- Chapter 2.6: Directional derivatives and the gradient

- The formal definition of the directional derivative: if f is a scalar-valued function of n variables, and $\vec{v}$ is a unit-length vector in $\mathbb{R}^n$, then the directional derivative of f at a point $\vec{a}$ in the direction of $\vec{v}$ is

$$D_{\vec{v}}f(\vec{a}) = \lim_{h \to 0} \frac{f(\vec{a} + h\vec{v}) - f(\vec{a})}{h}$$

  if such a limit exists.

- Although we won't usually work from the definition, lets look at an example computed directly from the definition.

- Example: Compute the directional derivative of $f(x, y) = xy$ at $(1, 1)$ in the direction $\vec{v} = (\frac{1}{2}, \frac{\sqrt{3}}{2})$.

$$D_{\vec{v}} f(\vec{a}) = \lim_{h \to 0} \frac{f(\vec{a} + h\vec{v}) - f(\vec{a})}{h}$$

$$= \lim_{h \to 0} \frac{f(1 + h/2, 1 + h\frac{\sqrt{3}}{2}) - f(1, 1)}{h}$$

$$= \lim_{h \to 0} \frac{1 + h/2 + h\frac{\sqrt{3}}{2} + h^2\frac{\sqrt{3}}{4} - 1}{h}$$

$$= \lim_{h \to 0} \left(\frac{1}{2} + \frac{\sqrt{3}}{2} + h\frac{\sqrt{3}}{4} \right)$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2}$$

- We can avoid these tedious computations by thinking about the directional derivative in a slightly different way. For each $\vec{a}$ and $\vec{v}$, the function $F(h) = f(\vec{a} + h\vec{v})$ is just a scalar function of one variable - its the composition of f with the vector valued function $\vec{q}(h) = \vec{a} + h\vec{v}$:
$$F(h) = f \circ \vec{q}(h).$$
So we can also write the directional derivative as $F'(0)$, which by the chain rule is

$$\begin{aligned} D_{\vec{v}} f(\vec{a}) &= F'(0) \\ &= Df(\vec{q}(0)) \cdot Dq(0) \\ &= Df(\vec{q}(0)) \cdot \vec{v} \\ &= Df(\vec{a}) \cdot \vec{v} \\ &= \bigtriangledown f(\vec{a}) \cdot \vec{v} \end{aligned}$$

This tells us that the directional derivative is just the dot product of the gradient with the direction vector. This is good news, because it means that all the directional derivatives are determined by the partial derivatives in the coordinate directions.

- If we think about the directional derivative in terms of the gradient, we see that

$$\begin{aligned} \bigtriangledown f(\vec{a}) \cdot \vec{v} &= || \bigtriangledown f(\vec{a})|| ||\vec{v}|| \cos\theta \\ &= || \bigtriangledown f(\vec{a})|| \cos\theta \end{aligned}$$

  since we have to use a unit vector in our directional derivative formula. Here θ is the angle between the gradient and $\vec{v}$. This means that the largest value of the directional derivative is obtained by taking $\vec{v}$ parallel to the gradient. In other words, the gradient tells us the direction of greatest increase of a function.

- For a scalar function of two variables, we can rephrase the above as saying that the gradient tells us the steepest uphill direction at any point on our graph.

- The gradient is also useful in thinking about the level sets of functions. Lets look at an example.

Example. Consider the function $g(x, y) = x^6 + xy + y^6$.
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Suppose we are interested in understanding the level curves of g. Because g is constant along the level curves, its directional derivative along a tangent line to a level curve should be zero. For this to happen, $|| \bigtriangledown g(\vec{a})|| \cos\theta = 0$ so either the gradient is the zero vector or it is orthogonal (at a right angle) to the level curve.

The gradient of g is $\bigtriangledown g = (6x^5 + y, 6y^5 + x)$. Thus at the point $(0, 1/4)$, the gradient is $(1/4, 3/512)$. This tells us that the level curve at that point has a tangent line in the direction of $(-3/512, 1/4)$ - nearly vertical.

- Level sets in higher dimensions: if we consider a level set of a function of three variables, such as $f(x, y, z) = x^2 + 2y^2 + 3z^2$, then the gradient is still orthogonal to the level set. That is, it is the normal direction to the tangent plane of the level set at that point.

Lets compute the tangent plane to the level set $f = 6$ at the point $(1, 1, 1)$. The gradient is $\bigtriangledown f = (2x, 4y, 6z)$, so the normal vector is $(2, 4, 6)$. Thus the tangent plane is

$$\begin{aligned} &\bigtriangledown f((1, 1, 1)) \cdot ((x, y, z) - (1, 1, 1)) \\ &= (2, 4, 6) \cdot ((x - 1), (y - 1), z - 1) = 0 \end{aligned}$$

or

$$2x + 4y + 6z = 12$$