemma* in its most general form: For any partition

$$(\mathbb{Z}/p\mathbb{Z})^\times = S\coprod -S$$

one has

$$\left(\begin{array}{c} x \\ p \end{array}\right) = (-1)^{\#(xS\cap -S)}$$

for all $x\in(\mathbb{Z}/p\mathbb{Z})^\times$. The Gauss Lemma applied in the present situation gives

$$(-1)^{v_p(\sin\mathbf{a}_{pq})} = (-1)^{\deg\left(H\Theta_p\left(\left[\frac{q}{p}\right]-\left[\frac{1}{p}\right]\right)\right)} = \left(\begin{array}{c} q \\ p \end{array}\right).$$

Analogously one has

$$(-1)^{v_q(\sin\mathbf{a}_{pq})} = \left(\begin{array}{c} p \\ q \end{array}\right).$$

Thus we recover the result [Seo 2001, Prop. 2.3] mentioned in the introduction. At the end of the day our method of proof differs little from Seo's. In effect, the only difference is in the choice of representative of the Das class to which the Gauss Lemma is applied: we work with the specially constructed representative $\mathbf{a}$, whereas Seo works directly with the canonical representative $\mathbf{a}_{pq}$.

5.3. **Calculation of α_{pq} in the case $2=p$.** Assume that $2=p$.

5.3.1. One has $\mathbf{c}_{\sigma_2} = \mathbf{c}_1 = 0$ in this case and hence

$$\alpha_{2q} = \operatorname{sign}\sigma_2 \sin\mathbf{c}_{\sigma_q}.$$

5.3.2. We may assume that the set T has been chosen in such a way that the lifting operator $H = \mathbf{H}_T$ has the following properties:

$$
\begin{aligned}
H\sigma_q^i \left[\tfrac{4}{q}\right] &= \begin{cases} \sigma_q^i \left[\tfrac{4}{q}\right] & \text{for } 0 \le i < (q-1)/2, \\ 0 & \text{for } (q-1)/2 \le i < q-1, \end{cases} \\
H \left[\tfrac{\nu}{4}\right] &= \begin{cases} \left[\tfrac{\nu}{4}\right] & \text{for } \nu = q, \\ 0 & \text{for } \nu = -q, \end{cases} \\
H\sigma_q^i \left[\tfrac{1}{2} + \tfrac{2}{q}\right] &= \begin{cases} \sigma_q^i \left[\tfrac{1}{2} + \tfrac{2}{q}\right] & \text{for } 0 \le i < (q-1)/2, \\ 0 & \text{for } (q-1)/2 \le i < q-1, \end{cases} \\
H\sigma_q^i \left[\tfrac{\nu}{4} + \tfrac{1}{q}\right] &= \begin{cases} 0 & \text{for } 0 \le i < q-1 \text{ and } \nu = 1, \\ \sigma_q^i \left[\tfrac{\nu}{4} + \tfrac{1}{q}\right] & \text{for } 0 \le i < q-1 \text{ and } \nu = -1. \end{cases}
\end{aligned}
$$

5.3.3. Put

$$\Theta_q := \left(\sigma_q^0 + \cdots + \sigma_q^{(q-3)/2}\right).$$

The relations

$$(1-\sigma_q)\Theta_q[a+b] = \left(1 - \sigma_q^{\frac{q-1}{2}}\right)[a+b] = [a+b] - [a-b] \qquad \left(a \in \frac{1}{4}\mathbb{Z},\ b \in \frac{1}{q}\mathbb{Z}/\mathbb{Z}\right)$$

are crucial to our calculations.

5.3.4. Recall that in the case $p = 2$ one has

$$\mathbf{x} = [0] + \left[\frac{1}{2}\right].$$

We have

$$
\begin{aligned}
Y_q\mathbf{x} &= -\Theta_q\left(\left[\tfrac{4}{q}\right]+\left[-\tfrac{4}{q}\right]+\left[\tfrac{1}{2}+\tfrac{2}{q}\right]+\left[\tfrac{1}{2}-\tfrac{2}{q}\right]\right)\\
HY_q\mathbf{x} &= -\Theta_q\left(\left[\tfrac{4}{q}\right]+\left[\tfrac{1}{2}+\tfrac{2}{q}\right]\right),\\
H(1-\sigma_q)HY_q\mathbf{x} &= -\left(\left[\tfrac{4}{q}\right]+\left[\tfrac{1}{2}+\tfrac{2}{q}\right]\right)\\
Y_2H(1-\sigma_q)HY_q\mathbf{x} &= -\left(\left[\tfrac{4}{q}\right]-\left[\tfrac{2}{q}\right]-\left[\tfrac{1}{4}+\tfrac{1}{q}\right]-\left[-\tfrac{1}{4}+\tfrac{1}{q}\right]\right)\\
Y_2\mathbf{x} &= -\left[\tfrac{q}{4}\right]-\left[-\tfrac{q}{4}\right]\\
HY_2\mathbf{x} &= -\left[\tfrac{q}{4}\right],\\
(Y_2HY_q-Y_qHY_2)\mathbf{x} &= -\Theta_q\left(\left[\tfrac{4}{q}\right]-\left[\tfrac{2}{q}\right]-\left[\tfrac{1}{4}+\tfrac{1}{q}\right]-\left[-\tfrac{1}{4}+\tfrac{1}{q}\right]\right)\\
&\quad+\left[\tfrac{q}{4}\right]-\left[\tfrac{1}{4}\right]-\Theta_q\left(\left[\tfrac{1}{4}+\tfrac{1}{q}\right]+\left[\tfrac{1}{4}-\tfrac{1}{q}\right]\right),\\
\mathbf{a} &= \Theta_q\left[-\tfrac{1}{4}+\tfrac{1}{q}\right]-H\Theta_q\left(\left[\tfrac{4}{q}\right]-\left[\tfrac{2}{q}\right]\right)+H\left(\left[\tfrac{q}{4}\right]-\left[\tfrac{1}{4}\right]\right),\\
H(1-\sigma_q)\mathbf{a} &= \left[-\tfrac{1}{4}+\tfrac{1}{q}\right]-\left[-\tfrac{1}{4}-\tfrac{1}{q}\right]-H(1-\sigma_q)H\Theta_q\left(\left[\tfrac{4}{q}\right]-\left[\tfrac{2}{q}\right]\right),\\
HY_2H(1-\sigma_q)HY_q\mathbf{x} &= \left[-\tfrac{1}{4}+\tfrac{1}{q}\right]-H\left(\left[\tfrac{4}{q}\right]-\left[\tfrac{2}{q}\right]\right)\\
\mathbf{c}_{\sigma_q} &= H(1-\sigma_q)\mathbf{a}-HY_2H(1-\sigma_q)HY_q\mathbf{x}+HY_qH(1-\sigma_q)HY_2\mathbf{x}\\
&= -\left[-\tfrac{1}{4}-\tfrac{1}{q}\right]+H(1-(1-\sigma_q)H\Theta_q)\left(\left[\tfrac{4}{q}\right]-\left[\tfrac{2}{q}\right]\right).
\end{aligned}
$$

5.3.5. Finally, one has

$$\frac{1}{\sin \mathbf{c}_{\sigma_q}} \equiv \sin\pi\left(\frac{1}{4}+\frac{1}{q}\right) \equiv \frac{1}{\sqrt{2}} \bmod \left(\mathbb{R}\cap\mathbb{Q}\left(\mathbf{e}\left(\frac{1}{4q}\right)\right)\right)^\times$$

because the sines and cosines of all integral multiples of the angle π/q belong to the field $\mathbb{R}\cap\mathbb{Q}\left(\mathbf{e}\left(\frac{1}{4q}\right)\right)$, and hence $\alpha_{2q} = \operatorname{sign}\sigma_2\sqrt{2} = -1$. The proof of the Main Formula is complete.

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