FINITENESS OF RELATIVE EQUILIBRIA OF THE FOUR-BODY PROBLEM

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Abstract. We show that the number of relative equilibria of the Newtonian four-body problem is finite, up to symmetry. In fact, we show that this number is always between 32 and 8472. The proof is based on symbolic and exact integer computations which are carried out by computer.

1. Introduction

The Newtonian n-body problem [37] is the study of the dynamics of n point particles with masses $m_i > 0$ and positions $x_i \in \mathbf{R}^d$, moving according to Newton's laws of motion:

$$m_j \ddot{x}_j = \sum_{i \neq j} \frac{m_i m_j (x_i - x_j)}{r_{ij}^3} \qquad 1 \leq j \leq n \tag{1}$$

where r_{ij} is the distance betweeen x_i and x_j.

A relative equilibrium motion is a solution of (1) in $\mathbf{R}^2$ of the form $x_i(t) = R(t)x_i(0)$ where $R(t)$ is a uniform rotation with angular velocity $\nu \neq 0$ around some point $c \in \mathbf{R}^2$. Such a solution is possible if and only if the initial positions $x_i(0)$ satisfy the algebraic equations

$$\lambda(x_j - c) = \sum_{i \neq j} \frac{m_i (x_i - x_j)}{r_{ij}^3} \qquad 1 \leq j \leq n \tag{2}$$

where $\lambda = -\nu^2 < 0$. It is easy to see that the center of rotation, c, must be the center of mass. A solution of these equations is called a relative equilibrium configuration or just a *relative equilibrium.* Each relative equilibrium actually gives rise to a family of elliptical periodic motions ranging from rigid circular rotation to homothetic collapse. They are also important in the study of the topology of the integral manifolds [42, 7, 1, 30]. More generally, a solution of (2) with $x_i \in \mathbf{R}^d$ is called a *central configuration*; so a relative equilibrium is just a planar central configuration. Equations (2) are invariant under rotations, translations, and dilations in the plane. Two relative equilibria are considered equivalent if they are related by these symmetry operations.

The relative equilibria of the three-body problem have long been known. Up to symmetry, there are always exactly five relative equilibria. Two of these are Lagrange's equilateral triangles [23] and the other three are collinear configurations discovered by Euler [15].

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For the collinear case an exact count of the relative equilibria of n bodies was found by Moulton [35]. There is a unique collinear relative equilibrium for any ordering of the masses, so taking into account rotations in the plane, there are $n!/2$ collinear equivalence classes. These can be thought of as generalizing Euler's collinear three-body relative equilibria. The equilateral triangle solutions of Lagrange also generalize to tetrahedra for the four-body problem [25] and to regular simplices of n bodies in $n-1$ dimensions [39], although these are central configurations rather than relative equilibria.

However, already in the four-body problem there is sufficient complexity to prevent a complete classification of the non-collinear relative equilibria. In fact, even an exact count is known only for the equal mass case and for certain cases where some of the masses are assumed to be sufficiently small [2, 48, 44]. The goal of this paper is to show that the number is always finite and to give mass-independent bounds. The finiteness problem for relative equilibria of the n-body problem was proposed by Chazy [8] and Wintner [46] and was listed by Smale as problem 6 on his list of problems for this century [43]. We will settle this problem for $n = 4$:

Theorem 1. *If the masses are positive, then there are only a finite number of equivalence classes of relative equilibria for the Newtonian four-body problem.*

In fact, we will show that there are at least 32 at most 8472 such equivalence classes, including the 12 collinear ones.

Finiteness of the number of relative equilibria had already been shown for generic choices of the four masses (without explicit genericity tests) [22, 31, 33]. Previous upper bounds, assuming finiteness, have been much larger [22, 31, 26]. Our bound is probably still far from sharp; numerical experiments [41] suggest that the number of relative equilibria is always between 32 and 50 (the number attained for the equal mass case [2]). A lower bound of 32 can be obtained by observing that in addition to the 12 collinear relative equilibria there are always at least 6 convex configurations and 14 configurations with one body inside the triangle formed by the other three (see section 6). If negative masses are allowed in the planar 5-body problem, a continuum of relative equilibria can exist [38]. It is not known whether this is possible for $n = 4$.

Here is an outline of the finiteness proof. The relative equilibrium equations can be expressed as polynomials in the six mutual distances r_{ij}, $1 \leq i < j \leq 4$ with coefficients depending on the parameters m_i. This choice of variables eliminates the rotation and translation symmetry and, after further normalizing the scale to eliminate dilations, the problem is to show that the resulting system of polynomial equations has finitely many solutions such that $r_{ij} \neq 0$ for all i, j. Some interesting ideas from algebraic geometry give testable criteria for this. These criteria are expressed in terms of the Newton polytopes of the polynomials. Essentially, the original system of equations is replaced by many simpler "reduced systems" for which one would like to show that there are *no* solutions with all the variables nonzero. To find all of the reduced systems and analyze them, we must resort to computer calculations. However, these computations are all of a kind which can be carried out exactly using arbitrary precision integer arithmetic. Namely, we perform algebraic operations on integer vectors and polynomials with integer and symbolic coefficients, and we need to find the convex hulls of polytopes defined by points in the integer lattice. A Mathematica notebook giving more details about the computations can be found at [18].

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2. Mutual Distance Equations for Relative Equilibria

In this section we describe two sets of algebraic equations satisfied by the mutual distances r_{ij} of every relative equilibrium configuration. The first set, due to Albouy and Chenciner [3], already contains $\binom{n}{2}$ equations for the $\binom{n}{2}$ distances. It seems probable that these equations already determine a finite number of non-zero solutions, but we are unable to show this with our methods. So it is necessary to append a second set of equations, due to Dziobek [12].

2.1. **The Albouy-Chenciner Equations.** We begin with the Cartesian equations (2) for central configuration equations of n points in $\mathbf{R}^d$. Multiplying the j-th equation by m_j and summing gives $mc = \sum_{j=1}^n m_j x_j$ where $m = \sum_{j=1}^n m_j$. We will always assume that $m \neq 0$. Then after setting $\lambda = m\lambda'$, the central configuration equations become:

$$\sum_{i=1}^{n} m_i S_{ij}(x_i - x_j) = 0 \qquad 1 \leq j \leq n \tag{3}$$

where

$$S_{ij} = \frac{1}{r_{ij}^3} + \lambda' \quad (i \neq j) \qquad\qquad S_{ii} = 0. \tag{4}$$

Introduce a $d \times n$ configuration matrix X whose columns are the position vectors x_i. This can be viewed as representing a linear map from a non-physical space of dimension n (n being the number of bodies) to the physical space $\mathbf{R}^d$ where the points are located. Then (3) is equivalent to the equation:

$$XA = 0 \tag{5}$$

where A is the $n \times n$ matrix with entries:

$$A_{ij} = m_i S_{ij} \quad (i \neq j) \qquad\qquad A_{jj} = -\sum_{i \neq j} A_{ij}. \tag{6}$$

The idea of Albouy and Chenciner is to replace X by some quantity which is invariant under rotations and translations of the position vectors x_i in $\mathbf{R}^d$. Any such quantity can be expressed as a function of the mutual distances r_{ij}. Now translation of all the positions x_i by a vector $u \in \mathbf{R}^d$ transforms X to $X + uL$ where L is the $1 \times n$ vector whose components are all 1. It follows that one can achieve translation invariance simply by restricting the linear map defined by X to the plane $P = \{v \in \mathbf{R}^n : Lv = v_1 + \cdots + v_n = 0\}$. By definition, the column sums of A are all zero. It follows that $A : \mathbf{R}^n \to P$ and it restricts to $A : P \to P$. So XA in (5) can be viewed as a linear map of P into $\mathbf{R}^d$.

Rotation invariance is obtained by passing to Gram matrices. For any configuration matrix X, the Gram matrix $G = X^t X$ is the $n \times n$ matrix whose entries are the Euclidean inner products $x_i \cdot x_j$. G is obviously rotation invariant. To maintain translation invariance, one can view G as representing a symmetric bilinear form $\beta(v, w) = v^t G w$ on P. The form β on P determines and is determined by the mutual distances r_{ij}. To see this, note that for any constants k_i, adding the vector $k_i L$ to the i-th row of G and the vector $k_i L^t$ to the i-th column produces a new matrix representing β. Choosing $k_i = -\frac{1}{2}|x_i|^2$ shows that β is represented by the matrix, B, whose entries are

$$B_{ij} = x_i \cdot x_j - \tfrac{1}{2}|x_i|^2 - \tfrac{1}{2}|x_j|^2 = -\tfrac{1}{2}r_{ij}^2. \tag{7}$$

Multiplying both sides of (5) by X^t gives $GA = 0$. The matrix GA can be viewed as representing a (non-symmetric) bilinear form on P, in which case it is permissible to replace G by B. Taking the symmetric part gives the Albouy-Chenciner equations for central configurations:

$$BA + A^tB = 0. \tag{8}$$

Let e_i denote the standard basis vectors in $\mathbf{R}^n$ and define $e_{ij} = e_i - e_j$. Then (8) is equivalent to the equations

$$e_{ij}^t(BA + A^tB)e_{ij} = 0 \qquad 1 \le i < j \le n \tag{9}$$

To see this let $\gamma(v, w) = v^tCw$ be the symmetric bilinear form on P associated to the matrix $C = BA + A^tB$. Then (9) means that $\gamma(e_{ij}, e_{ij}) = 0$ for $1 \le i < j \le n$. To show that $\gamma = 0$ it suffices to show that γ vanishes on the basis $e_{1i}, 2 \le i \le n$ of P. By the polarization identity

$$2\gamma(e_{1i}, e_{1j}) = \gamma(e_{ij}, e_{ij}) - \gamma(e_{1i}, e_{1i}) - \gamma(e_{1j}, e_{1j}),$$

this follows from (9).

Equations (9) provide $\binom{n}{2}$ constraints on the $\binom{n}{2}$ mutual distances r_{ij} of a central configuration. Conversely, it can be shown that if the quantities r_{ij} are the mutual distances of some configuration in some $\mathbf{R}^d$ and if they satisfy (10), then the configuration is central [3]. It is remarkable that the equations themselves are independent of d, so they determine the central configurations in all dimensions at once.

To find the equations explicitly, note that

$$\gamma(e_{ij}, e_{ij}) = 2e_{ij}^t BA\, e_{ij} = 2(BA_{ii} + BA_{jj} - BA_{ij} - BA_{ji}).$$

where BA_{ij} denotes the entries of the matrix BA. From (6) and (7) we find

$$\sum_{k=1}^{n} m_k \left[S_{ik}(r_{jk}^2 - r_{ik}^2 - r_{ij}^2) + S_{jk}(r_{ik}^2 - r_{jk}^2 - r_{ij}^2)\right] = 0 \tag{10}$$

for $1 \le i < j \le n$, where S_{ik}, S_{jk} are given by (4).

At this point we can normalize the equations to eliminate the dilation symmetry. In fact, if we scale all the distances by setting $\tilde{r}_{ij} = cr_{ij}$ where $c \ne 0$ is a constant, and if we further set $\tilde{\lambda}' = c^{-3}\lambda'$, we obtain another solution. We will choose c to achieve the normalization

$$\lambda' = -1.$$

2.2. **Dziobek's Equations.** For planar central configurations of $n = 4$ bodies, there is another set of equations which we will find useful. In this case, each of the equations in (2) contains three nonzero terms and the vectors $x_i - x_j$ which appear there are two-dimensional. Taking wedge products of the i-th equation with one of the vectors $x_i - x_k$, yields an equation relating two of the areas of the triangles containing the point x_i (the term involving $x_i - x_k$ drops out). Let A_l denote the oriented (signed) area of the triangle *not* containing x_l, where we consider $A_l > 0$ if and only if the subscripts of the three bodies at the vertices occur in counterclockwise order and let $\Delta_l = (-1)^{l+1} A_l$. Then the equations are

$$m_k S_{ik} \Delta_l = m_l S_{il} \Delta_k \tag{11}$$

where i, k, l are any three distinct indices in $\{1, 2, 3, 4\}$.

It is important to note that for a non-collinear relative equilibrium, the four triangles detemined by the bodies have nonzero areas. Otherwise, we would have three bodies along a line not containing the fourth. However, this contradicts the perpendicular bisector theorem [32]. Using this, it follows easily from (11) that

$$S_{12}S_{34} = S_{13}S_{24} = S_{14}S_{23}. \tag{12}$$

These will be called Dziobek's equations [12]. Because of the assumptions, they hold only for planar, non-collinear configurations. This suffices for our proof since the $4!/2 = 12$ collinear relative equilibria are already well understood.

Although we will devote most of our effort to the Albouy-Chenciner and Dziobek equations described above, another interesting set of equations will be needed to establish an upper bound for the number of relative equilibria. It is not difficult to show that the mutual distances r_{ij} and areas Δ_i of any configuration of four non-collinear points in the plane satisfy a system of equations

$$\begin{aligned}
&\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 = 0 \\
&\Delta_0 + r_{12}^2\,\Delta_2 + r_{13}^2\,\Delta_3 + r_{14}^2\,\Delta_4 = 0 \\
&\Delta_0 + r_{12}^2\,\Delta_1 + r_{23}^2\,\Delta_3 + r_{24}^2\,\Delta_4 = 0 \\
&\Delta_0 + r_{13}^2\,\Delta_1 + r_{23}^2\,\Delta_2 + r_{34}^2\,\Delta_4 = 0 \\
&\Delta_0 + r_{14}^2\,\Delta_1 + r_{24}^2\,\Delta_2 + r_{34}^2\,\Delta_3 = 0
\end{aligned}$$

for some constant Δ_0 [2, 33]. In fact the Δ_i are determined by these equations up to a common factor. For a relative equilibrium configuration it follows from (11) that

$$m_i m_j S_{ij} = \mu \Delta_i \Delta_j \qquad 1 \le i < j \le 4.$$

for some constant $\mu \ne 0$. If we define new variables z_i, k such that $m_i z_i = \sqrt{\mu}\,\Delta_i$ and $k = \sqrt{\mu}\,\Delta_0$ then we obtain a solution of the following system of 11 equations in 11 unknowns:

$$\begin{aligned}
&f_0 = m_1 z_1 + m_2 z_2 + m_3 z_3 + m_4 z_4 = 0 \\
&f_1 = m_2 z_2 r_{12}^2 + m_3 z_3 r_{13}^2 + m_4 z_4 r_{14}^2 + k = 0 \\
&f_2 = m_1 z_1 r_{12}^2 + m_3 z_3 r_{23}^2 + m_4 z_4 r_{24}^2 + k = 0 \\
&f_3 = m_1 z_1 r_{13}^2 + m_2 z_2 r_{23}^2 + m_4 z_4 r_{34}^2 + k = 0 \\
&f_4 = m_1 z_1 r_{14}^2 + m_2 z_2 r_{24}^2 + m_3 z_3 r_{34}^2 + k = 0 \\
&S_{ij} = z_i z_j \quad 1 \le i < j \le 4.
\end{aligned} \tag{13}$$

If r_{ij} are the mutual distances of any non-collinear, relative equilibrium, there exist z_i, k such that (13) hold (the constant μ may be negative and in this case z_i, k will be complex). Since we are interested in non-collinear relative equlibria, the argument about areas sketched above shows that we are justified in looking only at complex solutions of (13) with $z_i \ne 0$.

3. BKK Theory

In a remarkable paper [6], D.N. Bernstein considers the basic problem of solving a system of n equations in n variables. Related work of Khovanskii and Kushnirenko appeared around the same time and the resulting circle of ideas is often referenced by the initials BKK [20, 21]. For us, the most important part of this theory is that it provides testable criteria for determining whether or not a given system of polynomials has finitely many solutions. These criteria also apply to the case where the number of equations differs from the number of unknowns. We will begin by formulating a theorem based on these ideas which is sufficient for the applications we have in mind.

Consider a system of m polynomial equations in n complex variables:

$$f_i(x_1, \dots, x_n) = \sum_k c_k x_1^{k_1} \dots x_n^{k_n} = 0 \qquad i = 1, \dots, m \tag{14}$$

where the exponent vector $k = (k_1, \dots, k_n)$ runs over a finite set $S_i \subset \mathbf{Z}^n$ (called the support of f_i). We want to study solutions such that $x_i \neq 0$ for all i, i.e., $x = (x_1, \dots, x_n) \in \mathbf{T}$ where $\mathbf{T}$ or $\mathbf{T}^n$ denotes the "algebraic torus" $(\mathbf{C} \setminus 0)^n$. With this restriction, we can even allow the f_i to be Laurent polynomials, i.e., some of the exponents k_j could be negative. On the other hand, the denominators of Laurent polynomials can always be cleared without introducing new solutions in $\mathbf{T}$.

Let V be the algebraic variety in $\mathbf{T}$ defined by the system of equations. From the algebraic character of the equations, it follows that the projection of V onto each coordinate axis is either a finite set or the complement of a finite set (see the proof of proposition 1 below). In the latter case, the projection is said to be *dominant*. If V is infinite, at least one such projection must be dominant and so there is at least an algebraic curve of solutions. The idea behind Bernstein's finiteness test is to replace this geometrical condition by an algebraic one involving fractional power series (Puiseux series). It turns out that if the projection of V onto the x_l-axis is dominant then after setting $x_l = t$, one can find Puiseux series $x_j(t)$, $j \neq l$, which satisfy the given system equations identically in t. To prove finiteness, we need to show that no such series can exist.

To describe these ideas in more detail, let $\mathcal{P} = \mathcal{P}(t)$ denote the field of convergent, complex Puiseux series in a variable t. More precisely, let t_q, $q = 1, 2, 3, \dots$ be new variables and let $\mathcal{M}(t_q)$ be the field of all Laurent series in t_q which are convergent in some punctured neighborhood of $t = 0$ (i.e., local meromorphic functions). We want t_q to represent $t^{\frac{1}{q}}$ so define injective maps $i : \mathcal{M}(t_q) \to \mathcal{M}(t_{qr})$ by setting $t_q = t_{qr}^r$ for all natural numbers q, r. Then $\mathcal{P}$ is the direct limit of this collection of fields and maps. Less formally, take the union of all these fields with the identification maps i understood. These identifications justify the use of fractional exponent notation where t_q^i is written $t^{\frac{i}{q}}$. An element of $\mathcal{P}$ will then be a complex series of the form

$$x(t) = \sum_{i=i_0}^{\infty} a_i t^{\frac{i}{q}}$$

for some q, where the leading index i_0 could be negative. If two such series are given, belonging to $\mathcal{M}(t_q), \mathcal{M}(t_r)$, they can each be identified with elements of the field $\mathcal{M}(t_{qr})$. Performing the algebraic operations there, it follows that $\mathcal{P}$ is a field. It is also important to note that $\mathcal{P}$ is algebraically closed [24, Ch.V.2] . We then have:

Proposition 1. *Suppose that a system of m polynomial equations $f_i(x) = 0$ defines an infinite variety $V \subset \mathbf{T}$. Then there is a nonzero rational vector $\alpha = (\alpha_1, \ldots, \alpha_n)$, a point $a = (a_1, \ldots, a_n) \in \mathbf{T}$, and Puiseux series $x_j(t) = a_j t^{\alpha_j} + \ldots$, $j = 1, \ldots, n$, convergent in some punctured neighborhood U of $t = 0$, such that $f_i(x_1(t), \ldots, x_n(t)) = 0$ in U, $i = 1, \ldots, m$. Moreover, if the projection from V onto the x_l-axis is dominant, there exists such a series solution with $x_l(t) = t$ and another with $x_l(t) = t^{-1}$.*

Proof. Let $S \subset \mathbf{C}^n$ be the affine algebraic variety defined by the equations. We are interested in $V = S \cap \mathbf{T}$. Since $\mathbf{T}$ is the complement of the affine variety $x_1 \ldots x_n = 0$, V is the difference set of two affine varieties, i.e., a quasi-projective variety [40]. By hypothesis, $V = S \cap \mathbf{T}$ is infinite. It follows that there is at least one coordinate, x_l, such that the projection $\pi_i(V)$ onto the x_l axis is dominant (or Zariski dense), i.e., there are no polynomials in x_l alone which vanish on V. For otherwise there would be only finitely many possible values for each of the coordinates and V itself would be a finite set. It follows that $\pi_i(V)$ is also dense in the classical topology of $\mathbf{C}$ and even that $\pi_i(V)$ omits at most a finite subset of $\mathbf{C}$.

After reindexing, one may assume that $l = n$. Set $x_n(t) = t$ or $x_n(t) = 1/t$ and let $F_j(x_1, \ldots, x_{n-1}) = f_j(x_1, \ldots, x_{n-1}, x_n(t))$. Then after clearing denominators if necessary, F_j can be viewed as an element of the ring $\mathcal{R} = \mathcal{P}[x_1, \ldots, x_{n-1}]$ of polynomials in $n - 1$ variables with coefficients in $\mathcal{P}$ (in fact, the coefficients will be polynomials in t). The equations $F_j = 0$ define an affine variety in the space $\mathcal{P}^{n-1}$ and it must be shown that this variety contains at least one point with all coordinates nonzero (a "point" of $\mathcal{P}^{n-1}$ is really a vector of convergent Puiseux series). To get the coordinates to be nonzero, introduce another variable x_0 and another equation

$$F_0(x_0, x_1, \ldots, x_{n-1}) = x_0 x_1 \ldots x_{n-1} - 1 = 0.$$

Now it suffices to show that the variety $W \subset \mathcal{P}^n$ defined by $F_0 = \ldots = F_m = 0$ is nonempty, for if $(x_0, \ldots, x_{n-1}) \in W$ then $x_i(t) \in \mathcal{P}$, $i = 1, \ldots, n - 1$, are the required nonzero series solutions.

By the weak Nullstellensatz (which applies in any algebraically closed field), W is empty if and only if an equation of the form

$$1 = F_0\, G_0 + F_1\, G_1 + \ldots + F_m\, G_m \tag{15}$$

holds in $\mathcal{R}$. However, since $\pi_n(V)$ is dense in $\mathbf{C}$, we have that for almost every $t \in \mathbf{C}$, there exist $x_i \in \mathbf{C}$, $1 \leq i \leq n - 1$, with $x_i \neq 0$ such that $f_j(x_1, \ldots, x_{n-1}, t) = 0$ for $j = 1, \ldots, m$. Choose such a t in a common neighborhood of convergence of all the Puiseux series occuring as coefficients in (15). Substituting t and the corresponding values of x_i into (15) makes all the $F_j = 0$, $j = 1, \ldots, m$. Setting $x_0 = (x_1 x_2 \cdots x_{n-1})^{-1} \in \mathbf{C}$ also makes $F_0 = 0$ and gives a contradiction. So $W \neq \emptyset$ and the required nonzero series $x_j(t)$ exist. QED

A vector of nonzero Puiseux series $x(t) = (x_1(t), \ldots, x_n(t))$ as in proposition 1 will be said to have *order* $\alpha = (\alpha_1, \ldots, \alpha_n)$. Then to prove that V is finite, it suffices to show that for every nonzero rational vector α there is no Puiseux series solution of order α. The next result shows that we can further restrict the normal vectors α to a half-space in $\mathbf{R}^n$.

Proposition 2. *Let H be the half-space $c \cdot \alpha \geq 0$ where $c = (c_1, \ldots, c_n)$ is an arbitrary, nonzero integer vector. If system (14) has no Puiseux series solutions of order α for all $\alpha \in H$, then it has finitely many solutions in $\mathbf{T}$.*

Proof. Suppose, for the sake of contradiction, that there are infinitely many solutions in $\mathbf{T}$. First consider the special case $c = (0, \ldots, 0, 1)$. Either the projection of the solution set onto the x_n-axis is dominant or it is not. If it is, then by proposition 1, one can find Puiseux series $x_j(t) = a_j t^{\alpha_j} + \ldots$, $1 \leq j \leq n-1$, and $x_n(t) = t$ which solve the equations and the exponent vector $\alpha = (\alpha_1, \ldots, \alpha_{n-1}, 1)$ satisfies $c \cdot \alpha = 1 > 0$. On the other hand, if the projection onto the x_n-axis is not dominant, then it consists of finitely many points. In this case, the Puiseux series solution guaranteed by proposition 1 will have $x_n(t) = a_n$ constant. The corresponding exponent vector $\alpha = (\alpha_1, \ldots, \alpha_{n-1}, 0)$ satisfies $c \cdot \alpha = 0$. In either case, we get $c \cdot \alpha \geq 0$.

To complete the proof we will reduce the general case to the one just considered by a trick which can also be found in Bernstein's paper. We make a change of variables of the form $x_i = y_1^{C_{1i}} y_2^{C_{2i}} \ldots y_n^{C_{ni}}$ where y_j are new variables and C_{ij} are integers. If the matrix C with entries C_{ij} is unimodular, there will be an integer inverse matrix D and $y_i = x_1^{D_{1i}} x_2^{D_{2i}} \ldots x_n^{D_{ni}}$. These formulas define an isomorphism of the algebraic tori in the x and y variables which we will denote by writing $x = y^C, y = x^D$. Let $g_i(y_1, \ldots, y_n)$ be obtained from $f_i(x_1, \ldots, x_n)$ by this change of variables, $1 \leq i \leq m$. A monomial $x_1^{k_1} \ldots x_n^{k_n}$ in $f_i(x)$ becomes a monomial $y_1^{l_1} \ldots y_n^{l_n}$ in $g_i(y)$ where the exponent vectors k and l are related by $l = Ck$. If $y(t)$ is a convergent Puiseux series of order β, then $x(t) = y(t)^C$ will be a convergent Puiseux series of order α where $\beta = \alpha C$.

Now suppose a nonzero integer vector $c = (c_1, \ldots, c_n)$ is given. Assume without loss of generality that the components c_i have no nontrivial common factors. Then there is a unimodular integer matrix C whose n-th column is c [27, Sec.21]. Making the transformation above we get a systems of Laurent polynomial equations $g_1(y) = \ldots = g_m(y) = 0$ which also has infinitely many solutions in $\mathbf{T}$. By the special case considered at the beginning of the proof, this new system has a Puiseux series solution of order β with $\beta \cdot (0, \ldots, 0, 1) \geq 0$. It follows that $x(t) = y(t)^C \in \mathbf{T}$ is a Puiseux solution of the original equations of order α where $\beta = \alpha C$. Finally, by choice of the matrix C, we have $\alpha \cdot c = \alpha C \cdot (0, \ldots, 0, 1) = \beta \cdot (0, \ldots, 0, 1) \geq 0$ as required. QED

To apply this test we need a way to show that Puiseux solutions of a given order do not exist. Bernstein provides a simpler test by focusing on the leading terms of the solutions. Substituting $x_j(t)$ into equations (14) and reading off the coefficients of the lowest order terms in t gives a *reduced system*

$$f_{i\alpha}(a_1, \ldots, a_n) = \sum_{\alpha \cdot k = \mu_i} c_k x_1^{k_1} \ldots x_n^{k_n} = 0 \qquad i = 1, \ldots, m \tag{16}$$

where $\mu_i = \min_{l \in S_i} \alpha \cdot l$.

The equation $\alpha \cdot k = \mu_i$ which determines which terms of f_i appear in the reduced equation has a beautiful geometrical interpretation. Let P_i be the *Newton polytope* of f_i, i.e., the convex hull of the support S_i. Then $\alpha \cdot k = \mu_i$ defines a supporting hyperplane of P_i for which α is an inward normal vector. The hyperplane defines a face of the Newton polytope and the exponent vectors k which appear in the reduced equation are the vertices of P_i which lie on this face.

Since the coefficients of the leading terms of any series solution must vanish, we have a simple test for nonexistence of a solution of order α:

Proposition 3. *Let α be a nonzero rational vector. If the reduced system (16) has no solutions in $\mathbf{T}$ then there does not exist a Puiseux series solution of the full system (14) of order α.*

As Bernstein remarks, a system of equations gives rise to only finitely many distinct reduced systems. In other words, different vectors α can induce the same reduced system. Recall that the terms of the reduced polynomial $f_{i\alpha}$ correspond to vertices in the face of P_i determined by the supporting hyperplane with inward normal α. While the facets (faces of codimension 1) of a polytope have a unique inward normal up to scalar multiplication, the lower dimensional faces will have infinitely many. However, they all induce the same reduced system. The vectors $\alpha \in \mathbf{R}^n$ can be partitioned into finitely many sets according to which face of P_i they induce. The sets of the partition turn out to be convex cones and the partition is called the normal fan of P_i. To be sure that every reduced system is examined one needs to construct the common refinement of the normal fans of $P_1, \ldots, P_m$, i.e., the partition obtained by intersecting the sets of the partitions for the individual P_i. It turns out that this refined partition is just the normal fan of the Minkowski sum polytope $P_1 + \ldots + P_m = \{x \in \mathbf{R}^n : x = x_1 + \ldots + x_n, x_i \in P_i\}$ [50].

We can now outline our method for proving that a given system of polynomial equations has finitely many solution in $\mathbf{T}$. First we find the Newton polytopes P_i and compute their Minkowski sum, P. For each face of P (defined by an inward normal vector α), we see whether the corresponding reduced equations $f_{i\alpha} = 0$ have solutions in the algebraic torus $\mathbf{T}$. If not, then α and all other inward normal vectors to that face are ruled out as possible orders of Puiseux solutions. If the reduced equations do have solutions, we can still try to show that no Puiseux solutions of order α exist by examining higher-order terms in the series. If all vectors α in a half-space of the form $c \cdot \alpha \geq 0$ can be eliminated in this way, then the system has finitely many solutions in $\mathbf{T}$.

For our system of equations for relative equilibria, almost all of the reduced equations have no solutions in $\mathbf{T}$ provided the masses are positive. Only a few exceptional cases have to be handled by going to higher-order terms.

We will use one more remarkable fact from BKK theory to get an upper bound for the number of relative equilibria. Note that the Newton polytopes of a system do not depend on the actual coefficients of the polynomials, only on which coefficients are nonzero. So one can consider the set of all polynomial systems which have the same P_i but with different values for the coefficients. Then Bernstein shows that for systems of n equations in n unknowns and for generic choices of the coefficients, the number of solutions in the algebraic torus $\mathbf{T}$ is equal to a certain geometric invariant of the Newton polytopes called the mixed volume (the solutions have to be counted with appropriate multiplicities). Moreover, even when the number of solutions in $\mathbf{T}$ is infinite, the mixed volume gives an upper bound for the number of isolated solutions. The definition of mixed volume depends on some geometrical constructions on polytopes. We have used the Minkowski sum of polytopes P_i above. More generally, one can take a linear combination of polytopes $\lambda_1 P_1 + \ldots + \lambda_m P_m = \{x \in \mathbf{R}^n : x = \lambda_1 x_1 + \ldots + \lambda_n x_n, x_i \in P_i\}$. The Euclidean volume of this linear combination is a homogeneous polynomial function of the λ_i of degree n. If the volume $V(\lambda_1 P_1 + \ldots + \lambda_m P_m)$ is expanded as a sum of powers of the λ_i then the mixed volume is just the coefficient of the product $\lambda_1 \cdots \lambda_m$ [11]. It is not easy to compute the mixed volume from this definition, but there are other algorithms which have been implemented on the computer [19, 14].

4. The Newtonian Four-Body Problem

In this section we carry out the procedure outlined above for the relative equilibrium equations of the Newtonian four-body problem. There are six mutual distances r_{ij}. The six Albouy-Chenciner equations are rational functions in these variables. Since we are looking for solutions with $r_{ij} \neq 0$ we can clear denominators to obtain polynomial equations. Similarly, Dziobek's equations (12) are equivalent to polynomial equations in the mutual distances. To obtain a system of equations which is symmetrical under permutation of the four bodies, we view (12) as three equations:

$$S_{12}S_{34} - S_{14}S_{23} = S_{12}S_{34} - S_{13}S_{24} = S_{13}S_{24} - S_{14}S_{23} = 0.$$

So we have a system of nine equations satisified by the six mutual distances of any relative equilibrium.

A typical one of the six Albouy-Chenciner equations is:

$$\begin{aligned}(17)\quad & 2(m_1+m_2)r_{13}^3r_{14}^3r_{23}^3r_{24}^3 - 2mr_{12}^3r_{13}^3r_{14}^3r_{23}^3r_{24}^3 \\ &+ m_3r_{12}^3r_{13}^3r_{14}^3r_{24}^3 - m_3r_{12}r_{13}^5r_{14}^3r_{24}^3 + m_3r_{12}r_{13}^3r_{14}^3r_{23}^2r_{24}^3 \\ &+ m_3r_{12}^3r_{14}^3r_{23}^3r_{24}^3 + m_3r_{12}r_{13}^2r_{14}^3r_{23}^3r_{24}^3 - m_3r_{12}r_{14}^3r_{23}^5r_{24}^3 \\ &+ m_4r_{12}^3r_{13}^3r_{14}^3r_{23}^3 - m_4r_{12}r_{13}^3r_{14}^5r_{23}^3 + m_4r_{12}r_{13}^3r_{14}^3r_{23}^3r_{24}^2 \\ &+ m_4r_{12}^3r_{13}^3r_{23}^3r_{24}^3 + m_4r_{12}r_{13}^3r_{14}^2r_{23}^3r_{24}^3 - m_4r_{12}r_{13}^3r_{23}^3r_{24}^5 = 0\end{aligned}$$

where $m = m_1 + m_2 + m_3 + m_4$. The others are similar. Since there are fourteen distinct monomials, the corresponding Newton polytope in $\mathbf{R}^6$ has fourteen vertices. For example, if we order our six variables $r = (r_{12}, r_{13}, r_{14}, r_{23}, r_{24}, r_{34})$ then the vector of exponents of the first term of the polynomial above is $k = (0, 3, 3, 3, 3, 0) \in \mathbf{R}^6$ and this point will be one of the vertices of the Newton polytope.

The Dziobek equations are

$$\begin{aligned} r_{14}^3r_{23}^3 - r_{12}^3r_{34}^3 - r_{12}^3r_{14}^3r_{23}^3 + r_{12}^3r_{14}^3r_{34}^3 + r_{12}^3r_{23}^3r_{34}^3 - r_{14}^3r_{23}^3r_{34}^3 &= 0 \\ r_{13}^3r_{24}^3 - r_{12}^3r_{34}^3 - r_{12}^3r_{13}^3r_{24}^3 + r_{12}^3r_{13}^3r_{34}^3 + r_{12}^3r_{24}^3r_{34}^3 - r_{13}^3r_{24}^3r_{34}^3 &= 0 \\ r_{14}^3r_{23}^3 - r_{13}^3r_{24}^3 - r_{13}^3r_{14}^3r_{23}^3 + r_{13}^3r_{14}^3r_{24}^3 + r_{13}^3r_{23}^3r_{24}^3 - r_{14}^3r_{23}^3r_{24}^3 &= 0 \end{aligned}$$

Their Newton polytopes have six vertices.

Inclusion of the Dziobek equations may seem superfluous since it seems likely that the six Albouy-Chenciner equations already have finitely many solutions. From a computational point of view, their inclusion entails advantages and disadvantages. The Minkowski sum polytope has more faces, and hence there are more reduced systems to check. But the reduced systems are simpler. Many of them are trivial and those which are not are at least tractable. We were not able to successfully apply our methods to the six Albouy-Chenciner equations.

4.1. **Computation of the Minkowski Sum Polytope.** The Minkowski sum polytope, $P \subset \mathbf{R}^6$, of the nine Newton polytopes of our equations is quite complicated. A straightforward approach to such computations, which works well in simpler examples, is simply to add the vertices of the summands and then find a minimal set of vertices and facets with a program such as Porta 1.3.2 [9] or lrs [4].

In this case, however, it was necessary to develop an indirect approach which will be described in this section.

Porta uses the double-description method of Motzkin, Raiffa, Thompson, and Thrall [36]. It is implemented with multiple precision integer arithmetic. This algorithm is among the most efficient for high-dimensional, non-simple polytopes. On the other hand, the program lrs uses a reverse search linear programming approach, also implemented with exact arithmetic [5]. However, computing our polytope P directly using these programs proved infeasible on our current hardware. Our idea was to exploit permutation symmetry to reduce the computations to a size which these programs could handle.

The permutation group S_4 acts on the subscripts of the m_i and the r_{ij} (with the convention that $r_{ji} = r_{ij}$). Under this action the nine equation of our system will be permuted. In fact the six Albouy-Chenciner equation are permuted among themselves as are the three Dziobek equations. Let P_1 be the Minkowski sum of the Newton polytopes of the Albouy-Chenciner equations and P_2 the sum of the Newton polytopes for the Dziobek equations. Then the group action above induces actions on the polytopes P_1, P_2 and P. It was possible to directly compute P_1 and P_2 using Porta. We want to use these partial sums together with the symmetry to find $P = P_1 + P_2$.

We begin by constructing a list of possible inward normals for the facets of P. From polytope theory we know that every face of P is the Minkowski sum of a face from P_1 and a face from P_2. In particular, some of the facets of P can be found by taking the sum of a facet from one of the P_i with a vertex from the other. As an inward normal for this facet of P we can use the normal for the facet of P_i used to construct it. So we add all of the inward normals from the P_i to our list.

Next we want to compute the normals for the other facets of P. These new facets F_i arise from sums of lower-dimensional faces of P_1 and P_2, i.e., $F_i = g_i + h_i$ where g_i is a s-dimensional face of P_1, h_i is a t-dimensional face of P_2. To find all possible normal vectors of new facets it suffices to compute the sums of faces with $1 \leq s \leq 4$, $1 \leq t \leq 4$, and $s+t = 5$. The g_i and h_i can be found by computing the face lattices of P_1 and P_2. The f-vectors (the number of faces $f_0, \ldots, f_5$ of each dimension) for P_1 and P_2 are $(2881, 12942, 22504, 18657, 7178, 964)$ and $(54, 210, 357, 312, 135, 24)$ respectively. Once the facet F_i is in hand, it is elementary to compute its normal. The symmetry of P_1 and P_2 can be exploited by only computing sums where the g_i are representatives from each orbit of the symmetry action. This also means that the entire face lattice of P_1 need not be computed. Since there are very few faces of P_1 which have nontrivial stabilizers, the S_4 symmetry reduces the computational complexity by a factor of almost 24.

This procedure yields a large list of possible facet normals, most of which are spurious. For each candidate normal, the vertices from the 'raw' Minkowski sum $P_1 + P_2$ which minimized the inner product with the normal were found. This 'raw' Minkowski sum, computed by taking all possible sums of vertices of P_1 and P_2, has 134784 points. Using the fact that the dimension of the set of minimizing vertices has to be 5 for a true facet normal we could weed out the spurious normals. Besides the 973 distinct facet normals already present in P_1 and P_2, there are 2007 more arising from lower dimensional combinations as described above so the polytope P has 2980 facets.

The vertices of P are the extreme points among the 134784 points found above, i.e., those which are not in the convex hulls of other points on the list. To find them we used the following procedure. First we constructed other, asymmetrical raw Minkowski sums by writing $P = P_1' + P_2'$ where P_1' is the Minkowski sum of seven of the nine polytopes and P_2' is the sum of the other two. Since the polytope P is symmetric, any point of this raw sum for which the complete S_4 orbit is not contained in the sum cannot be a true vertex. By decomposing P in several different ways and intersecting the resulting lists, we found a much smaller set of candidate vertices. Eliminating any of these candidates which was not a point of intersection of six different facets reduced the list to 13836 points. Some of these are still not true vertices because the normals to these incident facets do not span $\mathbf{R}^6$. Removing these redundant points produced a list of 12828 vertices.

The final list of vertices and normals was verified as follows. Each of the 2980 normals α determines an inequality $\alpha \cdot k \geq \mu$ where μ is the minimum value of $\alpha \cdot k$ on the big list of 134784 points. On the other hand, a Porta computation showed that the convex hull of the smaller list of 12828 vertices is defined by exactly the same inequalities. It follows that the convex hull of the smaller list is really the polytope P and that the inequalities define P.

As a final verification, lrs was used to transform the polytope between a vertex representation and a facet representation. Starting from our list of 12828 vertices the program produced our list of 2980 facet inequalities and vice versa.

4.2. **Analysis of the Reduced Equations.** The “big Minkowski” polytope computed in the last section has 2980 facets and 13836 vertices. The next step of our procedure is to analyze the reduced systems corresponding to the various faces of P. The simplest faces are those for which one or more of the reduced equations consists of a single term. We will refer to such a face or reduced system as *trivial*. A trivial reduced system can have a solution with all variables nonzero only if the coefficient of the monomial vanishes. Since the single term in question is one of the terms of the original equations, we can guarantee that all possible trivial reduced systems will have no nonzero solutions by assuming that all of the original coefficients are nonzero. For the equations above, this amounts to assuming that the masses satisfy $m_i \neq 0$, $m \neq 0$ and $m_i + m_j \neq 0$ for distinct indices $i, j \in \{1, 2, 3, 4\}$. Using Mathematica [47] we found that all but 53 of the 2980 facets lead to trivial reduced systems of equations. The nontrivial ones are shown in Table 1.

Each of these inequalities corresponds to a facet of P with inward normal vector α given by the coefficients of $(k_1, \ldots, k_6)$. For example, inequality 22 has inward normal $\alpha = (0, 0, -1, -1, 0, 0)$. We will analyze the reduced equations only for the facets with normal vectors in the half-space $c \cdot \alpha \geq 0$ where $c = (-1, -1, -1, -1, -1, -1)$, i.e., for the facets corresponding to inequalities 22–53.

For inequality 22, we get the reduced equations by selecting from each of the nine polynomials f_i of our system, those monomials whose exponent vectors minimize $\alpha \cdot k = -k_3 - k_4$ among all the monomials in f_i. After cancelling out factors which are powers of the variables r_{ij}, the result is the following system of nine reduced equations $f_{i\alpha}$:

$$m_4 r_{13}^3 r_{14}^2 + m_3 r_{23}^2 r_{24}^3 = m_4 r_{12}^3 r_{14}^2 + m_2 r_{23}^2 r_{34}^3 = 0$$
$$m_3 r_{12}^3 r_{23}^2 + m_1 r_{14}^2 r_{34}^3 = m_2 r_{13}^3 r_{23}^2 + m_1 r_{14}^2 r_{24}^3 = 0$$
$$m_4 r_{12}^3 r_{13}^3 r_{24}^3 + m_4 r_{12}^3 r_{13}^3 r_{34}^3 + m_1 r_{12}^3 r_{24}^3 r_{34}^3 + m_1 r_{13}^3 r_{24}^3 r_{34}^3 - 2m r_{12}^3 r_{13}^3 r_{24}^3 r_{34}^3 = 0$$

1.	$k_1 + k_2 + k_3 + k_4 + k_5 + k_6 \geq 90$	28.	$-k_1 - k_2 - k_3 >= -63$
2.	$2k_1 + 2k_2 + 2k_4 - k_5 - k_6 \geq 28$	29.	$-k_4 - k_5 - k_6 \geq -69$
3.	$2k_1 + 2k_2 - k_3 + 2k_4 - k_6 \geq 28$	30.	$-k_2 - k_3 - k_6 \geq -69$
4.	$2k_1 + 2k_2 - k_3 + 2k_4 - k_5 \geq 28$	31.	$-k_1 - k_3 - k_5 \geq -69$
5.	$2k_1 + 2k_3 - k_4 + 2k_5 - k_6 \geq 28$	32.	$-k_1 - k_2 - k_4 \geq -69$
6.	$2k_1 - k_2 + 2k_3 + 2k_5 - k_6 \geq 28$	33.	$-k_2 - k_3 - k_4 - k_5 >= -81$
7.	$2k_1 - k_2 + 2k_3 - k_4 + 2k_5 \geq 28$	34.	$-k_1 - k_3 - k_4 - k_6 >= -81$
8.	$2k_2 + 2k_3 - k_4 - k_5 + 2k_6 \geq 28$	35.	$-k_1 - k_2 - k_5 - k_6 >= -81$
9.	$-k_2 - k_3 + 2k_4 + 2k_5 + 2k_6 \geq 28$	36.	$-2k_3 - 3k_5 - 3k_6 >= -170$
10.	$-k_1 + 2k_2 + 2k_3 - k_5 + 2k_6 \geq 28$	37.	$-3k_3 - 2k_5 - 3k_6 >= -170$
11.	$-k_1 + 2k_2 + 2k_3 - k_4 + 2k_6 \geq 28$	38.	$-3k_3 - 3k_5 - 2k_6 >= -170$
12.	$-k_1 - k_3 + 2k_4 + 2k_5 + 2k_6 \geq 28$	39.	$-2k_2 - 3k_4 - 3k_6 >= -170$
13.	$-k_1 - k_2 + 2k_4 + 2k_5 + 2k_6 \geq 28$	40.	$-3k_2 - 2k_4 - 3k_6 \geq -170$
14.	$k_1 + k_2 + k_4 \geq 36$	41.	$-3k_2 - 3k_4 - 2k_6 \geq -170$
15.	$k_1 + k_3 + k_5 \geq 36$	42.	$-2k_1 - 3k_4 - 3k_5 \geq -170$
16.	$k_2 + k_3 + k_6 \geq 36$	43.	$-2k_1 - 3k_2 - 3k_3 \geq -170$
17.	$k_4 + k_5 + k_6 \geq 36$	44.	$-3k_1 - 2k_4 - 3k_5 \geq -170$
18.	$k_1 + k_2 + k_3 \geq 30$	45.	$-3k_1 - 3k_4 - 2k_5 \geq -170$
19.	$k_1 + k_4 + k_5 \geq 30$	46.	$-3k_1 - 2k_2 - 3k_3 \geq -170$
20.	$k_2 + k_4 + k_6 \geq 30$	47.	$-3k_1 - 3k_2 - 2k_3 \geq -170$
21.	$k_3 + k_5 + k_6 \geq 30$	48.	$-2k_2 - 3k_3 - 3k_4 - 2k_5 \geq -208$
22.	$-k_3 - k_4 \geq -50$	49.	$-3k_2 - 2k_3 - 2k_4 - 3k_5 \geq -208$
23.	$-k_2 - k_5 \geq -50$	50.	$-2k_1 - 3k_3 - 3k_4 - 2k_6 \geq -208$
24.	$-k_1 - k_6 \geq -50$	51.	$-2k_1 - 3k_2 - 3k_5 - 2k_6 \geq -208$
25.	$-k_3 - k_5 - k_6 \geq -63$	52.	$-3k_1 - 2k_3 - 2k_4 - 3k_6 \geq -208$
26.	$-k_2 - k_4 - k_6 \geq -63$	53.	$-3k_1 - 2k_2 - 2k_5 - 3k_6 \geq -208$
27.	$-k_1 - k_4 - k_5 \geq -63$		

TABLE 1. Nontrivial facets for the Newtonian Minkowski sum polytope

$$m_3 r_{12}^3 r_{13}^3 r_{24}^3 + m_2 r_{12}^3 r_{13}^3 r_{34}^3 + m_3 r_{12}^3 r_{24}^3 r_{34}^3 + m_2 r_{13}^3 r_{24}^3 r_{34}^3 - 2m r_{12}^3 r_{13}^3 r_{24}^3 r_{34}^3 = 0$$
$$r_{13}^3 r_{24}^3 - r_{12}^3 r_{13}^3 r_{24}^3 - r_{12}^3 r_{34}^3 + r_{12}^3 r_{13}^3 r_{34}^3 + r_{12}^3 r_{24}^3 r_{34}^3 - r_{13}^3 r_{24}^3 r_{34}^3 = 0$$
$$r_{12}^3 + r_{34}^3 - 1 = r_{13}^3 + r_{24}^3 - 1 = 0$$

All such systems of reduced equations have a certain homogeneity. In this case, the monomials which appear in each equations have exponent vectors k with the same value of $-k_3 - k_4$. It follows that if $r = (r_{12}, r_{13}, r_{14}, r_{23}, r_{24}, r_{34}) \in \mathbf{T}$ is any solution, then for any $s \in \mathbf{C} \setminus 0$, we get another solution $(r_{12}, r_{13}, s^{-1} r_{14}, s^{-1} r_{23}, r_{24}, r_{34})$. In general, for the reduced equations $f_{i\alpha}$ we can rescale solutions via

$$(s^{\alpha_1} r_{12}, s^{\alpha_2} r_{13}, s^{\alpha_3} r_{14}, s^{\alpha_4} r_{23}, s^{\alpha_5} r_{24}, s^{\alpha_6} r_{34}).$$

Using this, we can arrange to have the following normalization condition hold:

$$r_{12} r_{13} r_{14} r_{23} r_{24} r_{34} = 1. \tag{18}$$

To see this note that under the rescaling above, the product of the r_{ij} changes by a factor for $s^{|\alpha|}$, where $|\alpha| = \alpha_1 + \ldots + \alpha_6$. As long as $|\alpha| \neq 0$ and $r_{ij} \neq 0$, we can rescale to make the product be 1.

After appending the normalization condition (18) to the reduced system we get ten equations. The equations generate an ideal in the polynomial ring with variables r_{ij} and m_i. If this ideal contains polynomials in the mass variables alone, then these polynomials give necessary conditions on the masses for the existence of values $r_{ij} \in \mathbf{C}$ making the equations valid. In particular if these mass polynomials are not zero for a given choice of m_i then for those m_i, the reduced equations have no solutions $r \in \mathbf{T}$ as required. Such polynomials in the masses alone can be found by computing a Gröbner basis for the ideal with an appropriately chosen monomial ordering (for an introduction to Gröbner basis theory, see [10]). Our Gröbner basis calculations were carried out using Mathematica and checked using Macaulay 2 [17]. For the reduced equations above, the mass polynomial $m_1m_4 - m_2m_3$ turns out to be in the ideal. Thus if this polynomial is nonzero, the reduced system has no solutions in $\mathbf{T}$. It follows from symmetry that the inequalities 23 and 24 will lead to similar mass polynomials $m_im_j - m_km_l$ with permuted subscripts. When writing such mass polynomials, the subscripts i, j, k, l will always represent distinct elements of $\{1, 2, 3, 4\}$. Unfortunately, these polynomials can vanish for positive masses and so the vector α remains a possible order of a Puiseux solution for such masses. However, this possibility will be ruled out in section 5.

In addition to the reduced systems arising from the nontrivial facets in the table, we have to study the reduced systems arising from the the lower-dimensional faces of the Minkowski sum. These faces can be constructed by intersecting two or more incident facets of P. If a set of facets are incident then the face obtained by intersecting them can be defined by an inequality of the form $\alpha \cdot k \geq \mu$ where α is any inward pointing normal vector to the face.. The choice of the inward normal α is not unique, but the reduced equations obtained are independent of which normal is used. We used the sum of the normals of the incident facets. The terms appearing in these reduced polynomials $f_{i\alpha}$ are always a subset of the terms which were present in the reduced polynomials of the corresponding facets. It follows that if a certain facet is trivial (some equation reduces to a single term) then all of the subfaces of this facet are also trivial. So in our study of faces we need only consider intersections of the 53 nontrivial facets. Thus any nontrivial face can be described by giving a list of indices from 1–53, describing the facets whose intersection yields the given face.

We have used permutation symmetry extensively to cut down the number of reduced systems which need to be analyzed. The action of the permutation group S_4 on the variables m_i and r_{ij} induces an action on the set of faces of the polytope P. It is clear that we need only check one representative face from each orbit of S_4. We choose the representative whose list of facet indices is minimal with respect to lexicographic order. Call such a list of indices a *minimal representative* for its orbit. Then the list of minimal face representatives can be computed inductively by dimension as follows. First construct a list L_1 of minimal representatives of the facets, by choosing from each S_4 orbit the facet with the smallest index. Next suppose that the lists $L_1, \ldots, L_k$ have already been constructed, where L_k is the list of k-tuples $I = (i_1, \ldots, i_k)$, $i_1 < i_2 < \ldots < i_k$, of minimal representatives for nontrivial faces which are defined by intersecting k incident facets. Then L_{k+1} can be constructed by considering all possible extensions of the k-tuples $I \in L_k$ to $(k + 1)$-tuples $I' = (i_1, \ldots, i_k, i_{k+1})$ with $i_{k+1} > i_k$ and then eliminating those extensions which describe non-incident facets, those which lead to trivial reduced systems and those which not minimal in their group orbits. To see this, note that if $I' \in L_{k+1}$ then the k-tuple, I, obtained by dropping the last component must represent k incident facets which determine a nontrivial reduced system. We also need to see that I is a minimal representative of its group orbit. If not, then there is some k-tuple $J = g \cdot I = (j_1, \dots, j_k)$, $g \in S_4$ with $J < I$ in lexicographical order; here we denote the action of S_4 by a dot. Then we will show that $J' = g \cdot I' < I'$, contradicting the minimality of I'. Now $J' = (j'_1, \dots, j'_{k+1})$ consists of the k-tuple J together with the index $g \cdot i_{k+1}$ inserted in its proper order. No matter where this new index is inserted, we will have $(j'_1, \dots, j'_k) \leq J < I$ and so $J' < I'$ as required.

We can further cut down the list by using our requirement that each face we analyze must have a normal vector α lying in our chosen half-space $c \cdot \alpha \geq 0$. Now if a face is obtained by intersecting two or more facets which *all* have inward normals with $c \cdot \alpha < 0$ then every normal vector α for the face will also satisfy this inequality. This is because the normal vectors for faces are convex combinations of the normals of their defining facets. Hence we can eliminate from our list all faces obtained by intersecting only facets with indices between 1 and 21.

After using symmetry and the half-space condition, only the faces determined by the following lists of inequalities have to be analyzed:

$$\{22\} \quad \{25\} \quad \{29\} \quad \{33\} \quad \{36\} \quad \{48\}$$
$$\{2,25\} \quad \{2,36\} \quad \{14,25\} \quad \{14,36\} \quad \{18,29\} \quad \{22,48\}$$
$$\{25,36\} \quad \{33,48\} \quad \{2,14,25\} \quad \{2,14,36\}$$
$$\{2,25,36\} \quad \{14,25,36\} \quad \{2,14,25,36\}$$

Any other combination of inequalities has one of the following properties: either the facets are not incident in the polytope P or they are incident but determine a trivial reduced system of equations or they can be obtained from a combination on this list using symmetry.

For each of the corresponding reduced systems, we append the normalization condition (18) and use the Gröbner basis method to find polynomials in the masses which are contained in the resulting ideal. The ideals $\{25\}$, $\{29\}$, $\{36\}$, $\{48\}$, $\{2,36\}$, $\{18,29\}$, $\{22,48\}$, $\{25,36\}$, $\{33,48\}$ contain one of the polynomials m_i, $m_i + m_j$ or m which we are already assuming to be nonzero. Ideals $\{2,25\}$, $\{14,25\}$, $\{14,36\}$, $\{2,14,25\}$, $\{2,14,36\}$, $\{2,25,36\}$, $\{14,25,36\}$, $\{2,14,25,36\}$ contain sums of three masses $m_i + m_j + m_k$. This leaves only facets 22 and 33.

As noted above, the ideals of facet 22 and its symmetrical counterparts contain polynomials of the form $m_i m_j - m_k m_l$. The ideal of facet 33 is more complicated. It contains $(m_i + m_j)^2 (m_k + m_l)^2 (m_i^3 + m_j^3)(m_k^3 + m_l^3) Q$ where

$$Q = (m_i^3 - m_j^3)^2 (m_k^3 - m_l^3)^2 + 4 m_i^3 m_j^3 (m_k^3 - m_l^3)^2 + 4 m_k^3 m_l^3 (m_i^3 - m_j^3)^2. \tag{19}$$

This polynomial also vanishes for some positive masses and so the corresponding normal vector α is still a possible order for a Puiseux solution.

At this point, our analysis of reduced systems has shown that if

$$m_i \neq 0 \qquad m_i + m_j \neq 0 \qquad m_i + m_j + m_k \neq 0 \qquad m = m_1 + m_2 + m_3 + m_4 \neq 0$$

then almost all of the vectors α in the half-space $c \cdot \alpha \geq 0$ are ruled out as possible orders of a Puiseux series solution of our system. The only remaining possibilities are positive rational multiples of the vectors $\alpha_{22} = (0, 0, -1, -1, 0, 0)$ and $\alpha_{33} =$

$(0,-1,-1,-1,-1,0)$ and their symmetrical analogues. In the next section these orders are also shown to be impossible.

5. Analysis of the Exceptional Cases

Assuming that $m_i > 0$, we will show that there do not exist Puiseux series solutions of our system of nine equations whose order α is a positive multiple of $\alpha_{22} = (0,0,-1,-1,0,0)$ and $\alpha_{33} = (0,-1,-1,-1,-1,0)$. This will complete the proof of theorem 1.

If the masses are chosen so that the corresponding mass polynomials vanish then one can find solutions of the reduced equations in $\mathbf{T}$. But these solutions represent only the leading coefficients of a possible Puiseux series solution. If there really exists such a solution, it must be possible to find consistent values for the higher order terms in the series as well.

5.1. **Facet 22.** Let us assume that $m_1m_4 = m_2m_3$ and that $m_i > 0$. Because our equations are homogeneous in the masses we can further assume that $m_4 = 1$. We wish to analyze the possible Puiseux series which begin with a positive rational multiple of the exponent vector $\alpha_{22} = (0,0,-1,-1,0,0)$. Recall that our variables are ordered as $r = (r_{12}, r_{13}, r_{14}, r_{23}, r_{24}, r_{34})$. If such a series exists, then the projection of the variety V onto the r_{14} and r_{23} axes does not consist of just a finite set, so it must be dominant. By proposition 1, there will be a Puiseux series solution with $r_{14}(t) = 1/t$. Then the exponent vector will be exactly α_{22}. It follows that the leading term of $r_{23}(t)$ is also of degree -1 while all the other series begin at order 0.

Making the substitution $r = (x_{12},\ x_{13},\ 1/t,\ x_{23}/t,\ x_{24},\ x_{34})$ and clearing the denominators gives a system of nine polynomial equations $F(t,x) = 0$ in five unknowns $x = (x_{12}, x_{13}, x_{23}, x_{24}, x_{34})$ with coefficients which are polynomials in t. We can expand the system as $F(t,x) = F_0(x) + F_1(x)t + F_2(x)t^2 + \ldots = 0$ (however, it turns out that $F_1(x) = 0$). A Puiseux series solution of our original system with order α would give rise to a Puiseux solution $x(t)$ of the new system with order 0.

The equation $F_0(x(0)) = 0$ is equivalent to the reduced equations for α_{22}. These determine the constant terms of the $x_{ij}(t)$, although not quite uniquely. Namely $x_{12}^3(0) = \frac{m_2}{1+m_2}$, $x_{34}^3(0) = \frac{1}{1+m_2}$, $x_{13}^3(0) = \frac{m_3}{1+m_3}$, $x_{24}^3(0) = \frac{1}{1+m_3}$, and $x_{23}(0) = \pm i$.

Using Mathematica, it is not hard to show that the rank of DF_0 at $x(0)$ is 5. By the implicit function theorem, the x_{ij} must be power series in t. Since the expansion of F continues at order 2 we set $x_{ij}(t) = x_{ij}(0) + u_{ij}t^2 + v_{ij}t^3 + \ldots$. Substitution into F leads to systems of linear equations for u and v. It turns out that the equations for u are consistent and determine u uniquely. However, the resulting equations for v are inconsistent so no Puiseux solution of this order exists.

5.2. **Facet 33.** Suppose now that (19) vanishes for positive masses. A representative case is when $m_1 = m_2$ and $m_3 = m_4 = 1$. We want to rule out Puiseux series whose order is a positive rational multiple of the exponent vector $\alpha_{33} = (0,-1,-1,-1,-1,0)$. As above, we may assume the order is exactly α_{33} and set $r_{13}(t) = 1/t$. Setting $r = (x_{12},\ 1/\mathrm{t},\ x_{14}/t,\ x_{23}/t,\ x_{24}/t,\ x_{34})$ gives a system $F(t,x) = F_0(x) + F_1(x)t + F_2(x)t^2 + \ldots = 0$ for $x = (x_{12}, x_{14}, x_{23}, x_{24}, x_{34})$. We want to rule out Puiseux solutions of order 0.

The reduced equations give the leading terms $x_{12}^3(0) = \frac{m_1}{1+m_1}$, $x_{34}^3(0) = \frac{1}{1+m_1}$, $x_{14}^3(0) = \pm i$, $x_{23}^3(0) = \mp i$, and $x_{24}(0) = -1$. Again the rank of $DF_0(x(0))$ is 5.

Although the polynomial $F_1(x)$ is nonzero, it turns out that $F_1(x(0)) = 0$ so once again, the possible series solutions take the form $x_{ij}(t) = x_{ij}(0) + u_{ij}t^2 + v_{ij}t^3 + \ldots$. As before, u is uniquely determined but then the equations determining v are found to be inconsistent. This completes the proof of theorem 1.

6. Upper Bound and Lower Bounds

6.1. **Lower Bound.** There are always exactly 12 collinear relative equilibria [35]. MacMillan and Bartky [28] prove the existence result of one convex relative equilibrium for each of the 6 rotationally distinct cyclic orderings of the 4 masses. The strictly concave case, where one particle is contained in the convex hull of the other three points, is more complicated. In [16] it is shown that if all four masses are different then there are at least 16 concave relative equilibria giving a lower bound of 34 for the total number of relative equilibria (the lower bound of 8 concave configurations stated in [16] counts reflected configurations as equivalent).

In fact this result can easily be refined to give the following proposition, the proof of which contains some additional geometric information about these configurations.

Proposition 4. *There are always at least 14 concave relative equilibria in the four-body problem. Unless exactly three of the masses are equal, there are at least 16 concave relative equilibria.*

Proof. The proof is based on the main result in [16] which can be formulated as follows. First we introduce a way of labeling a concave configuration of four masses. One mass, call it m_C, lies inside the triangle formed by the other three. For a scalene triangle, the sides and their lengths can be uniquely labeled L, I, and S such that $L > I > S$. By reflecting the configuration, if necessary, we can always assume that the edges occur in the counterclockwise order. We label the masses opposite the edges L, I, and S as m_L, m_I, and m_S respectively, so these will also be counterclockwise around the triangle. For an isosceles, but not equilateral, triangle, one or the other of L, S will be uniquely determined and the other two labels can be chosen to achieve counterclockwise order. For an equilateral triangle, just choose the labels to get the desired order (it is known that this case can only occur if $m_L = m_I = m_S$).

With these conventions, the main result of [16] shows that there exists a concave relative equilibrium of this form provided

$$m_I \geq m_L \qquad m_I \geq m_S \tag{20}$$

Of course the reflection of this configuration will be a rotationally distinct relative equilibrium with m_L, m_I, m_S in clockwise order.

Given four masses $m_1, \ldots, m_4$ we obtain several different concave configurations by assigning which masses are to play the roles of m_C, m_I, m_L, m_S. Assume without loss of generality that $m_1 \geq m_2 \geq m_3 \geq m_4$. Then the following 8 permutations will satisfy (20):

$$\begin{aligned}(m_C, m_I, m_L, m_S) = &(1,2,3,4) \quad (2,1,3,4) \quad (3,1,2,4) \quad (4,1,2,3)\\ &(1,2,4,3) \quad (2,1,4,3) \quad (3,1,4,2) \quad (4,1,3,2)\end{aligned}$$

Now if no three of the masses are equal, it is easy to see that the 16 relative equilibria determined by these 8 configurations and their reflections are pairwise inequivalent under our rotational symmetry. First note that if two configurations are equivalent, then they must have the same central mass and the masses on the triangle must be in the same cyclic order. For a fixed m_C there are just two permutations in the list above and these have the triangular points in opposite orders. Finally, these two configurations cannot be equivalent to one another's reflections. To see this, note that since the triangle is not equilateral, at least one of the edges L or S is uniquely determined by the geometry (independent of our ordering convention above). In that case, the two equivalent configurations would have to have the same choice for m_L or m_S. But two different permutations from the list which have the same m_C, always differ in both m_L and m_S.

Since Albouy [2] has shown that there are 50 relative equilibria when all four masses are equal, it only remains to consider the case of exactly three equal masses. If $m_1 = m_2 = m_3 > m_4$ then the six permutations in the table with $m_C \neq m_4$ still give 12 different relative equilibria. But the other two entries might be realized by equilateral triangles with m_4 at the center. These are equivalent to one another's reflections and we get only 14 in all. The case $m_1 > m_2 = m_3 = m_4$ is similar. QED

A complementary result of Xia [49] gives a relatively simple proof of a lower bound of 16 concave relative equilibria if certain nondegeneracy assumptions on the function IU^2 hold, otherwise at least 8. Here U is the potential $\sum_{i<j} m_i m_j r_{ij}^{-1}$ and I is the moment of inertia $m^{-1} \sum_{i<j} m_i m_j r_{ij}^2$.

To summarize, we can say that there are always at least 32 relative equilibria in the four-body problem. Unless exactly three of the masses are equal and the function IU^2 is degenerate, there are at least 34 relative equilibria.

6.2. **Upper Bound.** We have shown that for positive masses, the nine Albouy-Chenciner and Dziobek equations have finitely many solutions $r = (r_{12}, \ldots, r_{34})$ in the complex algebraic torus. Since the number of equations used exceeds the number of unknowns, it is not possible to directly apply the mixed volume bounds of BKK theory to this system. However, we will now apply it to a system of 10 equations and 10 unknowns derived from the equations (13) in section 2.2.

As noted in section 2.2, any non-collinear relative equilibrium determines a solution of (13) with $r_{ij} \neq 0$ and $z_i \neq 0$. However, it is possible that $k = 0$. To sidestep this difficulty we replace the equations $f_1 = \ldots = f_4 = 0$ by the differences $f_1 - f_4 = f_2 - f_4 = f_3 - f_4 = 0$ to obtain a system of 10 equations for (r, z). Then an upper bound for the number of complex solutions $(r, z) \in \mathbf{T}^{10}$ gives an upper bound for the number of equivalence classes of relative equilibria. In fact, we are only interested in solutions with r_{ij} real and positive.

The main point is that there is a two-to-one mapping from the set of solutions $(r, z) \in \mathbf{T}^{10}$ of this new system to solutions $r \in \mathbf{T}^6$ of the Albouy-Chenciner and Dziobek equations. To see this, suppose we have a solution (r, z) of our ten equations with $r_{ij} \neq 0$. We will show that r satisifies the nine Albouy-Chenciner and Dziobek equations. First of all, Dziobek's equations (12) follow immediately from the equations $S_{ij} = z_i z_j$. Next we will show how to derive the Albouy-Chenciner equation with $(i, j) = (1, 2)$; the others can be derived in a similar way. Setting $(i, j) = (1, 2)$ in (10) and replacing S_{ij} by $z_i z_j$ gives:

$$(21) \quad -2(m_1 + m_2) z_1 z_2 r_{12}^2 + m_3 z_1 z_3 (r_{23}^2 - r_{13}^2 - r_{12}^2) + m_3 z_2 z_3 (r_{13}^2 - r_{23}^2 - r_{12}^2) \\ + m_4 z_1 z_4 (r_{24}^2 - r_{14}^2 - r_{12}^2) + m_4 z_2 z_3 (r_{14}^2 - r_{24}^2 - r_{12}^2).$$

A bit of algebra shows that this expression is equal to

$$-(z_1 + z_2)r_{12}^2 f_0 + (z_2 - z_1)(f_1 - f_2)$$

so our system of 10 equations implies the Albouy-Chenciner equations.

On the other hand, we will now show that given any $r \in \mathbf{T}^6$, there exist at most two ways to find $z \in \mathbf{T}^4$ such that $z_i z_j = S_{ij}$ hold. Since all the z_i are to be nonzero, we must have $S_{ij} \neq 0$. Then the ratios of the z_i are uniquely determined by equations of the form $z_i/z_j = S_{ik}/S_{jk}$. Setting $z_i = c\zeta_i$ where $\zeta_i = z_i/z_4$, we have $c^2\zeta_1\zeta_2 = S_{12}$ which determines c up to sign. Thus z is determined up to sign.

If $m_i > 0$, we have shown that the Albouy-Chenciner and Dziobek system has finitely many solutions $r \in \mathbf{T}^6$ and it follows that our system of 10 equations has finitely many solutions in $\mathbf{T}^{10}$. Then BKK theory shows that the number of solutions is bounded above by the mixed volume of the Newton polytopes P_i, $i = 1, \ldots, 10$, of these equations. Using the program Mixvol we found that the mixed volume is 25380 [13]. This bounds the complex solutions with all variables nonzero. However, at most one third of these solutions can have real values for the mutual distances r_{ij}. To see this, note that if $(r, z) \in \mathbf{T}^{10}$ is any solution then so is $(\omega r, z)$ where ω is any third root of unity. So we can have at most 8460 real solutions (r, z) (probably the number of real solutions is much less). Since there are exactly 12 collinear relative equilibria which are not included here, our upper bound for the number of relative equilibria of any four positive masses is 8472.

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