Math 5535 Practice Final

(1) Sketch the phase portrait for the system $x' = xy$, $y' = 1 - x - y - xy$, including nullclines, fixed points and their stability. Be sure to include any calculations you use.

Solution: The vertical nullclines are the coordinate axes, and the horizontal nullcline is given by the rational function $y = \frac{1-x}{1+x}$. The fixed points are at $(1, 0)$ and $(0, 1)$. The Jacobian matrix is $\begin{bmatrix} y & -(y+1) \\ x & -(x+1) \end{bmatrix}$. The eigenvalues of the two fixed points are $-1, -1$ and $-1, 1$, respectively. So $(1, 0)$ is stable, $(0, 1)$ is a saddle (unstable). It is important to note that $(1, 0)$ is degenerate, with only one eigenvector.

(2) Suppose we have two two-dimensional systems $X' = F(X)$ and $X' = G(X)$ where the vector functions $F(X)$ and $G(X)$ are orthogonal, i.e. $F(X) \cdot G(X) = 0$ for any $X = (x_1, x_2)$. Suppose further that the first system has a closed orbit. Show that the second system must have at least one equilibrium point.

Solution: This can be easily done using index theory. Alternatively, we can use the Poincare-Bendixson theorem. The region Ω bounded by the given closed orbit is positively invariant for the flow of either $G(X)$ or $-G(X)$. The existence of a fixed point does not depend on the sign of $G(X)$ so without loss of generality we can assume that Ω is positively invariant. Then by Theorem 6.2.2 there must be a fixed point in Ω.

(3) Find a period-doubling pitchfork bifurcation that occurs at a negative parameter value ($a < 0$) for the logistic map $f_a(x) = ax(1 - x)$.

Solution: To find the equation for the period-two points, we compute $\frac{f_a^2(x)-x}{f_a(x)-x} = 1 + a - ax - a^2x + a^2x^2 = 0$. Period-two orbits will appear when the discriminant of this quadratic equation changes from negative to positive. The discriminant is $a^2 - 2a - 3 = (a - 3)(a + 1)$, so sign changes occur at $a = 3$ and $a = -1$.

(4) The plot below shows the diagonal and the second iterate $f(f(x))$ of a cubic map $f(x)$. What can you determine about the nine intersections $q_1 < q_2 < \ldots < q_9$ of $f(f(x))$ with the diagonal?

Solution: Since f is cubic, it can have at most nine solutions to the equation $f(f(x)) - x = 0$. Three of these must be solutions to $f(x) - x = 0$, i.e. fixed points. The other six are three period-two orbits. The slopes at the fixed points must be positive since there we have $f(f(x))' = f'(f(x))f'(x) = (f'(x))^2$. The slope of the graph at the two points of a period-two orbit must be equal (again by the chain rule). Since there are 5 positive slope points and 4 negative slope points, there must be one positive slope period-two pair and two negative slope period-two pairs.

It is also worth noting that there must be a fixed point inside the interval bounded by any period-two orbit.

The matching steepness of the slope at points q_1 and q_9 shows that they must be a period-two pair, forcing q_3, q_5, and q_7 to be fixed points.

From slope-matching it can be seen that q_2 and q_4 are an orbit pair and thus q_6 and q_8 are as well.

(5) Analyze the behavior of the following linear three-dimensional system:

$$X' = AX, \text{ where } A = \begin{bmatrix} -1 & 2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution: Because A is block-diagonal, we can immediately see that one of the eigenvalues is 1, with eigenvector $(0, 0, 1)$. The characteristic polynomial of the upper-left 2 by 2 block is $(-1-\lambda)(-2-\lambda)+4 = \lambda^2+3\lambda+6$. These eigenvalues are complex, with negative real part.

So the plane $(x, y, 0)$ is invariant, with orbits that spiral in to the origin. The z-axis is also invariant, with orbits going exponentially off to infinity. Other orbits will spiral in towards the $z-axis$ while exponentially increasing their $|z|$ value.

(6) Consider the system $x' = y + mx$, $y' = -x + my - x^2y$. There is a Hopf bifurcation at $m = 0$. Describe this bifurcation, and include an analysis of the system at the bifurcation point. (See problem 6.4.2).

Solution: The eigenvalues of the linearized system at the origin are $m \pm I$, so the origin switches from stable to unstable as m increases past 0. A stable periodic orbit is created out of the origin as m becomes positive. When $m = 0$, we analyze the stability using the Lyapunov function $L = (x^2 + y^2)/2$. Since $L' = -x^2y^2$, and the trajectories move off of the axes, we can apply Theorem 5.3.7 to conclude that the origin is weakly stable when $m = 0$.