MULTIZETA VALUES FOR $\mathbb{F}_q[t]$, THEIR PERIOD INTERPRETATION AND RELATIONS BETWEEN THEM

GREG W. ANDERSON AND DINESH S. THAKUR

ABSTRACT. We provide a period interpretation for multizeta values (in the function field context) in terms of explicit iterated extensions of tensor powers of Carlitz motives (mixed Carlitz-Tate t-motives). We give examples of combinatorially involved relations that these multizeta values satisfy.

0. INTRODUCTION

The multizeta values introduced and studied originally by Euler have been pursued again recently with renewed interest because of their emergence in studies in mathematics and mathematical physics connecting diverse viewpoints. They occur naturally as coefficients of the Drinfeld associator, and thus have connections to quantum groups, knot invariants and mathematical physics. They also occur in the Grothendieck-Ihara program to study the absolute Galois group through the fundamental group of the projective line minus three points and related studies of iterated extensions of Tate motives, Feynman path integral renormalizations, etc. We refer the reader to papers on this subject by Broadhurst, Cartier, Deligne, Drinfeld, Écalle, Furusho, Goncharov, Hoffman, Kreimer, Racinet, Terasoma, Waldschmidt, Zagier, Zudilin to mention just a few names.

Having learned about these rich interconnections at the Arizona Winter school, the second author, in 2002, defined and studied two types of multizeta values [T04, Sec 5.10] for function fields, one complex-valued (generalizing special values of Artin-Weil zeta functions) and the other with values in Laurent series over finite fields (generalizing Carlitz zeta values). For the $\mathbb{F}_q[t]$ case, the first type was completely evaluated in [T04] (see [M06] for a study in the higher genus case), for both types some identities were established, and for the second type failure of the shuffle identities was noted. Because of the failure, some other variants of the second type were also investigated in [T04]. Here we deal only with the second type, and we restrict attention exclusively to $\mathbb{F}_q[t]$.

We now outline the main points of the paper. (i) We introduce 'degenerate multizeta values', thus bringing back the failed sum shuffle relations, and in turn making the span of the degenerate and original (here called non-degenerate) multizetas into an algebra. (ii) We construct explicit mixed Carlitz-Tate t-motives with periods evaluating to multizeta values (generalizing the results for zeta values obtained in [AT90]), and we construct enough of them to account for all multizeta values. (iii) We give some examples of new combinatorially involved multizeta identities, discuss their t-motivic nature, and indicate contrasts with the classical case. (For a more systematic account of such identities, see [T08?].) Finally, (iv) without

Date: January 10, 2009.

proving anything, we discuss the transcendence problems to which our work can be applied. It is our hope that, just as the construction given in [AT90] helped determine [CY07] all the relations among the Carlitz zeta values, the constructions given in this paper will help to determine all relations among multizeta values. This hope is a key motivation for the paper.

Recent ongoing work [T08?] suggests that the introduction of degenerate multizeta values is not really necessary in the sense that the original multizeta value span is also an algebra, on account of new kinds of combinatorially involved identities. We hope that eventually the latter will be understood as analogs of integral shuffles once the relevant theory is developed.

Finally, let us state our main motivation: to provide a few clues toward the discovery of the t-motivic analog of the fundamental group of the projective line minus three points. Let's call it provisionally t-π_1. We hope to discover or inspire others to discover it through detailed investigation of its various children. The mixed Carlitz-Tate t-motives constructed here are presumably pro-nilpotent children of t-π_1. In other work [AT08?] we have found what plausibly could be metabelian children of t-π_1, i.e., reasonable analogs of Deligne-Soulé cocycles and Ihara power series, and we have linked these objects with the special points constructed in [AT90] and [A96]. We hope that the t-π_1-heuristic leads next to the analog of the Drinfeld associator.

1. Multizeta values over $\mathbb{F}_q[t]$

1.1. **Notation.**

$$
\begin{array}{rcl}
\mathbb{Z} & = & \{\text{integers}\} \\
\mathbb{Z}_+ & = & \{\text{positive integers}\} \\
\mathbb{Z}_{\geq 0} & = & \{\text{non-negative integers}\} \\
T, \tilde{t} & = & \text{independent variables} \\
q & = & \text{a power of a prime } p \\
t & = & -\tilde{t}^{q-1} \\
A & = & \mathbb{F}_q[t] \\
A_+ & = & \text{monics in } A \\
K & = & \mathbb{F}_q(t) \\
\tilde{K} & = & \mathbb{F}_q(\tilde{t}) \\
K_\infty & = & \mathbb{F}_q((1/t)) = \text{completion of } K \text{ at } \infty \\
\tilde{K}_\infty & = & \mathbb{F}_q((1/\tilde{t})) = \text{completion of } \tilde{K} \text{ at } \infty \\
\mathcal{O}_\infty & = & \mathbb{F}_q[[1/\tilde{t}]] \subset \tilde{K}_\infty \\
\mathbb{C}_\infty & = & \text{completion of algebraic closure of } \tilde{K}_\infty \\
[n] & = & t^{q^n} - t \\
D_n & = & \prod_{i=0}^{n-1}(t^{q^n} - t^{q^i}) \\
\ell_n & = & \prod_{i=1}^{n}(t - t^{q^i}) \;=\; (-1)^n L_n \\
\deg & = & \text{function assigning to } x \in K_\infty \text{ its degree in } t
\end{array}
$$

1.2. **Carlitz zeta values and power sums.**

1.2.1. For $s \in \mathbb{Z}_+$, put

$$\zeta(s) = \sum_{a \in A_+} \frac{1}{a^s} \in K_\infty.$$

These are the *Carlitz zeta values.* See [G96, T04] and references there for more.

1.2.2. It is convenient to break the Carlitz zeta values into power sums grouped by degree: Given integers $s > 0$ and $d \geq 0$ put

$$S_d(s) = \sum_{\substack{a \in A_+ \\ \deg a = d}} \frac{1}{a^s}.$$

(This is $S_d(-s)$ in the notation of [T04]).

1.3. **Multizeta values and multiple power sums.**

1.3.1. *Multizeta values.* For $s = (s_1, \ldots, s_r) \in \mathbb{Z}_+^r$, following [T04, Sec. 5.10], we define the multizeta value

$$\zeta(s) = \zeta(s_1, \cdots, s_r) = \sum_{d_1 > \cdots > d_r \geq 0} S_{d_1}(s_1) \cdots S_{d_r}(s_r) = \sum \frac{1}{a_1^{s_1} \cdots a_r^{s_r}} \in K_\infty,$$

where the second sum is over $(a_1, \ldots, a_r) \in A_+^r$ with $\deg a_i$ strictly decreasing. We say that this multizeta value has *depth* r and *weight* $\sum s_i$.

1.3.2. *Remark.* In [T04, Sec. 5.10] the notation $\zeta_d(s)$ was used, where the subscript d was just meant to call the word "degree" back to mind.

1.3.3. *Multiple power sums.* As with the Carlitz zeta function, it is convenient to break the sum defining a multizeta value down according to degrees. To that end we introduce the following notation. Given r-tuples $d = (d_1, \ldots, d_r) \in \mathbb{Z}_{\geq 0}{}^r$ and $s = (s_1, \ldots, s_r) \in \mathbb{Z}_+^r$, put

$$S_d(s) = S_d(s_1, \cdots, s_r) = \prod_{i=1}^{r} S_{d_i}(s_i).$$

Then we have

$$\zeta(s) = \sum S_d(s),$$

where the sum runs through all $d = (d_1, \ldots, d_r) \in (\mathbb{Z}_{\geq 0})^r$ with the d_i's strictly decreasing.

1.3.4. *Motivation for degenerate multizeta values.* Now classically, sum shuffle identities prove that the multizeta values span an algebra over $\mathbb{Z}$. The algebra structure is something we obviously want to preserve on the function field side. But as shown in [T04] (see also Section 3 below), in the function field situation, you cannot usually shuffle a's in A_+. Ultimately the problem is that whereas $\mathbb{Z}$ has a natural total order, A has none. But all is not lost. We may choose to shuffle degrees rather than individual elements of A_+. Then, at the expense of somewhat broadening the definition of multizeta values, we once again get a full set of sum shuffle identities, enough to prove that the $\mathbb{F}_p$-linear combinations of $\mathbb{F}_q[t]$-multizeta values form an algebra over $\mathbb{F}_p$ (see 3.7.5 below for the explicit identity proving this).

These considerations bring us to the following definition generalizing that given in 1.3.1.

1.3.5. *Degenerate multizeta values.* For each subset I of $\{1, 2, \cdots, r-1\}$ and $s \in \mathbb{Z}_+^r$, we define

$$\zeta_I(s) := \sum S_d(s) \in K_\infty$$

where we sum over $d = (d_1, \cdots, d_r) \in (\mathbb{Z}_{\geq 0})^r$ with monotonically decreasing d_i's having jumps exactly in positions in I. Of course for $I = \{1, \ldots, r-1\}$ we have $\zeta(s) = \zeta_I(s)$. For $I \neq \{1, \ldots, r-1\}$ we call $\zeta_I(s)$ a *degenerate* multizeta value. For contrast we say that $\zeta(s)$ is a *nondegenerate* multizeta value.

1.3.6. *Remarks.* (i) See 3.6 for an explanation of why we may not really need these degenerate multizeta values to get an algebra.

(ii) Alternatively, we could consider a variant of degenerate multizeta values defined by conditions of the type $d_1 > d_2 \geq d_3 > \cdots$ in place of conditions of the type $d_1 > d_2 = d_3 > \cdots$ as above. These are clearly simple $\mathbb{F}_p$-linear combinations of the degenerate multizeta values already defined.

2. Period interpretation

2.1. **Twisting.** Given

$$f = \sum_{i=0}^{\infty} f_i T^i \in \mathbb{C}_\infty[[T]],$$

put

$$f^{(n)} = \sum_i f_i^{q^n} T^i \in \mathbb{C}_\infty[[T]].$$

Extend this rule entry-wise to matrices with entries in $\mathbb{C}_\infty[[T]]$.

2.2. **The function Ω.**

$$\Omega(T) = \tilde{t}^{-q} \prod_{i=1}^{\infty} (1 - T/t^{q^i}) \in \mathcal{O}_\infty[[T]].$$

This has an infinite radius of convergence, and satisfies the functional equation

$$\Omega^{(-1)}(T) = (T - t)\Omega(T).$$

The quantity

$$\tilde{\pi} = 1/\Omega(t)$$

is a period of the Carlitz module ([AT90, p. 179] or [T04]) and could be thought of as the analog of $2\pi i$. ($\tilde{\pi}$ is exactly the same notation as in [T04].)

2.3. **Carlitz gamma and factorial.** Given $n \in \mathbb{Z}_{\geq 0}$, we define the *Carlitz gamma* Γ and *factorial* Π by

$$\Gamma_{n+1} := \Pi_n := \prod_i D_i^{n_i},$$

where

$$n = \sum_{i=0}^{\infty} n_i q^i \quad (0 \leq n_i \leq q)$$

is the base q expansion of n.

2.4. **The polynomials H_s and the period representation of Carlitz zeta.**

2.4.1. *Interpolation of power sums.* From [AT90, 3.7.4], we know there exists for each $s \in \mathbb{Z}_{\geq 0}$ a unique polynomial $H_s = H_s(T) \in A[T]$ such that

$$(H_{s-1}\Omega^s)^{(d)}|_{T=t} = \Gamma_s S_d(s)/\tilde{\pi}^s \tag{1}$$

for all $d \in \mathbb{Z}_{\geq 0}$ and $s \in \mathbb{Z}_+$. It is known that $H_s\Omega^{s+1} \in \pi\mathcal{O}_\infty[[T]]$ for all $s \in \mathbb{Z}_{\geq 0}$.

2.4.2. Explicitly, $H_s(T)$ is defined by the generating series identity

$$\sum_{s=0}^{\infty} \frac{H_s(T)}{\Gamma_{s+1}|_{t=T}} x^s = \left(1 - \sum_{i=0}^{\infty} \prod_{j=1}^{i} \frac{(T^{q^i} - t^{q^j})}{(T^{q^i} - T^{q^{j-1}})} x^{q^i}\right)^{-1}. \tag{2}$$

2.4.3. *Remarks.* (i) Our $H_s(T)$ is the same as the two variable polynomial $H_s(y, T)$ of [AT90] evaluated at $y = t$. (ii) Our notation here neatly (or confusingly, depending on your point of view) interpolates between [ABP04] and [T04] in the sense that the latter two references use t and T in exactly opposite ways. And note that, instead of T, the letter θ was used in [A86]. Unfortunately the reader of this literature just has to live with a jumble of t's, T's and the occasional θ.

2.4.4. *Delayed interpolation of power sums.* Combining the interpolation formula above with the functional equation in 2.2, we see that, for $w \geq 1$,

$$(H_{s-1}^{(-w)}[(T-t)^{(0)} \cdots (T-t)^{(-w+1)}]^s \Omega^s)^{(d+w)}|_{T=t} = \Gamma_s S_d(s)/\tilde{\pi}^s.$$

2.4.5. In [AT90, 3.8.2], using the interpolation formula (1), we constructed an algebraic point on the s-th tensor power $C^{\otimes s}$ of the Carlitz module C whose logarithm connected to the Carlitz zeta value $\zeta(s)$. This is equivalent to constructing an extension over A of $C^{\otimes n}$ by the trivial module $C^{\otimes 0}$ having $\Gamma_s\zeta(s)$ as its period.

2.5. **Iterated extensions and multizeta values as periods.** Let us first see the mechanism of how iterated sums as in the definition of multizeta sums come up naturally as entries in triangular matrices satisfying shtuka $F - 1$ equations. The original t-motive formalism [A86] or the equivalent dual t-motive formalism used in [ABP04] will then realize these matrices as period matrices of appropriate iterated extensions of tensor powers of the Carlitz module. These should be viewed as analogs of iterated extensions of Tate motives $\mathbb{Z}(s)$.

2.5.1. Let $s = (s_1, \ldots, s_r) \in \mathbb{Z}_+^r$,. Let $[[X]]$ denote the diagonal matrix

$$[X^{s_1+\cdots+s_r}, X^{s_2+\cdots+s_r}, \cdots, X^{s_r}, X^0].$$

Put $\Lambda = [[\Omega]]$ and $D = [[(T-t)]]$ and

$$Q = \begin{bmatrix} 1 & & & \\ Q_{21} & 1 & & \\ & \ddots & \ddots & \\ & & Q_{r+1,r} & 1 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & & & \\ L_{21} & \ddots & & \\ \vdots & \ddots & \ddots & \\ L_{r+1,1} & \ldots & L_{r+1,r} & 1 \end{bmatrix},$$

$$\Phi = Q^{(-1)}D, \quad \Psi = \Lambda L.$$

For the moment we leave L_{ij} and Q_{ij} undefined. These we will determine presently.

2.5.2. We have then $\Lambda^{(-1)} = D\Lambda$ and so the uniformizability relation (see below)

$$\Phi\Psi = \Psi^{(-1)}, \tag{3}$$

is equivalent to

$$(\Lambda^{-1}Q\Lambda)L^{(1)} = L. \tag{4}$$

2.5.3. We have

$$\Lambda^{-1}Q\Lambda = \begin{bmatrix} 1 & & & \\ \Omega^{s_1}Q_{21} & 1 & & \\ & \ddots & \ddots & \\ & & \Omega^{s_r}Q_{r+1,r} & 1 \end{bmatrix},$$

and hence the relation (4) translates into recursions

$$L_{ij} = \Omega^{s_{i-1}}Q_{i,i-1}L^{(1)}_{i-1,j} + L^{(1)}_{ij}, \quad i > j+1$$

$$L_{i,i-1} = \Omega^{s_{i-1}}Q_{i,i-1} + L^{(1)}_{i,i-1}.$$

2.5.4. Let us now solve the recursions. We have

$$\begin{array}{cc} L_{21} - L^{(1)}_{21} = \Omega^{s_1}Q_{21}, & L_{21} = \sum_{i=0}^{\infty}(\Omega^{s_1}Q_{21})^{(i)}, \\ \vdots & \vdots \\ L_{r+1,1} - L^{(1)}_{r+1,1} = \Omega^{s_r}Q_{r+1,r}L^{(1)}_{r,1}, & L_{r+1,1} = \sum_{i=0}^{\infty}(\Omega^{s_r}Q_{r+1,r}L^{(1)}_{r,1})^{(i)}, \end{array}$$

with entries on the right telescoping to entries on the left, and then

$$L_{r+1,1} = \sum_{i_1 > \cdots > i_r \geq 0} (\Omega^{s_r}Q_{r+1,r})^{(i_r)} \cdots (\Omega^{s_1}Q_{21})^{(i_1)},$$

as a consequence of the the lines on the right.

We thus see (after evaluation at $t = T$) the iterated sum expression of the type defining multizeta values. This calculation due to the first author was the starting point of this paper.

By the theory of t-motives [A86, G96, T04] and the equivalent theory of dual t-motives [ABP04, P08], the period matrix for the dual t-motive defined by Φ, with Ψ related to it as in the uniformizability relation (3) is given by Ψ^{-1} evaluated at $T = t$. (For the reader familiar with t-motives but not dual t-motives, we just note here that in passing from t-motives to dual t-motives, the residues in the recipe for t-motive periods in [A86], [T04, 7.4] are changed to evaluation.)

2.5.5. Since $\Psi = \Lambda L$, we have $\Psi_{ij} = \Omega^{s_i+\cdots+s_r}L_{ij}$.

2.5.6. If we let

$$Q_{i+1,i} = H_{s_i-1}(T)$$

then we see that $L_{r+1,1}$ evaluated at $T = t$ is

$$\Gamma_{s_1}\cdots\Gamma_{s_r}\zeta(s_1,\cdots,s_r)/\tilde{\pi}^{s_1+\cdots+s_r}.$$

2.5.7. In fact, for $i > k \geq 1$, we see that Ψ_{ik} evaluated at $T = t$ is

$$\Gamma_{s_k}\cdots\Gamma_{s_{i-1}}\zeta(s_k,\cdots,s_{i-1})/\tilde{\pi}^{s_k+\cdots+s_r}.$$

2.5.8. Let us give a simpler recipe for the entries of $\Psi = (\psi_{ij})$ and the period matrix $\Psi^{-1} = (p_{ij})$.

Let us write temporarily $Z_{ij\cdots}$ for $\Gamma_{s_i}\Gamma_{s_j}\cdots\zeta(s_i, s_j, \cdots)$. Given an expression F in $Z_{ij\cdots}$, by $F(k)$ we denote the corresponding expression obtained from F by increasing all the indices by k. For example, $F = Z_1Z_2 - Z_{12}$ would give $F(1) = Z_2Z_3 - Z_{23}$.

The j-th column of Ψ is $\tilde{\pi}^{-(s_j+\cdots+s_r)}$ times $[0, \cdots, 0, 1, Z_j, Z_{j(j+1)}, \cdots, Z_{j(j+1)\cdots r}]$.

Let us 'normalize' as follows:

$$\psi'_{ij} = \psi_{ij}/\tilde{\pi}^{-(s_j+\cdots+s_r)}, \quad p'_{ij} = p_{ij}/\tilde{\pi}^{s_i+\cdots+s_r}.$$

Then $p'_{(i+k)(j+k)} = p_{ij}(k)'$ and $\psi'_{(i+k)(j+k)} = \psi'_{ij}(k)$. So entries of these normalized matrices are immediately obtained from those of the first column by shifting and adding indices. We have already described the first column for (normalized) Ψ. Let us describe it for the (normalized) period matrix by giving its first column.

$$\begin{aligned} p'_{i1} &= -(p'_{i-1,1}(1)Z_1 + p'_{i-2,1}(2)Z_{12} + \cdots + p'_{11}(i-1)Z_{1,2,\ldots,i-1}) \\ &= -(p'_{i2}Z_1 + \cdots + p'_{ii}Z_{1,2,\ldots,i-1}). \end{aligned}$$

In particular, note that the bottom-left entry is $-Z_{12\cdots r}$ plus (or minus) products of lower depth Z entries (without any $\tilde{\pi}$-powers) and $\prod \Gamma_{s_i}$ is a common factor. For example, in rank 2, we get Z_1. In rank 3, we get $Z_1Z_2 - Z_{12} = \Gamma_{s_1}\Gamma_{s_2}[\zeta(s_1)\zeta(s_2) - \zeta(s_1, s_2)]$. In rank 4, we get $Z_1Z_{23} + Z_{12}Z_3 - Z_1Z_2Z_3 - Z_{123}$.

2.5.9. Since $H_s^{(-1)}(T) \in A[T]$, as follows from its explicit description above, the entries of Φ are defined over $A[T]$. Thus the non-degenerate multizeta value $\zeta(s)$ (times $\prod \Gamma_{s_i}$) occurs as an entry of the inverse of the period matrix of mixed Carlitz-Tate t-motive over A.

Let us summarize what we have proved:

Theorem 1. *With $Q_{i,j}$ defined as in 2.5.6, the (dual) t-motive over A defined in 2.5.1 is a rank $r+1$ motive with the inverse of the period matrix given as in 2.5.7, in particular containing the depth r multizeta value $\zeta(s_1, \ldots, s_r)$.*

2.5.10. Recall the degenerate multizeta values defined in 1.3.5. To take care of these values in a slight expansion of the framework of the preceding theorem, we modify as follows. We use the polynomials in the delayed interpolation formula of 2.4.4 in place of the polynomials H_{s-1} whenever there are no jumps of degrees. In this way we get t-motives which have arbitrary degenerate multizeta values (multiplied by appropriate gamma factors) as periods. However, the matrices Φ arising this way have entries in $\mathbb{F}_q[t^{1/q^w}][T]$. In other words, we need to make an inseparable base extension to realize those values.

2.5.11. *Remarks.* (i) Bloch and Terasoma inform us that in the classical case, it is not known whether multizeta values of depth r can be achieved as periods of rank $r+1$ motives, although given a multizeta value there is a combinatorial way using the description of Grothendieck-Teichmüller group to find the rank. In the function field situation we can say at least that depth r multizeta values appear in the *inverse* of the period matrix of a dual t-motive of rank $r+1$.

(ii) From the calculations above, we see that multizeta value of depth r plus a linear combination of lower depth multizeta values appears as an entry of the period matrix. To make each multizeta value of depth r itself appear as an entry of the period matrix (rather than an entry of the inverse), we need only form tensor products and direct sums of the basic dual t-motives already constructed. For example, (ignoring gamma factors by working over K rather than A), since $\zeta(s_1)\zeta(s_2) - \zeta(s_1, s_2)$ occurs in rank 3, and $\zeta(s)$ in rank 2, we can get $\zeta(s_1, s_2)$ in rank 7 (taking direct sum of rank 3 with tensor product of the two rank 2).

(iii) For some classical treatments in the number field case, the results and the period calculations exist only over $\mathbb{Q}$ and not over $\mathbb{Z}$. In contrast, our approach in the function field situation naturally gives results over $\mathbb{F}_q[t]$.

(iv) It was shown by Abhyankar that in finite characteristic any curve (not necessarily defined over an algebraic extension of the prime field) can be realized as an étale cover of the affine line, and so one has in principle a Belyi-type embedding of the absolute Galois group of $\mathbb{F}_q(t)$ into the outer automorphism group of the algebraic fundamental group of the affine line over $\overline{\mathbb{F}_q(t)}$. But on account of wild ramification the latter group is overwhelmingly complicated! Accordingly, we have bypassed the Belyi-embedding approach to deal directly with the relevant mixed motives. (Indeed, it is not clear that t-π_1 need be a fundamental group.) In the depth one case, a better justification for our *ad hoc* approach is obtained through construction and study of close analogs [AT08?] of the Deligne-Soulé cocycles, the Ihara power series and related structures. (In fact, the power series (2) in a suitable sense defines a "pro-t-motive" giving rise to the Ihara power series analog.) The higher depth situation is a work in progress and will be dealt with elsewhere.

3. Relations between multizeta values

First, until 3.6, we focus on some basic facts about non-degenerate multizeta values and refer to [T08?] for more general results.

3.1. **Relations to the p^{th}-power map.** Since we are in characteristic p, the definition immediately implies that the multizeta value at (ps_i) is the p-th power of the corresponding multizeta value at (s_i).

3.2. **Zeta at 'even' s.** Carlitz proved [T04, 5.2.1] an analog of Euler's result that if s is 'even' in the sense that $q-1$ divides s, then $\zeta(s)/\tilde{\pi}^s$ is in K.

3.3. **Low s relations.** Using $S_d(kp^n) = 1/\ell_d^{kp^n}$, for $0 < k < q$ [T04, 5.9.1], it was noted in [T04, 5.10.6] that the non-degenerate multizeta values with weight not more than q satisfy classical sum-shuffle identities. More generally, $\prod_j \zeta(s_i(j))$ is a sum of multizeta values as in classical sum shuffle identities, if all sums over j of any $s_i(j)$'s is not more than q.

So any classical sum-shuffle relation with fixed $s_i(j)$'s works for q large enough.

Note here though that because of the characteristic, the look and consequences of these relations can be different. We have, by 3.3, $\zeta(1)\zeta(1) = \zeta(2) + 2\zeta(1,1)$ working for all $\mathbb{F}_q[t]$ as q varies, but the the right side reduces to $\zeta(2)$ when $p=2$ and the relation does not give any information on $\zeta(1,1)$. Also, when $p=2$, but not in general, the sum shuffle $\zeta(k)\zeta(k) = \zeta(2k) + 2\zeta(k,k)$ works by 3.1, but the right side reduces to $\zeta(2k)$ and the relation does not give any information on $\zeta(k,k)$.

3.3.1. More generally, if $s_i = ap^m$, $s_{i+1} = bp^n$, $s_i + s_{i+1} = cp^k$, with $a, b, c \leq q$, then $S_d(s_i)S_d(s_{i+1}) = S_d(s_i + s_{i+1})$, and thus a degenerate multizeta value, with no jump at an index i, equals the multizeta value of one depth lower obtained by putting $s_i + s_{i+1}$ in combined i^{th} and $(i+1)^{st}$ place.

3.4. **Failure of naive sum shuffle.** In [T04, Thm. 5.10.12], it is shown that when $q = 3$, $\zeta(2,2)/\tilde{\pi}^4$ is not in K. So when $q = 3$, using Carlitz's result above for 'even' s, we see that the naive analog of classical sum shuffle $\zeta(2)^2 = 2\zeta(2,2) + \zeta(4)$ fails. (See below for a much simpler proof of the last fact).

3.5. **Failure of naive integral shuffle.** Classically, there are integral shuffle identities between multizeta values coming from their iterated integral expressions and thus connecting immediately to mixed motives. For example, in the usual iterated integral notation we have

$$\int w_1 w_0 \int w_1 w_0 = 2 \int w_1 w_0 w_1 w_0 + 4 \int w_1 w_1 w_0 w_0$$

which with $w_0 = dz/z$ and $w_1 = dz/(1-z)$ gives classically the identity $\zeta(2)^2 = 2\zeta(2,2) + 4\zeta(3,1)$. In our case, the same identity does not work, for example, if $p = 2$, because the right side then is zero, but the left side is nonzero and is in fact transcendental.

3.6. **Different identities with similar consequences!** Classically, the multizeta values form a graded algebra. More precisely, on account of the sum shuffle identities, the product of multizeta values is a sum of multizeta values all of the same weight equal to the the sum of the weights of multizeta values in the product. We were driven to introduce the degenerate multizeta values into our picture in order to preserve this key classical piece of structure in the function field setting. But it seems (see [T08?] for details) that the nondegenerate multizeta values still form an algebra, on account of relations which have an appearance quite different from the shuffle type identities. We only state here the simplest case of how the failures of shuffle identities mentioned above are salvaged by different identities: we have $\zeta(2)^2 = \zeta(4)$, when $p = 2$ (as can be seen from 3.1) or when $q = 3$ (as can be seen from 3.2 by direct Bernoulli calculation or by direct calculation using generating functions). The sum shuffle above then fails for the simple reason that $\zeta(2,2)$ is non-zero as can be seen by a straight calculation of degrees.

3.7. **Sum-shuffle relations using degenerate multizeta values.** We now make explicit the sum-shuffle relations already mentioned in 1.3.4. The following notation seems to be best suited for this.

3.7.1. A *linear preorder* ρ in a set X is a relation satisfying the following conditions for all $x, y, z \in X$:

- $x\rho y$ and $y\rho z$ implies $x\rho z$ (transitivity)
- $x\rho y$ or $y\rho x$ (comparability) (It implies $x\rho x$ (reflexivity)).

Consider the following further axiom:

- $x\rho y$ and $y\rho x$ implies $x = y$ (antisymmetry)

A linear preorder which has also the antisymmetry property is a total ordering. In general, for any linear preorder ρ, the relation "$x\rho y$ and $y\rho x$" is an equivalence relation. (In other words, a linear preorder is the same thing as an equivalence relation along with a total order on the equivalence classes.)

3.7.2. Given a linear preorder ρ in $\{1, \ldots, r\}$, we define the subset $(\mathbb{Z}_{\geq 0})^r_\rho \subset (\mathbb{Z}_{\geq 0})^r$ to be that whose members are the r-tuples $(n_1, \ldots, n_r)$ such that $i\rho j$ if and only if $n_i \leq n_j$ for all $i, j \in \{1, \ldots, r\}$.

3.7.3. Given a linear preorder ρ in $\{1, \ldots, r\}$ and $s \in \mathbb{Z}_+^r$ put

$$\zeta_\rho(s) = \sum_{d \in (\mathbb{Z}_{\geq 0})^r_\rho} S_d(s).$$

These are exactly the multizeta values studied in this paper, albeit expressed in a more elaborate notation in which the same multizeta value may have several different presentations.

3.7.4. When we sum over all linear preorders, we get (analog of the classical sum shuffle identity)

$$\prod_i \zeta(s_i) = \sum_\rho \zeta_\rho(s)$$

for all $s = (s_1, \cdots, s_r) \in \mathbb{Z}_+^r$.

3.7.5. More generally, we now give the explicit identity showing that $\mathbb{F}_p$-linear combinations of the $\zeta_\rho(s)$ form an algebra. For $\nu = 0, 1$ fix $s^{(\nu)} \in \mathbb{Z}_+^{r_\nu}$ and a linear preorder ρ_ν on $\{1, \ldots, r_\nu\}$. Put $r = r_0 + r_1$ and let $s \in \mathbb{Z}_+^r$ be the concatenation of $s^{(0)}$ and $s^{(1)}$. Then we have

$$\zeta_{\rho_0}(s^{(0)})\zeta_{\rho_1}(s^{(1)}) = \sum_\rho \zeta_\rho(s)$$

where the sum is extended over all linear pre-orders ρ of $\{1, \ldots, r\}$ such that ρ restricts on $\{1, \ldots, r_0\}$ to the linear pre-order ρ_0, and on $\{r_0 + 1, \ldots, r_1\}$ to the linear pre-order ρ_1 shifted by r_0.

3.8. **Relations from power sum relations involving digit conditions.** We refer to [T08?] for many combinatorially interesting non-classical relations between the non-degenerate multizeta values. Here we only give some examples to illustrate many more complicated relations that exist once we allow degenerate multizetas. Let z denote the 'totally degenerate' (with all degree equalities) multizeta.

The claims in the following examples follow by direct manipulations from

$$S_d(aq + b) = \frac{1}{\ell_d^{aq+b}} \left(1 + \sum_{j=1}^{a} (-1)^j \binom{b + j - 1}{j} \frac{[d]^{jq}}{[1]^j} \right), \quad \text{if } 0 < a, b < q,$$

which follows easily from e.g., [T04, 5.6.3] or [T08?] or 2.4 above.

3.8.1. By low s relations above, $z(s_1, \cdots, s_r) = z(\sum s_i)$, if $s_i = k_i p^{n_i}$, $\sum s_i = kp^n$, with $k_i, k \leq q$. For example,

$$z(p) = z(p - i, i) = z(p - j - k, j, k) = z(1, ..., 1)$$

and similarly we can change depth and replace parts in non-totally degenerate multizetas at the places degeneration is allowed by such method.

3.8.2. Let $p = 2$. If $0 < b < q/3$, b is of form $4k - 1$, then $S_d(q + b)^3 = S_d(3q + 3b)$, whereas if $0 < b < q/2$ and b is of form $4k + 1$, then $S_d(q + b)^3 = S_d(3q + b)S_d(1)^{2b}$. Summing over d gives relations between degenerate multizetas.

3.8.3. Let $p \equiv 1 \mod 4$, choose q, k and $0 < b < q$ with $b^2 \equiv -1 \mod p$ and $q > kp - 2 - 2b > 0$. Then

$$S_d(q+b)^2 + S_d(q+1)^2 S_d(1)^{2b-2} \\ +S_d(q+(kp-2-2b))S_d(1)^{q-kp+2+4b} + (p-3)S_d(1)^{2q+2b} = 0.$$

Summing over d we again get multizeta identities.

As should be clear from these examples, we can find a wide variety of relations involving various weights and depths.

3.8.4. *Motivic nature of relations and transcendence properties.* Recent works have proved [ABP04, P08] the analog of Grothendieck's conjecture that the relations between periods should come from motivic relations. Using this, together with the description [AT90] (see 2.4) of Carlitz zeta values, it has been proved in [CY07] that all the algebraic relations between the Carlitz zeta values come from 3.1 and 3.2, and this is still true [CPY08?] even if you consider all with varying q (in the same characteristic) together. Similarly, all algebraic relations between Carlitz zeta values and gamma values at proper fractions for the gamma function for $\mathbb{F}_q[t]$ [T04, Chap. 4] are known, by recent results of [CPY08b?, CPTY08?]. (For history of earlier transcendence results, see [T04, 10.3, 10.5]).

As for the higher depth multizeta values, some transcendence results in very special circumstances were proved in [T04, 10.5], using manipulations of sums and techniques of [AT90, Y91, Y97]. By applying to the new identities in [T08?] the transcendence results of [P08] on logarithms, more results were obtained. It should be possible now to prove many more transcendence and independence results for multizeta values using the period interpretation given in this paper, along with the general transcendence results of [P08] that are now available.

In contrast to the classical case, it is a nice description of the full set of identities in higher depths, not whether they would be motivic, which is the hard part in our case! A good full description of all identities is still being sought. See [T08?] for some progress.

Acknowledgments: The second author thanks the Ellentuck Fund and the von Neumann Fund for their support during his stay at the Institute for Advanced Study, Princeton during Spring 2008, where this paper was finalized. He is supported by NSA grant 98230-08-1-0049. We thank the referee for the suggestions to improve the exposition.

References

[A86] G. W. Anderson, *t-motives.* Duke Math. J. **53**(1985), 457-502.

[A96] G. W. Anderson, *Log-algebraicity of twisted A-harmonic series and special values of L-series in characteristic p.* J. Number Theory **60**(1996), 165–209.

[ABP04] G. W. Anderson, W. D. Brownawell, M. A. Papanikolas, *Determination of the algebraic relations among special* Γ*-values in positive characteristic.* Ann. of Math. (2) **160**(2004), 237–313.

[AT90] G. W. Anderson and D. S. Thakur, *Tensor powers of the Carlitz module and zeta values*, Ann. of Math. (2) **132** (1990), 159–191.

[AT08?] G. W. Anderson and D. S. Thakur, *Ihara power series for* $\mathbb{F}_q[t]$, Preprint.

[CY07] C.-Y. Chang and J. Yu, *Determination of algebraic relations among special zeta values in positive characteristic*, Adv. Math. **216**(2007) 321-345.

[CPY08?] C.-Y. Chang, M. Papanikolas, J. Yu, *Algebraic independence of Carlitz zeta values with varying constant fields*, Submitted for publication.

[CPY08b?] C.-Y. Chang, M. Papanikolas, J. Yu, *Determination of algebraic relations among special gamma values and zeta values in positive characteristic*, Submitted for publication.

[CPTY08?] C.-Y. Chang, M. Papanikolas, D. Thakur, J. Yu, *Algebraic independence of arithmetic gamma values and Carlitz zeta vales*, Submitted for publication.

[G96] D. Goss, *Basic structures of function field arithmetic.* Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], **35**, Springer-Verlag, Berlin, 1996.

[M06] R. Masri, *Multiple zeta values over global function fields*, Proc. Symp. Pure Math **75**(2006), 157-175.

[P08] M. Papanikolas, *Tannakian duality for Anderson-Drinfeld motives and algebraic independence of Carlitz logarithms*, Invent. Math. **171**(2008), 123-174.

[T04] D. Thakur, *Function Field Arithmetic,* World Scientific Pub., 2004.

[T08?] D. Thakur, *Relations between multizeta values for* $\mathbb{F}_q[t]$, Submitted for publication

[Y91] J. Yu, *Transcendence and special zeta values in characteristic p,* Ann. of Math. (2) **134**(1991), no. 1, 1–23.

[Y97] J. Yu, *Analytic homomorphisms into Drinfeld modules,* Ann. of Math. (2) **145**(1997), no. 2, 215–233.

University of Minnesota, Mpls., MN 55455
E-mail address: gwanders@umn.edu

University of Arizona, Tucson, AZ 85721
E-mail address: thakur@math.arizona.edu