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Multidimensional Nonlinear Diffusion Arising in Population Genetics*

D. G. ARONSON AND H. F. WEINBERGER

School of Mathematics, University of Minnesota, 127 Vincent Hall, Minneapolis, Minnesota 55455

TO STANISLAW ULAM ON HIS SIXTY-FIFTH BIRTHDAY

1. INTRODUCTION

In this paper we shall study the behavior for large values of the time variable t of the semilinear diffusion equation

$$\partial u/\partial t = \Delta u + f(u), \tag{1.1}$$

where

$$\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^{\,2}}$$

is the Laplace operator in $\mathbb{R}^n$. Throughout this work we assume that $f(0) = f(1) = 0$ and consider only solutions $u(x, t)$ with values in $[0, 1]$. We are primarily interested in the propagation of perturbations from the rest state $u \equiv 0$ and in certain threshold phenomena. The problems which we consider are suggested by the classical theory of population genetics. In an earlier paper [1] we have discussed the genetic background in some detail and have studied propagation and threshold phenomena for diffusions in one space dimension. Here we are mainly concerned with diffusions in $\mathbb{R}^n$ for $n \geqslant 2$. Although many of the arguments employed in Ref. [1] are valid only for the case $n = 1$, we shall show that the essential features of the one-dimensional diffusions described in [1] are also present in the multidimensional case. In reference [1] we considered both the pure initial value problem and the initial-boundary value problem in a quarter-space; here we shall consider only the initial value problem.

Before describing our results in more detail we briefly summarize the background from population genetics.

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Consider a population of diploid individuals distributed, for example, on a planar habitat. Suppose that the gene at a specific locus in a specific chromosome pair occurs in two allelic forms denoted by a and A. The population is thus divided into three classes. The individuals in two of these classes, called homozygotes, have genotypes aa or AA, while those in the third class, called heterozygotes, have genotype aA. Assume that the population mates at random, producing offspring with a birth rate r, and that it diffuses through the habitat with diffusion constant equal to one. Moreover, assume that the death rate depends only on the genotype with respect to the alleles a and A. Let τ_1, τ_2, and τ_3 denote the death rates of the genotypes aa, aA, and AA, respectively. In general these death rates will be different so that some genotypes are more viable than others.

Let $v(x, t)$ denote the relative density of the allele A at the point x of the habitat at time t. In [1] it is shown that if the derivatives of the densities of the various genotypes are initially small, if r is very large, and if $\epsilon = |\tau_1 - \tau_2| + |\tau_2 - \tau_3|$ is very small, then for times which are small relative to ϵ^{-1} the relative density v is close to the solution of (1.1) with the same initial values as v and with

$$f(u) = u(1-u)\{(\tau_1 - \tau_2) - (\tau_1 - 2\tau_2 + \tau_3)u\}. \tag{1.2}$$

The use of Eq. (1.1) with f given by (1.2) in this context was first suggested by Fisher [3] on the basis of a heuristic argument.

Regardless of the values of the parameters τ_j, the function $f(u)$ given by (1.2) has the properties

$$f \in C^1[0, 1], \qquad f(0) = f(1) = 0. \tag{1.3}$$

Throughout this paper we shall always assume that (1.3) holds for the forcing term $f(u)$ in Eq. (1.1). Additional assumptions on $f(u)$ in (1.1) are suggested by the genetic model (1.2) for various relative values of the parameters τ_j. Since we can interchange the labels of a and A and hence the values of τ_1 and τ_3, there is no loss of generality in assuming that

$$\tau_1 \geqslant \tau_3 .$$

We shall be mainly concerned with the following cases.

Heterozygote Intermediate. Here $\tau_3 \leqslant \tau_2 < \tau_1$ so that the viability of the heterozygote lies between the viabilities of the homozygotes. The relevant properties of the function $f(u)$ are

$$f'(0) > 0, \qquad f(u) > 0 \qquad \text{for} \quad u \in (0, 1). \tag{1.4}$$

This is the case which is considered in the classical studies of Fisher [3] and Kolmogoroff, Petrovsky, and Piscounoff [13].

Heterozygote Superior. In this case $\tau_2 < \tau_3 \leqslant \tau_1$ so that the heterozygote is more viable than either homozygote. The relevant features of f are

$$\begin{gathered}\exists a \in (0,1) \ni f(u) > 0 \quad \text{for} \quad u \in (0,a) \quad \text{and} \quad f(u) < 0 \quad \text{for} \quad u \in (a,1) \\ f'(0) > 0, \qquad f'(1) > 0.\end{gathered} \tag{1.5}$$

Heterozygote Inferior. In this case $\tau_3 \leqslant \tau_1 < \tau_2$ so that the heterozygote is less viable than either homozygote. The relevant features of f in this case are

$$\begin{gathered}\exists a \in (0,1) \ni f(u) < 0 \quad \text{for} \quad u \in (0,a) \quad \text{and} \quad f(u) > 0 \quad \text{for} \quad u \in (a,1) \\ \int_0^1 f\,du > 0 \\ f'(0) < 0.\end{gathered} \tag{1.6}$$

Certain flame propagation problems in chemical reactor theory also lead to an equation of the form (1.1) where the forcing term f satisfies (1.3) and the generalization

$$\begin{gathered}\exists a \in (0,1) \ni f(u) \leqslant 0 \quad \text{for} \quad u \in (0,a) \quad \text{and} \quad f(u) > 0 \quad \text{for} \quad u \in (a,1) \\ \int_0^1 f\,du > 0.\end{gathered} \tag{1.6$'$}$$

of (1.6). We shall refer to this as the combustion case (cf. [6, 8–11]).

In terms of the genetic model we are motivated by the following problem. How does a given initial distribution of the allele A evolve in time? Is the allele A ultimately wiped out or does it persist? In the latter case, is the allele a ultimately eliminated or do both alleles coexist in an equilibrium distribution? In mathematical terms the problem is to determine the nature of the stability of the equilibrium states $u \equiv 0$, $u \equiv 1$, and any others which may occur.

In Section 3 we give a condition which guarantees that the rest state $u \equiv 0$ is unstable with respect to any nontrivial perturbation. We shall refer to this in picturesque language as the hair-trigger effect. Specifically, we show that if $f(u) > 0$ in $(0, \alpha)$ for some $\alpha \in (0, 1)$ and

$$\liminf_{u \searrow 0} u^{-(1+2/n)} f(u) > 0$$

then for any solution $u(x, t) \in [0, 1]$ of (1.1), $u(x, 0) \not\equiv 0$ implies

$$\liminf_{t \to +\infty} u(x, t) \geqslant \alpha.$$

A consequence of this result is the existence of the hair-trigger effect when f is in either the heterozygote intermediate case (1.4) or the heterozygote superior case (1.5). The hair-trigger effect in these cases for one-dimensional diffusions was established by different methods in [1]. If $f(u) = O(u^\beta)$ for

$\beta > 1 + 2/n$, then the rest state $u \equiv 0$ is stable with respect to suitably restricted perturbations and there is no hair-trigger effect. For example, suppose that f is given by (1.2). If $\tau_1 = \tau_2 > \tau_3$ then the mere presence of the allele a confers a fixed selective disadvantage and $f(u) = (\tau_1 - \tau_3)\, u^2(1 - u)$. In this case we have the hair-trigger effect for $n = 1$ and 2 but not for $n \geqslant 3$. To derive these results we make essential use of results due to Fujita [5], Hayakawa [7], and others [12] concerning the existence and nonexistence of global solutions of the initial value problem for the semilinear equation

$$\partial p/\partial t = \Delta p + kp^{\beta}.$$

A proof of the main nonexistence result is given in the appendix to this paper.

In Section 4 we investigate the existence of plane wave solutions of Eq. (1.1), that is, the existence of solutions of the form $u(x, t) = q(x \cdot \nu - ct)$ for some $c \in \mathbb{R}$ and arbitrary unit vectors $\nu \in \mathbb{R}^n$. This problem is formally equivalent to one whose solution we outlined in Ref. [1], namely, the problem of finding traveling wave solutions of (1.1) when $n = 1$. Here we shall solve these problems in full detail. One of the main results in Section 4 is the existence of a minimal wave speed $c^* \in \mathbb{R}^+$ which is completely determined by the forcing term $f(u)$.

As in the one-dimensional case [1], c^* is an asymptotic speed of propagation of disturbances. For example, we show that if $u \in [0, 1]$ is a solution of (1.1) in $\mathbb{R}^n \times \mathbb{R}^+$ such that $u(x, 0)$ has bounded support then $u(x, t) \to 0$ as $t \to +\infty$ uniformly in the region $|x| \geqslant ct$ when $c > c^*$. On the other hand, if $u(x, 0)$ is such that

$$\liminf_{t \to +\infty} u(x, t) \geqslant \alpha > 0 \tag{1.7}$$

uniformly on compact subsets of $\mathbb{R}^n$, where $\alpha = 1$ in the heterozygote superior case and $\alpha = 1$ in the other cases, then

$$\liminf_{t \to +\infty} u(x, t) \geqslant \alpha$$

uniformly in the cone $|x| \leqslant ct$, provided $c \in [0, c^*)$. In view of the results established in Section 3 the condition (1.7) is automatically satisfied in the heterozygote intermediate and superior cases. However, in the heterozygote inferior and combustion cases there are threshold effects. For example, in the heterozygote inferior case $u(x, t) \to 0$ as $t \to \infty$ for some nontrivial data of bounded support, while for other data of bounded support $u(x, t) \to 1$ as $t \to +\infty$. In Section 6 we derive threshold criteria in both the heterozygote inferior and combustion cases. Throughout Sections 5 and 6 the results coincide with the one-dimensional results proved in [1]. The proofs are, however, different.

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2. Preliminaries

Throughout this paper we will be concerned with solutions of the initial value problem

$$\partial u/\partial t = \Delta u + f(u) \text{ in } \mathbb{R}^n \times \mathbb{R}^+ \tag{2.1a}$$

$$u(x, 0) = u_0(x) \text{ in } \mathbb{R}^n, \tag{2.1b}$$

where the forcing term f satisfies the conditions (1.3) and $u_0(x)$ is a given bounded $C(\mathbb{R}^n)$ function. A solution $u(x, t)$ of problem (2.1) is a $C(\mathbb{R}^n \times [0, +\infty))$ function satisfying (2.1b) and possessing derivatives $\partial u/\partial t$, $\partial u/\partial x_i$, $\partial^2 u/\partial x_i\, \partial x_j$ which are continuous in $\mathbb{R}^n \times \mathbb{R}^+$ and satisfy (2.1a).

Since we shall only deal with solutions u with values in $[0, 1]$, f need not be defined in $\mathbb{R}\setminus[0, 1]$. However, it will be convenient to assume temporarily that f is defined in all of $\mathbb{R}$ in such a way that $f \in C^1(\mathbb{R})$ and $f \equiv 0$ in $\mathbb{R}\setminus[-1, 2]$. Suppose that u_0 is bounded. Then by a standard application of the method of successive approximations one can construct a solution $u(x, t)$ of problem (2.1) which has the following properties. For every $T \in \mathbb{R}^+$ and $\delta \in (0, T)$ there exist numbers $A(T) \in \mathbb{R}^+$ and $B(\delta, T) \in \mathbb{R}^+$ such that

$$|u(x, t)| \leqslant A(T) \text{ in } \mathbb{R}^n \times [0, T]$$

and

$$|\nabla_x u(x, t)| \leqslant B(\delta, T) \text{ in } \mathbb{R}^n \times [\delta, T]. \tag{2.2}$$

Here both $A(T)$ and $B(\delta, T)$ depend on $\sup |u_0|$ and $\sup |f'|$. The solution constructed in this manner is unique in the class of functions which are bounded in $\mathbb{R}^n \times [0, T]$ for arbitrary $T \in \mathbb{R}^+$.

To obtain additional information about the solution of the problem (2.1) we will need the following comparison result. Since this result is used often in what follows we state it in sufficient generality to cover all of the applications which occur in this paper. To this end, let Ω denote a subdomain of $\mathbb{R}^n$ which may be all of $\mathbb{R}^n$ itself. In the statement of the following proposition the c_j are constants and g denotes a function which is defined and uniformly Lipschitz continuous on $\mathbb{R}$.

Proposition 2.1. *Let $u(x, t)$ and $v(x, t)$ denote bounded continuous functions defined in $\bar{\Omega} \times [0, T]$ which satisfy the inequality*

$$\frac{\partial u}{\partial t} - \Delta u - \sum_{j=1}^{n} c_j \frac{\partial u}{\partial x_j} - g(u) \geqslant \frac{\partial v}{\partial t} - \Delta v - \sum_{j=1}^{n} c_j \frac{\partial v}{\partial x_j} - g(v) \text{ in } \Omega \times (0, T].$$

If $u(x, 0) \geqslant v(x, 0)$ in Ω and $u(x, t) \geqslant v(x, t)$ in $\partial\Omega \times [0, T]$ in case $\Omega \not\equiv \mathbb{R}^n$, then $u \geqslant v$ in $\bar{\Omega} \times [0, T]$. If, in addition, $u(x, 0) > v(x, 0)$ in an open subset of Ω, then $u > v$ in $\Omega \times (0, T]$.

Proof. Set $w = u - v$. Then

$$\frac{\partial w}{\partial t} - \Delta w - \sum_{j=1}^{n} c_j \frac{\partial w}{\partial x_j} \geqslant g(u) - g(v) \text{ in } \Omega \times (0, T).$$

Define the function

$$k(x, t) = \begin{cases} \dfrac{(g \circ u)(x, t) - (g \circ v)(x, t)}{u(x, t) - v(x, t)} & \text{for } (x, t) \text{ such that } u(x, t) \neq v(x, t) \\ 0 & \text{for } (x, t) \text{ such that } u(x, t) = v(x, t). \end{cases}$$

Then since g is uniformly Lipschitz continuous on $\mathbb{R}$, $k(x, t)$ is bounded in $\Omega \times [0, T]$. Moreover, $g(u) - g(v) = kw$. Thus

$$\frac{\partial w}{\partial t} - \Delta w - \sum_{j=1}^{n} c_j \frac{\partial w}{\partial w_j} - kw \geqslant 0 \text{ in } \Omega \times (0, T]$$

and the assertion follows from the strong maximum principle for linear parabolic inequalities [4, p. 39].

Remark. In Proposition 2.1 the requirement that u and v be bounded is much stronger than necessary. However, if Ω is unbounded some growth condition as $|x| \to +\infty$ is needed. For example, it suffices to assume that u and v are $O(e^{c|x|^2})$ in $\Omega \times [0, T]$ for some $c > 0$.

In view of (1.3) both $u \equiv 0$ and $u \equiv 1$ are solutions of Eq. (1.1). If in the initial value problem (2.1) one has $u_0(x) \in [0, 1]$ in $\mathbb{R}^n$ then it follows from Proposition 2.1 that the solution satisfies $u(x, t) \in [0, 1]$ in $\mathbb{R}^n \times \mathbb{R}^+$. Note that in this case the solution is independent of the continuation of f into $\mathbb{R} \setminus [0, 1]$. From here on we shall restrict our attention to solutions with values in $[0, 1]$ and therefore no longer need to assume that f is defined outside $[0, 1]$.

The next result is a version of the basic lemma in Ref. [1]. We include the detailed proof for the sake of completeness.

PROPOSITION 2.2. *Let $q(x) \in [0, 1]$ denote a solution of the ordinary differential equation*

$$q'' + cq' + f(q) = 0 \tag{2.3}$$

in $\mathbb{R}^+$ with $q(0) = 1$ and let $v(x, t)$ denote the solution of the initial value problem

$$\begin{aligned} \frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial x^2} + c\frac{\partial v}{\partial x} + f(v) \text{ in } \mathbb{R} \times \mathbb{R}^+ \\ v(x, 0) &= 1 \qquad \text{in } \mathbb{R} \setminus \mathbb{R}^+, \\ &= q(x) \qquad \text{in } \mathbb{R}^+. \end{aligned} \tag{2.4}$$

In Eqs. (2.3) *and* (2.4), c *denotes a nonnegative constant. Then* $v(x, t)$ *is a nonincreasing function of* t *for each* $x \in \mathbb{R}$. *Moreover,*

$$\lim_{t \to +\infty} v(x, t) = \tau(x)$$

uniformly on each bounded interval, where $\tau(x) \in [0, 1]$ *is the largest solution of Eq.* (2.3) *in* $\mathbb{R}$ *which satisfies the inequality*

$$\begin{aligned} \tau(x) &\leqslant 1 \quad \text{in } \mathbb{R}\setminus\mathbb{R}^+ \\ &\leqslant q(x) \text{ in } \mathbb{R}^+. \end{aligned}$$

Proof. Since $v(x, 0) \in [0, 1]$, Proposition 2.1 implies that $v(x, t) \in [0, 1]$ in $\mathbb{R} \times \mathbb{R}^+$. In particular, $v(0, t) \leqslant 1 = q(0)$. Therefore, by Proposition 2.1, $v(x, t) \leqslant q(x)$ in $\mathbb{R}^+ \times \mathbb{R}^+$. It follows that $v(x, h) \leqslant v(x, 0)$ holds in $\mathbb{R}$ for any $h > 0$. We now apply Proposition 2.1 to $v(x, t + h)$ and $v(x, t)$ to obtain $v(x, t + h) \leqslant v(x, t)$ in $\mathbb{R} \times \mathbb{R}^+$ for any $h > 0$. Thus for each x, the function $v(x, t)$ is nonincreasing in t and bounded below by zero. Therefore

$$\lim_{t \to +\infty} v(x, t) = \tau(x)$$

exists. Note that $\tau \in [0, 1]$ since $v \in [0, 1]$. Moreover, $v(x, t) \leqslant q(x)$ in $\mathbb{R}^+ \times \mathbb{R}^+$ implies $\tau(x) \leqslant q(x)$ in $\mathbb{R}^+$.

For arbitrary $\eta > 0$ and $(x, t) \in \mathbb{R} \times \mathbb{R}^+$

$$\begin{aligned} v(x, t + \eta) = {} & \int_{\mathbb{R}} g(x - \xi, t)\, v(\xi, \eta)\, d\xi \\ & + \int_{\eta}^{t+\eta} \int_{\mathbb{R}} g(x - \xi, t + \eta - y)(f \circ v)(\xi, y)\, d\xi\, dy, \end{aligned} \tag{2.5}$$

where

$$g(x, t) = \frac{1}{2(\pi t)^{1/2}} \exp\{-(x - ct)^2 / 4t\}$$

is the fundamental solution of the linear equation

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + c \frac{\partial w}{\partial x}. \tag{2.6}$$

By means of the substitution $s = y - \eta$ in the second integral on the right-hand side of (2.5), $v(x, t + \eta)$ can be rewritten in the form

$$\begin{aligned} v(x, t + \eta) = {} & \int_{\mathbb{R}} g(x - \xi, t)\, v(\xi, \eta)\, d\xi \\ & + \int_0^t \int_{\mathbb{R}} g(x - \xi, t - s)(f \circ v)(\xi, s + \eta)\, d\xi\, ds. \end{aligned}$$

Since $v(\cdot, \eta) \to \tau(\cdot)$ as $\eta \to \infty$ it follows from the dominated convergence theorem that

$$\tau(x) = \int_{\mathbb{R}} g(x - \xi, t)\, \tau(\xi)\, d\xi + \int_0^t \int_{\mathbb{R}} g(x - \xi, t - s)(f \circ \tau)(\xi)\, d\xi\, ds \tag{2.7}$$

for all $x \in \mathbb{R}$ and arbitrary $t \in \mathbb{R}^+$.[1]

From the representation (2.7), we conclude that τ is continuous. Moreover,

$$\tau'(x) = \int_{\mathbb{R}} g_x(x - \xi, t)\, \tau(\xi)\, d\xi + \int_0^t \int_{\mathbb{R}} g_x(x - \xi, t - s)(f \circ \tau)(\xi)\, d\xi\, ds$$

and setting $t = 1$ we obtain the estimate

$$|\tau'(x)| \leqslant \frac{1}{\pi^{1/2}}\{1 + 2 \max_{[0,1]} |f|\}. \tag{2.8}$$

Since τ is continuous and since the convergence of the continuous functions v to τ is monotone it follows from Dini's theorem that $v \to \tau$ uniformly on bounded intervals. It remains to be shown that $\tau(x)$ satisfies the steady state equation (2.3).

For arbitrary $t > 0$ write

$$\tau(x) = \int_{\mathbb{R}} g(x - \xi, t)\, \tau(\xi)\, d\xi + \int_0^t \int_{\mathbb{R}} g(x - \xi, t - s)\, \delta(\xi; x)\, d\xi\, ds + t(f \circ \tau)(x)$$

and

$$\tau'(x) = \int_{\mathbb{R}} g_x(x - \xi, t)\, \tau(\xi)\, d\xi + \int_0^t \int_{\mathbb{R}} g_x(x, \xi, t - s)\, \delta(\xi; x)\, d\xi\, ds,$$

where

$$\delta(\xi; x) \equiv (f \circ \tau)(\xi) - (f \circ \tau)(x).$$

In view of (2.8), $|\delta(\xi; x)| \leqslant \text{const}\, |x - \xi|$. Using this we derive the representations

$$\tau''(x) = \int_{\mathbb{R}} g_{xx}(x - \xi, t)\, \tau(\xi)\, d\xi + \int_0^t \int_{\mathbb{R}} g_{xx}(x - \xi, t - s)\, \delta(\xi; x)\, d\xi\, ds$$

[1] We are indebted to Professor B. Frank Jones Jr. for suggesting this argument.

and

$$0 = \frac{d}{dt}\tau(x) = \int_{\mathbb{R}} g_t(x-\xi, t)\,\tau(\xi)\,d\xi$$

$$+ \int_0^t \int_{\mathbb{R}} g_t(x-\xi, t-s)\,\delta(\xi; x)\,d\xi\,ds + (f \circ \tau)(x)$$

valid for arbitrary $t > 0$. Since $g(x, t)$ is a solution of (2.6) for $t > 0$ it follows that $\tau'' + c\tau' + f(\tau) = 0$ in $\mathbb{R}$.

If $\sigma(x)$ is any solution of Eq. (2.3) with $\sigma \leqslant 1$ in $\mathbb{R}$ and $\sigma(x) \leqslant q(x)$ in $\mathbb{R}^+$, Proposition 2.1 implies that $v(x, t) \geqslant \sigma(x)$. Hence $\tau(x) \geqslant \sigma(x)$ so that τ is the largest solution with these properties.

3. The Hair-Trigger Effect

In this section we shall derive a sufficient condition for the instability of the rest state $u \equiv 0$ with respect to every nontrivial nonnegative perturbation. Specifically, we shall prove the following result.

THEOREM 3.1. *Let $f(u)$ satisfy* (1.3) *together with the conditions*

$$f(u) > 0 \text{ in } (0, \alpha) \qquad \text{for some } \alpha \in (0, 1] \tag{3.1}$$

and

$$\liminf_{u \searrow 0} u^{-(1+2/n)} f(u) > 0. \tag{3.2}$$

If $u \in [0, 1]$ is a solution of (1.1) *in $\mathbb{R}^n \times \mathbb{R}^+$ with $u \not\equiv 0$ then*

$$\liminf_{t \to +\infty} u(x, t) \geqslant \alpha$$

uniformly in bounded subsets of $\mathbb{R}^n$.

In the proof of Theorem 3.1 we use a known result on the nonexistence of global solutions of certain semilinear parabolic equations. The first results in this direction are due to Fujita [5]. The result we use is an extension of a result of Fujita which is due to Hayakawa [7] in the case $n = 1$ or 2. For general n it is a special case of a theorem proved by Kobayashi, Sirao, and Tanaka [12, 16].

LEMMA 3.1. *Consider the initial value problem*

$$\begin{aligned} \frac{\partial p}{\partial t} &= \Delta p + k p^{1+2/n} && \text{in } \mathbb{R}^n \times \mathbb{R}^+ \\ p(x, 0) &= p_0(x) && \text{in } \mathbb{R}^n. \end{aligned} \tag{3.3}$$

$p_0(x) \geqslant 0$ *and* $p_0(x) \not\equiv 0$ *then there exists a* $T \in \mathbb{R}^+$ *and a nonnegative function* $p(x, t)$ *such that for each* $T' \in (0, T)$ $p(x, t)$ *is the unique solution of problem* (3.3) *which is bounded in* $\mathbb{R}^n \times [0, T']$ *and*

$$\limsup_{t \nearrow T} \sup_{x \in \mathbb{R}^n} p(x, t) = +\infty. \tag{3.4}$$

To avoid interrupting the main flow of ideas in the proof of Theorem 3.1, we present the rather technical proof of Lemma 3.1 in the appendix to this paper. The next lemma provides the linkage between our instability problem and the nonexistence of global solutions of problem (3.3).

LEMMA 3.2. *Under the hypotheses of Theorem* 3.1.

$$\limsup_{t \to +\infty} \sup_{x \in \mathbb{R}^n} u(x, t) \geqslant \alpha. \tag{3.5}$$

Proof. To prove (3.5) it suffices to show that

$$\limsup_{t \to +\infty} \sup_{x \in \mathbb{R}^n} u(x, t) > \eta^* \tag{3.6}$$

for every $\eta^* \in (0, \alpha)$. Suppose that (3.6) does not hold for some $\eta^* \in (0, \alpha)$. Then for any $\eta \in (\eta^*, \alpha)$ these exists a $t_\eta \in \mathbb{R}^+$ such that

$$\sup_{x \in \mathbb{R}^n} u(x, t) \leqslant \eta \qquad \text{in} \quad [t_\eta, +\infty). \tag{3.7}$$

In view of (3.1) and (3.2) there exists a number $k(\eta) > 0$ such that

$$f(u) \geqslant k(\eta)\, u^{1+2/n}$$

for $u \in [0, \eta]$ and hence for $(x, t) \in \mathbb{R}^n \times [t_\eta, +\infty)$.

Let $p(x, t)$ denote the solution of problem (3.3) with $k = k(\eta)$ and $p_0(x)$ $u(x, t_\eta)$. Since $u \not\equiv 0$ it follows from Proposition 2.1 that $p_0(x) = u(x, t_\eta) > 0$. Moreover, Proposition 2.1 implies that $u(x, t + t_\eta) \geqslant p(x, t)$ in $\mathbb{R}^n \times [0, T)$. Thus, in view of (3.4),

$$\limsup_{t \nearrow T} \sup_{x \in \mathbb{R}^n} u(x, t + t_\eta) = +\infty$$

in contradiction to (3.7). Thus (3.6) holds for every $\eta^* \in (0, \alpha)$ and the lemma is proved.

The next lemma is an extension of a result due to Kanel' [11] and is the main step in the proof of Theorem 3.1. For $\delta > 0$ define

$$\begin{aligned} q_\delta(r) &= \delta(1 - r^2)^3 \qquad &&\text{for} \quad 0 \leqslant r \leqslant 1 \\ &= 0 &&\text{for} \quad r > 1. \end{aligned}$$

LEMMA 3.3. *Suppose that $f^*(v)$ satisfies (1.3) and (3.1), and that*

$$f^*(v) = kv^{1+2/n} \quad \text{for} \quad v \in [0, b],$$

where $k \in \mathbb{R}^+$ is a constant and $b \in (0, \alpha)$. If $v(x, t)$ denotes the solution of the initial value problem

$$\begin{aligned} \partial v/\partial t &= \Delta v + f^*(v) \quad &\text{in} \quad \mathbb{R}^n \times \mathbb{R}^+ \\ v(x, 0) &= q_\delta(|x|) \quad &\text{in} \quad \mathbb{R}^n \end{aligned} \tag{3.8}$$

with $0 < \delta < \min\{b, (3n/k)^{n/2}\}$, then

$$\lim_{t\to+\infty} v(0, t) \geqslant \alpha. \tag{3.9}$$

Proof. The second partial derivatives of the initial data $v(x, 0)$ are Lipschitz continuous in $\mathbb{R}^n$. By the Schauder-type theory for parabolic equations [4, p. 86] it follows that v, $\partial v/\partial t$, $\text{grad}_x v$, and Δv are continuous in $\mathbb{R}^n \times [0, +\infty)$. Moreover, the fact that $v(x, 0)$ depends only on the distance $|x|$ from the origin implies that $v(x, t)$ depends only on $|x|$ for each t.

Set $v(x, t) = V(r, t)$, where $r = |x|$. Then

$$V_t = V_{rr} + \frac{n-1}{r} V_r + f^*(V) \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^+,$$

and, in view of the smoothness of v, $V_r(0, t) = 0$ for all $t \in [0, +\infty)$. Let $W(r, t) \equiv V_r(r, t)$. Then $W(0, t) = 0$ for $t \in [0, +\infty)$, $W(r, 0) \leqslant 0$ in $\mathbb{R}^+$ and

$$W_t = W_{rr} + \frac{n-1}{r} W_r + \left\{ f^{*\prime}(V) - \frac{n-1}{r^2} \right\} W \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^+.$$

Note that the coefficient of W is bounded above. Hence the maximum principle [4, p. 38] applies and we conclude that $W = V_r \leqslant 0$ in $\mathbb{R}^+ \times \mathbb{R}^+$. Thus, for each t, the function $V(r, t)$ attains its maximum at $r = 0$. Therefore

$$0 \leqslant v(x, t) \leqslant v(0, t) \leqslant 1 \text{ in } \mathbb{R}^n \times \mathbb{R}^+. \tag{3.10}$$

Next we shall show that $v(0, t)$ is ultimately a monotonic function of t and hence that $\lim_{t\to+\infty} v(0, t)$ exists. To this end we set $z = e^{-lt}\, \partial v/\partial t$, where

$$l = \max_{[0,1]} f^{*\prime}(u).$$

Then z satisfies the equation

$$\partial z/\partial t = \Delta z + H(x, t)z \text{ in } \mathbb{R}^n \times \mathbb{R}^+,$$

where $H(x, t) \equiv (f^{*\prime} \circ v)(x, t) - l \leqslant 0$. Since $0 \leqslant q_\delta(x) \leqslant \delta < b$,

$$z(x, 0) = Z(|x|) \equiv \begin{cases} 6\delta(1 - |x|^2)\{(4 + n)|x|^2 - n \\ \quad + [k\delta^{2/n}/6](1 - |x|^2)^{2+6/n}\} & \text{for } |x| \leqslant 1 \\ 0 & \text{for } |x| > 1. \end{cases}$$

Note that z depends only on $|x|$ and t.

It is easily verified that since $\delta < (3n/k)^{n/2}$, there exists an $r_\delta \in (0, 1)$ such that $Z(r)$ is an increasing function in $[0, r_\delta]$, with $Z(r) < 0$ in $[0, r_\delta)$ and $Z(r) \geqslant 0$ in $[r_\delta, +\infty)$. Then the set

$$\mathscr{S}_0 \equiv \{t: t \in [0, +\infty), z(0, t) < 0\}$$

contains the point $t = 0$ and, in view of the continuity of z, all sufficiently small $t > 0$. We shall show that either $\mathscr{S}_0 = [0, +\infty)$ or $\mathscr{S}_0 = [0, t_1)$ for some $t_1 \in \mathbb{R}^+$.

In order to determine the structure of $\mathscr{S}_0$, we first prove that

$$\lim_{|x| \to +\infty} z(x, t) = 0 \tag{3.11}$$

uniformly in every bounded t-interval.

Let A be a constant with the property

$$|Z(r)| \leqslant A\, e^{-1/4}.$$

Since $H \leqslant 0$, the function

$$w = A\,(t + 1)^{-n/2} \exp\{-|x|^2/4(t + 1)\}$$

satisfies the inequality

$$(\partial w/\partial t) - \Delta w - Hw \geqslant 0 \text{ in } \mathbb{R}^n \times \mathbb{R}^+.$$

Moreover w is positive and

$$w(x, 0) \geqslant A\, e^{-1/4} \qquad \text{for } |x| \leqslant 1.$$

Since $Z(r) = 0$ for $r \geqslant 1$, it follows from Proposition 2.1 applied to w and $\pm z$ that

$$|z(x, t)| \leqslant w = A\,(t + 1)^{-n/2} \exp\{-|x|^2/4(t + 1)\}.$$

This proves (3.11).

Next we shall show that $\mathscr{S}_0$ is an interval. Suppose that for some $t^* \in \mathbb{R}^+$ we have $z(0, t^*) < 0$. For convenience we extend the domain of z to all of $\mathbb{R}^{n+1}$ by setting $z(x, t) = z(x, -t)$ for $(x, t) \in \mathbb{R}^n \times \mathbb{R}^-$. We set $\epsilon \equiv -z(0, t^*) > 0$. Let $\mathscr{N}$ denote the component of the open set

$$\mathscr{S} = \{(x, t): (x, t) \in \mathbb{R}^{n+1}, z(x, t) < -\epsilon/2\}$$

which contains the point $(0, t^*)$. For each $\sigma \in \mathbb{R}$ the set $\mathscr{N}_\sigma \equiv \mathscr{N} \cap \{t = \sigma\}$ is either empty or else it is open, spherically symmetric and, in view of (3.11), bounded. We shall show that

$$\mathscr{N}_t \neq \varnothing \text{ in } [0, t^*]. \tag{3.12}$$

By hypothesis, $\mathscr{N}_{t^*} \neq \varnothing$ and, by the continuity of z, $\mathscr{N}_t \neq \varnothing$ for all sufficiently large $t < t^*$. Suppose there exists a $\sigma \in [0, t^*)$ such that $\mathscr{N}_t \neq \varnothing$ in $(\sigma, t^*]$ and $\mathscr{N}_\sigma = \varnothing$. Then $\mathscr{N} \subset \mathbb{R}^n \times (\sigma, +\infty)$. Define the set

$$\mathscr{M} \equiv \{(x, t): (x, t) \in \mathscr{N} \cap \mathbb{R}^n \times (\sigma, t^*], z(x, t) \leqslant -\epsilon\}.$$

Then $(0, t^*) \in \mathscr{M}$ so that $\mathscr{M} \neq \varnothing$ and, according to (3.11), $\mathscr{M}$ is bounded. Suppose that (y, s) is a limit point of $\mathscr{M}$. Then $z(y, s) \leqslant -\epsilon$ and (y, s) belongs to some component, say $\hat{\mathscr{N}}$, of $\mathscr{S}$. Since $\hat{\mathscr{N}}$ is an open set, there exists a neighborhood $\mathscr{U}$ of (y, s) such that $\mathscr{U} \subset \hat{\mathscr{N}}$. On the other hand, since (y, s) is a limit point of $\mathscr{M}$, $\mathscr{U}$ contains points of $\mathscr{M}$ and hence of $\mathscr{N}$. Thus $\hat{\mathscr{N}} \cap \mathscr{N} \neq \varnothing$ and consequently $\hat{\mathscr{N}} \equiv \mathscr{N}$. Therefore $\mathscr{M}$ is closed and $\mathscr{M}$ is a compact subset of $\mathscr{N}^* \equiv \mathscr{N} \cap \mathbb{R}^n \times (\sigma, t^*]$.

Let $\lambda \equiv \inf_{\mathscr{N}^*} z$. Then $z(0, t^*) = -\epsilon$ implies that $\lambda \leqslant -\epsilon < 0$ and hence $\lambda = \inf_{\mathscr{M}} z$. Since $\mathscr{M}$ is a compact subset of $\mathscr{N}^*$ there exists a point $(\xi, \tau) \in \mathscr{M}$ such that $z(\xi, \tau) = \lambda$. Note that $\tau \in (\sigma, t^*]$ so that $\mathscr{N}_\tau \neq \varnothing$. By the strong maximum principle [4, p. 38; or 15, p. 174], $z(x, \tau) \equiv \lambda \leqslant -\epsilon$ for all x in the component of $\mathscr{N}_\tau$ which contains ξ. However, this contradicts the fact that $z(x, \tau) = -\epsilon/2$ on the boundary of any component of $\mathscr{N}_\tau$. Therefore we conclude that (3.12) holds.

In view of the choice of δ, $\mathscr{N}_0 = \{x: |x| < r_\epsilon\}$ for some $r_\epsilon \in (0, r_\delta)$. Moreover, since we have extended z by reflection in the hyperplane $t = 0$ it follows that $\mathscr{N}_t \neq \varnothing$ for $t \in [-t^*, t^*]$. Let $\mathscr{C}$ denote the intersection of $\mathscr{N}$ with any plane through the t-axis. Because of the radial symmetry of $\mathscr{N}$, $\mathscr{C}$ is a connected open subset of the two-dimensional (r, t)-space and $\mathscr{C}$ is symmetric about the t-axis. Therefore there exists a polygonal arc $\Gamma \subset \mathscr{C}$ which joins $(0, 0)$ to $(0, t^*)$. If Γ' is a subarc of Γ lying in the half-space $t \leqslant 0$ with endpoints on the r-axis then, since $\mathscr{C} \cap \{t = 0\} = (-r_\epsilon, r_\epsilon)$, Γ' can be replaced by the segment of the r-axis joining its endpoints. Thus we may assume without loss of generality that Γ lies in the upper half-plane $t \geqslant 0$. 2 dim'l

We now show that $t^* \in \mathscr{S}_0 \cap \mathbb{R}^+$ implies

$$z(0, t) < 0 \text{ in } [0, t^*]. \tag{3.13}$$

Since $z(0, 0) < 0$, $z(0, t^*) < 0$, and z is continuous, it follows that $z(0, t) < 0$ for all sufficiently small $t > 0$ and all sufficiently large $t < t^*$. Suppose for contradiction that (3.13) does not hold. Then there exists a $\sigma \in (0, t^*)$ such that

$$z(0, \sigma) \geqslant 0. \tag{3.14}$$

If (x, t) is such that $(|x|, t) \in \Gamma$ then $z(x, t) < -\epsilon/2$. Thus, in view of (3.14), $(0, \sigma) \notin \Gamma$. Since Γ joins $(0, 0)$ to $(0, t^*)$ and lies in the half-space $\mathbb{R} \times [0, +\infty)$ there exist real numbers s_j for $j = 1$ and 2 such that $0 \leqslant s_1 < \sigma < s_2$, $(0, s_j) \in \Gamma$ for $j = 1$ and 2, and the subarc Γ' of Γ joining $(0, s_1)$ to $(0, s_2)$ lies completely in one of the quarter-spaces $\mathbb{R}^+ \times [0, +\infty)$ or $\mathbb{R}^- \times [0, +\infty)$. Let $\mathcal{O}'$ denote the open subset of $\mathbb{R} \times \mathbb{R}^+$ bounded by Γ' and its reflection with respect to the t-axis, and define

$$\mathscr{P} \equiv \{(x, t): (x, t) \in \mathbb{R}^n \times \mathbb{R}^+, (|x|, t) \in \mathcal{O}'\}.$$

Note that $\mathscr{P}$ is a bounded open set and $(0, \sigma) \in \mathscr{P}$.

Let $\mu \equiv \sup\{z(x, t): (x, t) \in \bar{\mathscr{P}}\}$. Then (3.14) and $(0, \sigma) \in \mathscr{P}$ imply that $\mu \geqslant 0$. Since $\bar{\mathscr{P}}$ is compact, there exists a point $(\xi, \tau) \in \bar{\mathscr{P}}$ such that $z(\xi, \tau) = \mu$. By the construction, $z < -\epsilon/2$ on the boundary of $\mathscr{P}$. Therefore (ξ, τ) belongs to the open set $\mathscr{P}$. According to the strong maximum principle, $z(x, \tau) \equiv \mu \geqslant 0$ for all x in the component of the open set $\mathscr{P} \cap \{t = \tau\}$ which contains ξ. This contradicts the fact that $z(x, \tau) < -\epsilon/2$ on the boundary of every component of $\mathscr{P} \cap \{t = \tau\}$. We conclude that (3.13) holds.

In view of (3.13), $\mathscr{S}_0 = [0, t_1)$ for some $t_1 \in (0, +\infty]$. Thus either $z(0, t) < 0$ for all $t \geqslant 0$ or there exists a $t_1 \in \mathbb{R}^+$ such that $z(0, t) \geqslant 0$ for all $t \geqslant t_1$. In either event, $v(0, t)$ is ultimately a monotonic function of t and it follows from (3.10) that

$$\eta^* \equiv \lim_{t \to +\infty} v(0, t)$$

exists. Suppose that $\eta^* \in [0, \alpha)$. For each $\eta \in (\eta^*, \alpha)$ there exists a $t_\eta \in \mathbb{R}^+$ such that $v(0, t) < \eta$ for all $t \geqslant t_\eta$. Therefore, according to (3.10),

$$0 \leqslant v(x, t) \leqslant v(0, t) < \eta \text{ in } \mathbb{R}^n \times [t_\eta, +\infty).$$

Since this contradicts the conclusion of Lemma 3.2, it follows that $\eta^* \geqslant \alpha$.

Proof of Theorem 3.1. Since $f(u)$ satisfies (1.3), (3.1), and (3.2), it is not difficult to construct a function $f^*(u)$ which satisfies (1.3) and (3.1), and which is such that $f^*(u) \leqslant f(u)$ in $[0, 1]$ and $f^*(u) = ku^{1+2/n}$ in $[0, b]$ for some $k > 0$ and $b \in (0, \alpha)$. By Proposition 2.1, $u \not\equiv 0$ implies $u(x, h) > 0$ in $\mathbb{R}^n$ for any fixed $h \in \mathbb{R}^+$. Thus

$$m(\sigma) \equiv \inf\{u(x, h): x \in \mathbb{R}^n, |x| < \sigma + 1\} > 0$$

for each $\sigma \in \mathbb{R}^+$. Fix σ and let $\delta = \min\{b, (3n/k)^{2/n}, m(\sigma)\}$. If $v(x, t)$ denotes the solution of problem (3.8) corresponding to this δ and the f^* defined above, then it follows from Proposition 2.1 that $u(x, t + h) \geqslant v(x - y, t)$ for any $y \in \mathbb{R}^n$ such that $|y| \leqslant \sigma$. Therefore, by Lemma 3.3,

$$\liminf_{t\to+\infty} u(y, t + h) \geqslant \lim_{t\to+\infty} v(0, t) \geqslant \alpha$$

uniformly for all $y \in \mathbb{R}^n$ such that $|y| \leqslant \sigma$.

Remark. The results of Kobayashi, Sirao and Tanaka [12, 16] can be used to extend Theorem 3.1. Specifically, the condition (3.2) can be replaced by

$$\liminf_{u\searrow 0} f(u)/h(u) > 0$$

where, for example,

$$h(u) = u^{1+2/n}/(\log 1/u)(\log\log 1/u).$$

If, in addition to (1.3), the forcing term $f(u)$ in (1.1) satisfies the condition (1.4) of the heterozygote intermediate case or (1.5) of the heterozygote superior case then

$$\lim_{u\searrow 0} u^{-1}f(u) = f'(0) > 0.$$

Thus conditions (3.1) and (3.2) are satisfied and Theorem 3.1 applies to yield the following extension to $n > 1$ of results proved in [1].

COROLLARY 3.1. *Let $f(u)$ satisfy* (1.3) *and let $u \in [0, 1]$ be a solution of* (1.1) *in $\mathbb{R}^n \times \mathbb{R}^+$ such that $u \not\equiv 0$.*

(i) *If $f(u)$ satisfies* (1.4) *then*

$$\lim_{t\to\infty} u(x, t) = 1$$

uniformly on bounded subsets of $\mathbb{R}^n$.

(ii) *If $f(u)$ satisfies* (1.5) *and $u \not\equiv 1$ then*

$$\lim_{t\to\infty} u(x, t) = \alpha$$

uniformly on bounded subsets of $\mathbb{R}^n$.

Fujita [5] has observed that if $\beta > 1 + 2/n$, the initial value problem

$$\begin{aligned} \partial p/\partial t &= \Delta p + kp^{\beta} && \text{in } \mathbb{R}^n \times \mathbb{R}^+ \\ p(x, 0) &= p_0(x) && \text{in } \mathbb{R}^n \end{aligned} \tag{3.15}$$

admits global solutions for suitably restricted nontrivial initial data. For example, let

$$w(x, t; a, \beta, k) \equiv \left\{\left(\frac{n}{2} - \frac{1}{\beta - 1}\right)\Big/(kt + a)\right\}^{1/(\beta-1)} e^{-k|x|^2/4(kt+a)}$$

where $\beta > 1 + 2/n$, $a > 0$, and $k > 0$. Then

$$(\partial w/\partial t) - \Delta w - kw^\beta \geqslant 0 \text{ in } \mathbb{R}^n \times \mathbb{R}^+$$

and

$$\lim_{t\to+\infty} w(x, t; a, \beta, k) = 0$$

uniformly for $x \in \mathbb{R}^n$. Let $p(x, t)$ denote a solution of problem (3.15) with $p_0(x) \not\equiv 0$ satisfying $0 \leqslant p_0(x) \leqslant w(x, 0; a, \beta, k)$ for some suitable value of a. By Proposition 2.1, $0 \leqslant p(x, t) \leqslant w(x, t; a, \beta, k)$ in $\mathbb{R}^n \times \mathbb{R}^+$ and, in particular, $p(x, t) \to +0$ as $t \to +\infty$ uniformly in $\mathbb{R}^n$. Another simple application of Proposition 2.1 yields the following condition for the stability of the rest state $u \equiv 0$ for Eq. (1.1).

THEOREM 3.2. *Let $f(u)$ satisfy* (1.3), *and*

$$f(u) \leqslant ku^\beta \textit{ in } [0, 1] \tag{3.16}$$

for some constants $k > 0$ and $\beta > 1 + 2/n$. If $u \in [0, 1]$ is a solution of (1.1) *in $\mathbb{R}^n \times \mathbb{R}^+$ with $u(x, 0) \leqslant w(x, 0; a, \beta, k)$ in $\mathbb{R}^n$ for some $a > 0$, then*

$$\lim_{t\to+\infty} u(x, t) = 0$$

uniformly in $\mathbb{R}^n$.

In the genetic cases $f(u)$ is given by (1.2). If $\tau_1 = \tau_2 > \tau_3$ then $f(u) = ku^2(1 - u)$ with $k = \tau_1 - \tau_3$. This will occur if the advantageous allele A is recessive. Here

$$\lim_{u\searrow 0} u^{-2} f(u) = k > 0$$

and

$$f(u) \leqslant ku^2 \text{ in } [0, 1].$$

Thus we can apply Theorems 3.1 and 3.2 to obtain the following result.

COROLLARY 3.2. *Let $f(u) = ku^2(1 - u)$ for $k > 0$ and let $u \in [0, 1]$ be a solution of* (1.1) *in $\mathbb{R}^n \times \mathbb{R}^+$ such that $u \not\equiv 0$.*

(i) *If $n = 1$ or 2 then*

$$\lim_{t\to+\infty} u(x, t) = 1$$

uniformly on bounded subsets of $\mathbb{R}^n$.

(ii) *If* $n \geqslant 3$ *and* $u(x, 0) \leqslant w(x, 0; a, 2, k)$ *in* $\mathbb{R}^n$ *for some* $a > 0$ *then*

$$\lim_{t \to +\infty} u(x, t) = 0$$

uniformly in $\mathbb{R}^n$.

4. Plane Wave Solutions

A plane wave solution of Eq. (1.1) is a solution of the form $q(x \cdot \nu - ct)$, where ν is an arbitrary unit vector in $\mathbb{R}^n$, c a nonnegative number and

$$x \cdot \nu = \sum_{j=1}^{n} x_j \nu_j .$$

The function $q(x \cdot \nu - ct)$ is a solution of (1.1) if and only if $q(\xi)$ satisfies the ordinary differential equation

$$q'' + cq' + f(q) = 0 \tag{4.1}$$

in $\mathbb{R}$. We shall be interested only in waves which satisfy the auxilary conditions $q(\xi) \in [0, 1]$, $q(\xi) \not\equiv 0$, and

$$\lim_{\xi \to +\infty} q(\xi) = 0. \tag{4.2}$$

For a given forcing term $f(u)$ the problem of finding plane wave solutions of Eq. (1.1) is identical to the problem of finding traveling wave solutions of

$$u_t = u_{xx} + f(u).$$

The solution of this problem was outlined in Ref. [1]; here we shall present it in full detail.

Equation (4.1) is equivalent to the system of first-order equations

$$\begin{aligned} q' &= p \\ p' &= -cp - f(q) \end{aligned} \tag{4.3}$$

The functions $q(\xi), p(\xi)$ corresponding to a solution of (4.3) trace out a trajectory in the q, p-plane or, as it is usually called, the phase plane. Such a trajectory has slope

$$dp/dq = -c - f(q)/p \tag{4.4}$$

at any point where $p \neq 0$.

Assume that f satisfies (1.3). Then the points $(0, 0)$ and $(1, 0)$ are critical points for the system (4.3). Of course there may be other critical points, but they must all be of the form $(a, 0)$ with $f(a) = 0$. A nontrivial (plane or traveling) wave solution with values in $[0, 1]$ which satisfies (4.2) corresponds to a trajectory in the strip $\{(q, p): 0 < q < 1, p \in \mathbb{R}\}$ which joins $(0, 0)$ to another critical point $(a, 0)$ with $a \in (0, 1]$. Actually, as we shall see below, the nontrivial waves which we seek correspond to trajectories in the semistrip

$$S = \{(q, p): 0 < q < 1, p < 0\}.$$

If $c = 0$ then the coordinates of a point on any trajectory through $(0, 0)$ are related by the first integral of (4.1)

$$\tfrac{1}{2}p^2 + F(q) = 0,$$

where

$$F(q) = \int_0^q f(u)\, du.$$

In particular, if $F(q) \geqslant 0$ for all sufficiently small $q > 0$ then there is no trajectory in S through $(0, 0)$. The same is true if $f'(0) > 0$ and $c \in \mathbb{R}^+$ is such that $0 < c^2 < 4f'(0)$ since in this case the origin is either a center or a spiral point [14].

Fix $c \geqslant 0$. For each $\nu > 0$ there is a unique trajectory of (4.3) through the regular point $(0, -\nu)$. As long as it remains in the half-space $p < 0$ this trajectory has the representation $p = p_c(q; \nu)$, where $p_c(q; \nu)$ is the solution of (4.4) with $p_c(0; \nu) = -\nu$. Let

$$q_{c,\nu} = \sup\{\eta: \eta \in (0, 1], p_c(q; \nu) < 0 \text{ for } q \in [0, \eta)\}.$$

Then $q_{c,\nu} \in (0, 1]$ and the trajectory through $(0, -\nu)$ is in S for $q \in (0, q_{c,\nu})$. We shall use $R_{c,\nu}$ to denote the curve $p = p_c(q; \nu)$ for $q \in (0, q_{c,\nu})$. For convenience, if $q_{c,\nu} < 1$ we extend the domain of $p_c(q; \nu)$ to $[0, 1]$ by setting $p_c(q; \nu) = 0$ for $q \in [q_{c,\nu}, 1]$.

Every point in S is a regular point for (4.3) so that, for fixed $c \geqslant 0$, there is at most one curve $R_{c,\nu}$ through any point of S. Thus $0 < \nu < \mu$ implies $p_c(q; \mu) \leqslant p_c(q; \nu) \leqslant 0$ for each $q \in [0, 1]$ and

$$p_c(q) = \lim_{\nu \searrow 0} p_c(q; \nu)$$

exists in $[0, 1]$. Define

$$T_c \equiv S \cap \{(q, p): 0 < q < 1, p = p_c(q)\}.$$

Note that it is possible to have $T_c = \varnothing$. On the other hand, suppose that there exists a $q_c \in (0, 1]$ such that $p_c(q) < 0$ in $(0, q_c)$ and $p_c(q_c) = 0$ in case $q_c \neq 1$. Then it follows from (4.4) and the monotone convergence theorem that $p_c(q)$ is a solution of (4.4) in $(0, q_c)$ and

$$\lim_{q \searrow 0} p_c(q) = 0.$$

Thus, in this case, the set T_c is a trajectory of (4.3) through (0, 0). We redefine p_c outside the interval $[0, q_c]$, if necessary by setting $p_c(q) = 0$ for $q \in [q_c, 1]$ if $q_c < 1$.

The critical value of c for which there exist wave solutions will be defined in terms of the behavior of the trajectories T_c. To study this behavior we shall need the following elementary technical lemma.

LEMMA 4.1. *For $j = 1$ and 2, let $p_j(q)$ denote real valued continuous functions defined on $[a, b]$ which satisfy the differential equations*

$$p_j' = F_j(q, p_j)$$

in (a, b). If $p_1(a) > p_2(a)$ and if either

$$F_1(q, p_2(q)) > F_2(q, p_2(q)) \tag{4.5}$$

or

$$F_1(q, p_1(q)) > F_2(q, p_1(q)) \tag{4.6}$$

in (a, b), then $p_1(q) \geqslant p_2(q)$ in $[a, b]$.

Proof. Suppose there is a $\hat{q} \in (a, b]$ such that $p_1(\hat{q}) < p_2(\hat{q})$. Since $p_1(a) > p_2(a)$ there exists a $q^* \in (a, \hat{q})$ such that $p_1(q) > p_2(q)$ in $[a, q^*)$ and $p_1(q^*) = p_2(q^*)$. Consequently $p_1'(q^*) \leqslant p_2'(q^*)$. On the other hand, if (4.5) holds then

$$p_1'(q^*) = F_1(q^*, p_1(q^*)) = F_1(q^*, p_2(q^*)) > F_2(q^*, p_2(q^*)) = p_2'(q^*).$$

A similar computation yields the same result in case (4.6) holds. In both cases we have a contradiction and it follows that $p_1 \geqslant p_2$ in $[a, b]$.

PROPOSITION 4.1. *For each $c \in \mathbb{R}^+$ with $c^2 > 4f'(0)$ the set T_c is a trajectory of (4.3) in S through (0, 0) and it is extremal in the sense that no other trajectory through (0, 0) has points in S below T_c. Moreover*

$$p_c(q) \geqslant (1/c) \min_{[0,1]} f(u) - 2cq \tag{4.7}$$

in $[0, 1]$ and there exists a $\rho_c \in (0, 1]$ such that

$$p_c(q) \leqslant -(c/2)q \tag{4.8}$$

in $[0, \rho_c]$.

Proof. Set $p_1(q) = -(c/2)q$. Then $p_1' = F_1(q, p_1) \equiv -c/2$ and $p_1(0) = 0$. Set $p_2(q) = p_c(q; \nu)$ for an arbitrary $\nu \in \mathbb{R}^+$. Then $p_2' = F_2(q, p_2) \equiv -c - f(q)/p_2$ and $p_2(0) = -\nu$. By hypothesis, $f'(0) < c^2/4$ and $f \in C^1[0, 1]$. Hence there exists $\rho_c \in (0, 1]$ such that $u \in [0, \rho_c]$ implies $f'(u) < c^2/4$. If $q \in (0, \rho_c)$ then by the theorem of the mean

$$F_2(q, p_1(q)) = -c + 2f(q)/cq = -c + (2/c) f'(\theta q)$$

for some $\theta \in (0, 1)$. In particular, $\theta q \in (0, \rho_c)$ and

$$F_2(q, p_1(q)) < -c/2 = F_1(q, p_1(q)).$$

It therefore follows from Lemma 4.1 that $p_c(q; \nu) \leqslant -(c/2)q$ in $[0, \rho_c]$ and (4.8) is obtained by letting $\nu \searrow 0$. Moreover, it follows that T_c is a trajectory in S through $(0, 0)$ with $q_c \geqslant \rho_c$. The extremal property of T_c is an immediate consequence of its definition as the limit of the $R_{c,\nu}$ and the regularity of points of S.

To prove that (4.7) holds let $\mu \in \mathbb{R}^-$ be an arbitrary number such that

$$-c\mu < \min_{[0,1]} f(u).$$

For any $\nu \in (0, \mu)$ set $p_1(q) = p_c(q; \nu)$ and $p_2(q) = -\mu - 2cq$. Then $f(q) > -c\mu = cp_2(q) + 2c^2 q > cp_2(q)$ and $p_2(q) < 0$ in $(0, 1)$ imply that

$$F_1(q, p_2(q)) = -c - f(q)/p_2(q) > -2c = F_2(q, p_2(q))$$

in $(0, 1)$. Thus, by Lemma 4.1,

$$p_c(q; \nu) \geqslant -\mu - 2cq$$

in $(0, 1)$ for all $\nu \in (0, \mu)$. It follows that

$$p_c(q) \geqslant -\mu - 2cq$$

in $(0, 1)$ and we obtain (4.7) by letting $-\mu \to (1/c) \min_{[0,1]} f(u)$.

PROPOSITION 4.2. *Suppose that $c \in \mathbb{R}^-$ and $c^2 > 4\sigma$ where*

$$\sigma \equiv \sup\{f(u)/u \colon u \in (0, 1]\}.$$

Then $q_c = 1$ and $p_c(1) < 0$.

Taking into account the lower bound for $p_c(q)$ given by (4.7) we can rephrase Proposition 4.2 as follows. *The trajectory T_c connects $(0, 0)$ with a point on the negative half-line $q = 1$.*

Proof. Set $p_1(q) = -\frac{1}{2}(c + (c^2 - 4\sigma_0)^{1/2})q$ for arbitrary σ_0 such that $4\sigma < 4\sigma_0 < c^2$ and $p_2(q) = p_c(q; \nu)$ for arbitrary $\nu \in \mathbb{R}^+$. It is easily verified that

$$p_1' = F_1(q, p_1) \equiv -c - \sigma_0 q/p_1 .$$

Since $f(q) \leqslant q\sigma < q\sigma_0$ and $p_1(q) < 0$ in $(0, 1]$,

$$F_2(q, p_1(q)) = -c - f(q)/p_1(q) < -c - \sigma_0 q/p_1(q) = F_1(q, p_1(q)).$$

Thus, according to Lemma 4.1,

$$p_c(q; \nu) \leqslant -\tfrac{1}{2}(c + (c^2 - 4\sigma_0)^{1/2})q$$

Now let $\sigma_0 \searrow \sigma$ and $\nu \searrow 0$ to obtain

$$p_c(q) \leqslant -\tfrac{1}{2}(c + (c^2 - 4\sigma)^{1/2})q$$

in $[0, 1]$.

In view of Proposition 4.2 the number

$$c^* = \inf\{c\colon c > 0,\ c^2 > 4f'(0),\ q_c = 1,\ p_c(1) < 0\}$$

is well defined and satisfies

$$4f'(0) \leqslant (c^*)^2 \leqslant 4\sigma.$$

If $\sigma = f'(0)$ as in the classical work of Kolmogoroff, Petrovsky, and Piscounoff [13] then $c^* = 2\{f'(0)\}^{1/2}$.

Since $c^* \geqslant 0$ it follows that $c^* > 0$ provided that $f'(0) > 0$. More generally, we have the following result.

PROPOSITION 4.3. *If*

$$m \equiv \max_{[0,1]} F(q) \equiv \max_{[0,1]} \int_0^q f(u)\, du > 0$$

then $c^* > 0$.

Proof. Since $F'(0) = f(0) = 0$, the line $z = sq$ does not intersect the graph of $z = F(q)$ on $(0, 1]$ if $s \in \mathbb{R}^+$ is sufficiently large. On the other hand, since $m > 0$ the line and the graph do intersect for all sufficiently small values of s. Let

$$s_0 = \inf\{s\colon s \in \mathbb{R}^+,\ sq > F(q) \text{ for } q \in (0, 1]\}.$$

The graphs of $z = F(q)$ and $z = s_0 q$ intersect in $(0, 1)$ and, in particular, there exists $q_0 \in (0, 1)$ such that $F(q) < s_0 q$ in $(0, q_0)$ and $s_0 q_0 = F(q_0)$. Note that $F(q) < F(q_0)$ in $[0, q_0)$, $F(q_0) > 0$, and $s_0 = F'(q_0) = f(q_0) > 0$. For $c = 0$ the trajectory through $(q_0, 0)$ satisfies

$$\tfrac{1}{2}p^2(q) + F(q) = F(q_0).$$

In particular, it is in S for $q \in (0, q_0)$. Moreover $\frac{1}{2}p^2(0) = F(q_0) > 0$ so that this trajectory connects the regular point $(q_0, 0)$ to a regular point on the negative p-axis. By continuity, the same is true of the trajectories through $(q_0, 0)$ for all sufficiently small $c > 0$. Since there is a unique trajectory through each point of S, for such values of c no trajectory in S through $(0, 0)$ can intersect the negative half-line $q = 1$. Thus, in view of its definition, $c^* > 0$.

COROLLARY. *If, in addition to* (1.3), $f(u)$ *satisfies one of the conditions* (1.4), (1.5), (1.6), *or* (1.6′), *then* $c^* > 0$.

In the next section we shall show that c^* is an asymptotic speed of propagation of disturbances. The main result of this section is the existence of plane wave solutions of (1.1) with wave speed c^*. For this purpose we shall need more information about the extremal trajectories T_c.

PROPOSITION 4.4. *If* $c \in \mathbb{R}^+$ *is such that* $c^2 > 4f'(0)$ *then* T_c *has slope*

$$s_c = \tfrac{1}{2}(-c - \{c^2 - 4f'(0)\}^{1/2})$$

at $(0, 0)$ *and any other trajectory of* (4.3) *in* S *through* $(0, 0)$ *has slope*

$$r_c = \tfrac{1}{2}(-c + \{c^2 - 4f'(0)\}^{1/2}).$$

If $c^2 = 4f'(0)$ *then every nontrivial trajectory of* (4.3) *in* S *through* $(0, 0)$ *has slope* $s_c = -c/2 = r_c$.

Proof. Assume that $c \in \mathbb{R}^+$ with $c^2 \geq 4f'(0)$. From the general theory of two-dimensional autonomous systems as developed, for example, in Petrovskii's book it follows that every trajectory in the strip $\{(q, p): 0 < q < 1, p \in \mathbb{R}\}$ through $(0, 0)$ approaches $(0, 0)$ with slope r_c or s_c [14, pp. 178–179].[2] An immediate consequence of this statement is the second assertion of the Proposition. Suppose that $c^2 > 4f'(0)$. Then, again by the general theory, there is at most one trajectory in S through $(0, 0)$ with slope s_c at $(0, 0)$ [14, pp. 180–181]. According to Proposition 4.1, $p_c(q) \leq -cq/2$ in $[0, \rho_c]$ for some $\rho_c \in (0, 1]$ so that $p_c'(0) \leq -c/2$. Since $r_c > -c/2$ it follows that $p_c'(0) = s_c$ and thus T_c is the unique trajectory in S through $(0, 0)$ with slope s_c at the origin.

[2] As presented in reference [14] the results cited in this paragraph apply only in the case $c^2 > 4f'(0)$. However it is not difficult to check that the argument is also valid for $c^2 = 4f'(0)$ when the trajectories are constrained to lie in the strip $0 < q < 1$.

We shall use Proposition 4.4 together with the next lemma to establish the continuity of the trajectories T_c with respect to the parameter c.

LEMMA 4.2. *If $c, d \in \mathbb{R}^+$ are such that $4f'(0) \leqslant c^2 < d^2$ and $T_c \neq \varnothing$ then*

$$p_d(q) < p_c(q) \text{ in } (0, q_c].$$

Proof. In view of (4.4) we have the "pseudo first integral" relations

$$\frac{d}{dq}\{\tfrac{1}{2}p_c^2 + F(q)\} = -cp_c \text{ in } (0, q_c)$$

and

$$\frac{d}{dq}\{\tfrac{1}{2}p_d^2 + F(q)\} = -dp_d \text{ in } (0, q_d).$$

Since $c < d$ implies $s_c > s_d$ it follows from Proposition 4.4 that $p_d(q) < p_c(q)$ for all sufficiently small $q > 0$. Suppose there exists a $q^* \in (0, q_c]$ such that $p_d(q) < p_c(q)$ in $(0, q^*)$ and $p_d(q^*) = p_c(q^*)$. Then

$$0 = \tfrac{1}{2}\{p_d^2(q^*) - p_c^2(q^*)\} = \int_0^{q^*} \{cp_c(q) - dp_d(q)\}\, dq. \tag{4.9}$$

On the other hand, $p_d(q) < p_c(q) < 0$ in $(0, q^*)$ and $c < d$ implies $cp_c - dp_d > 0$ in $(0, q^*)$ which contradicts (4.9). Therefore $p_d(q) < p_c(q)$ in $(0, q_c]$ as asserted.

PROPOSITION 4.5. *If $d \in \mathbb{R}^+$ is such that $d^2 > 4f'(0)$ then*

$$\lim_{c \nearrow d} p_c(q) = p_d(q)$$

in $[0, 1]$.

Proof. We consider only values of $c \in \mathbb{R}^+$ such that $4f'(0) < c^2 < d^2$. In view of Lemma 4.2, for each fixed $q \in (0, 1]$ the family $\{p_c(q)\}$ is nonincreasing and bounded below by $p_d(q)$. Therefore

$$r(q) = \lim_{c \nearrow d} p_c(q)$$

exists in $[0, 1]$ and satisfies

$$p_c(q) \geqslant r(q) \geqslant p_d(q).$$

Thus, in particular, $p = r(q)$ represents a trajectory of (4.3) in S through $(0, 0)$. Moreover, by Proposition 4.4,

$$s_c \geqslant r'(0) = \lim_{q \searrow 0}\{r(q)/q\} \geqslant s_d\,.$$

Letting $c \nearrow d$, one sees that $r'(0) = s_d$ so that $r(q) \equiv p_d(q)$.

Remark. A similar argument shows that

$$\lim_{c \searrow d} p_c(q) = p_d(q).$$

Thus the family $\{T_c\}$ of extremal trajectories varies continuously with c for $c \in \mathbb{R}^+$ such that $4f'(0) < c^2$.

We come now to the main result of this section, namely, the existence of plane wave solutions. Before stating it we introduce some notation.

Suppose that $\eta \in (0, 1)$ and $f(\eta) \neq 0$. Then there exists a unique trajectory of (4.3) through the regular point $(\eta, 0)$. For this trajectory $q(0) = \eta$, $p(0) = 0$ and $p'(0) = -f(\eta) \neq 0$. It follows from (4.3) that a component of this trajectory is in S for q in a one-sided neighborhood of η and we shall denote this component by $U_{c,\eta}$.

In the remainder of the paper we shall use the following notation:

$$\begin{aligned} \alpha &= 1 \quad \text{if } f(u) \text{ satisfies (1.4), (1.6) or (1.6')} \\ &= a \quad \text{if } f(u) \text{ satisfies (1.5).} \end{aligned}$$

THEOREM 4.1. *Suppose that $f(u)$ satisfies* (1.3) *and one of the conditions* (1.4), (1.5), (1.6), *or* (1.6′). *Then there exists a strictly decreasing function $q^*(\xi)$ such that*

$$(d^2/d\xi^2)\, q^* + c^*(d/d\xi)\, q^* + f(q^*) = 0$$

in $\mathbb{R}$,

$$\lim_{\xi \to -\infty} q^*(\xi) = \alpha,$$

and

$$\lim_{\xi \to +\infty} q^*(\xi) = 0.$$

Moreover, for any unit vector $\nu \in \mathbb{R}^n$, the function $u(x, t) = q^*(x \cdot \nu - c^* t)$ is a plane wave solution of (1.1).

Proof. It suffices to show that for $c = c^*$ there exists a trajectory in S which leaves S at $(0, 0)$ and $(\alpha, 0)$. To this end we distinguish two cases.

Case 1. $(c^*)^2 > 4f'(0)$. In this case the following three assertions, which are proved below, imply that T_{c^*} is the required trajectory.

(a) $p_{c^*}(q_{c^*}) = 0$.

(b) $f(q_{c^*}) = 0$.

(c) q_{c^*} is not the right hand end point of an interval in which $f \equiv 0$.

If (a) does not hold then $q_{c^*} = 1$ and $p_{c^*}(1) = -\gamma < 0$. By Proposition 4.[illegible], $p_c(q) \searrow p_{c^*}(q)$ in $[0, 1]$ as $c \nearrow c^*$. Thus for all sufficiently large $c < c^*$, $q_c = 1$ and $p_c(1) < -\gamma/2$. Since this contradicts the definition of c^* it follows that (a) must hold.

Suppose that $f(q_{c^*}) \neq 0$. Then $q_{c^*} \in (0, 1)$. Since T_{c^*} approaches $(q_{c^*}, 0)$ from S with $q < q_{c^*}$, it follows from (4.4) that $f(q_{c^*}) > 0$. By continuity, there exists an $\eta \in (q_{c^*}, 1)$ such that $f(q) > 0$ in $[q_{c^*}, \eta]$. Thus the trajectory $U_{c^*,\eta}$ is in S for $q \in [q_{c^*}, \eta)$. Moreover, because of the extremal property of T_{c^*}, the trajectory $U_{c^*,\eta}$ lies strictly below T_{c^*} for $q \in [0, q_{c^*}]$. Therefore $U_{c^*,\eta}$ leaves S through a point on the negative p-axis. By the continuity with respect to parameters of trajectories which consist entirely of regular points, for all sufficiently small $c > c^*$ the trajectories $U_{c,\eta}$ in S through $(\eta, 0)$ also leave S at a point on the negative p-axis. For any such value of c the trajectory T_c cannot intersect the negative half-line $q = 1$ since in order to do so it would have to cross $U_{c,\eta}$ in S. This contradicts the definition of c^* and hence (b) must hold.

Finally suppose that (c) does not hold. Then there exists an $\eta \in [0, q_{c^*})$ such that $f(\eta) = 0$ and $f(q) \leqslant 0$ in (η, q_{c^*}). In view of (4.4)

$$\frac{d}{dq} p_{c^*}(q) = -c^* - \frac{f(q)}{p_{c^*}(q)} \leqslant -c^*$$

in (η, q_{c^*}) so that

$$p_{c^*}(q_{c^*}) \leqslant p_{c^*}(\eta) - c^*(q_{c^*} - \eta) < 0.$$

Since this contradicts (a) it follows that (c) holds.

Case 2. $(c^*)^2 = 4f'(0)$. By the Corollary to Proposition 4.3, $c^* > 0$. Thus $f'(0) > 0$ and $f(u)$ satisfies either (1.4) or (1.5). If $\eta \in (0, \alpha)$ then $f(q) > 0$ in $(0, \eta]$. Hence the trajectory $U_{c^*,\eta}$ cannot leave S through any point of the q-axis. By the argument used in Case 1 to prove (b), $U_{c^*,\eta}$ cannot leave S at any point of the negative p-axis. Therefore $U_{c^*,\eta}$ is a trajectory which leaves S at $(0, 0)$ and $(\eta, 0)$. The family $\{U_{c^*,\eta}\}$ decreases as $\eta \nearrow \alpha$ and it is bounded below by the trajectory in S through $(\alpha, -\nu)$ for every $\nu > 0$. Hence

$$T^* = \lim_{\eta \nearrow \alpha} U_{c^*,\eta}$$

exists and is a trajectory which leaves S at $(0, 0)$ and $(\alpha, 0)$.

COROLLARY. *Under the hypotheses of Theorem* 4.1

$$\lim_{\xi \to +\infty} \{q^*(\xi)\}^{1/\xi} = e^{s^*},$$

where $s^* = s_{c^*}$.

Proof. By Proposition 4.4, the slope at $(0, 0)$ of the trajectory corresponding to $q^*(\xi)$ is s^*. The assertion follows from L'Hôpital's rule since

$$\lim_{\xi \to +\infty} \frac{1}{\xi} \log q^*(\xi) = \lim_{\xi \to +\infty} \frac{q^{*\prime}(\xi)}{q^*(\xi)} = s^*.$$

Remark 1. In Case 2 of the proof the condition $(c^*)^2 = 4f'(0)$ is used only to conclude that $f(u)$ satisfies (1.4) or (1.5). Thus the argument actually shows that if $f(u)$ satisfies (1.4) or (1.5) then there exist plane waves for all $c \geqslant c^*$. Note that, in view of the definition of c^*, the plane wave solutions for $c > c^*$ cannot correspond to the extremal trajectory T_c.

Remark 2. If $(c^*)^2 = 4f'(0)$ then

$$T_{c^*} \equiv \lim_{\nu \searrow 0} R_{c^*,\nu}$$

is a trajectory in S through $(0, 0)$. The proof is similar to the proof of Proposition 4.1 with $U_{c^*,\eta}$ for any $\eta \in (0, \alpha)$ serving as the upper bound in place of the line $p = -cq/2$. The trajectory T_{c^*} is again an extremal trajectory so that T^* either coincides with T_{c^*} or lies strictly above it. Both cases can occur so that the plane wave does not necessarily correspond to the extremal trajectory.

We conclude this section with a result which will be used in the next section to construct comparison functions. In stating it we shall use the following notation. For $c \geqslant 0$ define

$$\begin{aligned} \gamma_c &= 0 && \text{if there exists no trajectory in } S \text{ through } (0, 0), \\ &= q_c && \text{if the extremal trajectory } T_c \text{ in } S \text{ through } (0, 0) \text{ exists.} \end{aligned}$$

We again let $\alpha = 1$ if f satisfies (1.4), (1.6), or (1.6′) and $\alpha = a$ if f satisfies (1.5).

LEMMA 4.3. *Suppose that $f(u)$ satisfies (1.3) and one of the conditions (1.4), (1.5), (1.6), or (1.6′). If $c \in (0, c^*)$ then $\gamma_c \in [0, \alpha)$ and for every $\eta \in (\gamma_c, \alpha)$ the trajectory $U_{c,\eta}$ leaves S at $(\eta, 0)$ and a point on the negative p-axis.*

Proof. If $\gamma_c = 0$ then $c^2 \leqslant 4f'(0)$. Since $c > 0$ it follows that $f'(0) > 0$ so that $f(u)$ satisfies (1.4) or (1.5). Therefore, in this case, $f(u) > 0$ in (γ_c, α).

Suppose now that $\gamma_c = q_c > 0$. If $q_c = 1$ then, in view of Lemma 4.2, $q_d = 1$ and $p_d(1) < 0$ for all $d > c$. Since $c < c^*$ this contradicts the definition of c^* and we conclude that $q_c < 1$. If $\alpha = a < 1$ then $f(u)$ satisfies (1.5) and, in particular, $f(u) < 0$ in $(a, 1)$. Thus, in view of (4.4), $q_c \notin (a, 1)$. If $q_c = a$ then, by Lemma 4.2, $d > c$ implies that $p_d(a) < 0$. Since $f(u) < 0$ in $(a, 1)$, it follows that $p_d(1) < 0$ for all $d > c$ which again contradicts the definition of c^*. Thus $c \in (0, c^*)$ implies that $q_c \in (0, \alpha)$. If $f(u)$ satisfies (1.6) or (1.6′) then $p_c(q) \leqslant -cq$ in $(0, a]$ so that $q_c > a$. Thus whenever $\gamma_c = q_c > 0$, $\gamma_c \in (0, \alpha)$ and $f(u) > 0$ in $[\gamma_c, \alpha)$.

For arbitrary $c \in (0, c^*)$ and $\eta \in (\gamma_c, \alpha)$ consider the trajectory $U_{c,\eta}$. On $U_{c,\eta}$, $q(0) = \eta$, $p(0) = 0$ and $p'(0) = -f(\eta) < 0$. Thus $U_{c,\eta}$ is in S for sufficiently large $q < \eta$. Indeed, $U_{c,\eta}$ cannot leave S through any point of the segment (γ_c, η) on the q-axis since $f(u) > 0$ there. Moreover, $U_{c,\eta}$ cannot leave S at the point $(\gamma_c, 0)$ since either $\gamma_c = 0$ and there are no trajectories in S through the origin or else $f(\gamma_c) > 0$. Finally, if $\gamma_c > 0$ then by Lemma 4.2 and the extremal property of T_c the trajectory $U_{c,\eta}$ must lie strictly below T_c for $q \in [0, \gamma_c]$.

Remark. Suppose that $c \in (0, c^*)$ and that T_c is nontrivial. Then, as shown in the proof of Lemma 4.3, $q_c \in (0, \alpha)$ and $f(q_c) > 0$. If $f(u)$ satisfies (1.4) or (1.5) then $f(u) > 0$ in $(0, q_c]$. If $f(u)$ satisfies (1.6) or (1.6′) then on any trajectory in S through the origin, $p(q) \leqslant -cq$ in $[0, a]$. Thus in all cases, no trajectory in S through $(0, 0)$ can leave S via another critical point. Therefore c^* is the minimal wave speed.

5. Propagation of Disturbances

In this section we shall investigate the propagation of disturbances from the rest state $u \equiv 0$. Roughly speaking, we shall show that any disturbance which is initially of bounded support and which becomes sufficiently large as $t \to \infty$ will be propagated with asymptotic speed c^*. In view of what we proved in Section 3, it follows that all disturbances of bounded support are propagated with asymptotic speed c^* if $f(u)$ satisfies the conditions of either the heterozygote intermediate case (1.4) or the heterozygote superior case (1.5). In the heterozygote inferior and combustion cases there are also threshold effects so that a given disturbance may not propagate at all. These effects are discussed in detail in the next section.

We begin by showing that for any f which satisfies the general hypothesis (1.3), a disturbance with bounded support cannot be propagated with a speed greater than c^*.

THEOREM 5.1. *Let $u \in [0, 1]$ be a solution of (1.1) in $\mathbb{R}^n \times \mathbb{R}^+$ such that $u(x, 0) \equiv 0$ outside the ball $|x| \leqslant \rho$, for some $\rho \in R^+$. Then for any $c > c^*$ and any $y \in \mathbb{R}^n$*

$$\lim_{t \to +\infty} \max_{|\zeta - y| \geqslant ct} u(\zeta, t) = 0.$$

Proof. If $c > c^*$ then, by the definition of c^*, the trajectory T_c leaves the semistrip S at the origin and at a point on the negative halfline $q = 1$. Let $\omega_c(\xi)$ denote the corresponding solution of Eq. (4.1) defined in $\mathbb{R}^+$ and parametrized so that $\omega_c(0) = 1$. Note that $\omega_c' < 0$ in $\mathbb{R}^+$ and $\omega_c(\xi) \to 0$ as $\xi \to +\infty$.

Let $v(\xi, t)$ denote the solution of the initial value problem

$$\begin{aligned} v_t &= v_{\xi\xi} + cv_\xi + f(v) \text{ in } \mathbb{R} \times \mathbb{R}^+ \\ v(\xi, 0) &= 1 \qquad\qquad \text{for } \xi < \rho, \\ &= \omega_c(\xi - \rho) \quad \text{for } \xi \geqslant \rho. \end{aligned}$$

By Proposition 2.2, $v(\xi, t) \searrow \tau(\xi)$ in $\mathbb{R}$ as $t \to +\infty$, where $\tau(\xi)$ is the largest solution of Eq. (4.1) in $\mathbb{R}$ with values in [0, 1] which satisfies

$$\tau(\xi) \leqslant \omega_c(\xi - \rho) \qquad \text{for} \quad \xi \geqslant \rho. \tag{5.1}$$

Suppose that $\tau(\xi) \not\equiv 0$. Then since $\tau(\xi) \in [0, 1]$ and $\tau(\xi) \to 0$ as $\xi \to +\infty$, $\tau(\xi)$ corresponds to a trajectory $\mathscr{J}$ in the strip $\{(q, p): 0 < q < 1, p \in \mathbb{R}\}$ through the point (0, 0). The slope of $\mathscr{J}$ at (0, 0) is either r_c or s_c (cf. the proof of Proposition 4.4). If the slope of $\mathscr{J}$ is r_c then, by L'Hôpital's rule,

$$\lim_{\xi \to +\infty} \{\tau(\xi)\}^{1/\xi} = e^{r_c}.$$

But according to Proposition 4.4, the slope of T_c at (0, 0) is s_c so that

$$\lim_{\xi \to +\infty} \{\omega_c(\xi)\}^{1/\xi} = e^{s_c}. \tag{5.2}$$

Since $s_c < r_c$ and $\tau(\xi) \leqslant \omega_c(\xi - \rho)$ this shows that the slope of $\mathscr{J}$ cannot be r_c. Hence the slope of $\mathscr{J}$ at (0, 0) must be s_c. It follows from Proposition 4.4 that $\mathscr{J} = T_c$ so that, in particular, $\tau(\xi)$ is a translate of $\omega_c(\xi)$. Hence there exists a $\xi_0 \in \mathbb{R}$ such that $\tau(\xi_0) = 1$ and $\tau'(\xi_0) < 0$. Then $\tau(\xi) > 1$ for all sufficiently large $\xi < \xi_0$. Since this contradicts the condition $\tau(\xi) \in [0, 1]$ in $\mathbb{R}$ we conclude that $\tau(\xi) \equiv 0$ and

$$\lim_{t \to +\infty} v(\xi, t) = 0 \tag{5.3}$$

in $\mathbb{R}$.

For arbitrary fixed $h > 0$ define

$$z(\xi, t) \equiv v(\xi + h, t) - v(\xi, t).$$

Then

$$z_t = z_{\xi\xi} + c z_\xi + (f' \circ v)(\hat{\xi}, t) z \text{ in } \mathbb{R} \times \mathbb{R}^+$$

for some $\hat{\xi} \in (\xi, \xi + h)$ and, because of the monotonicity of $\omega_c(\xi)$,

$$z(\xi, 0) \leqslant 0 \text{ in } \mathbb{R}.$$

By Proposition 2.1, $z(\xi, t) \leqslant 0$ in $\mathbb{R}$ and hence $v(\xi, t)$ is a nonincreasing function of ξ for each $t \in \mathbb{R}^+$.

Let ν be an arbitrary unit vector in $\mathbb{R}^n$ and define

$$w(x, t) \equiv v(x \cdot \nu - ct, t).$$

Then

$$w(x, 0) = v(x \cdot \nu, 0) \geqslant v(|x|, 0) \geqslant u(x, 0) \text{ in } \mathbb{R}^n$$

and

$$w_t = \Delta w + f(w) \text{ in } \mathbb{R}^n \times \mathbb{R}^+.$$

Therefore, according to Proposition 2.1,

$$u(x, t) \leqslant v(x \cdot \nu - ct, t) \text{ in } \mathbb{R}^n \times \mathbb{R}^+.$$

Since ν is arbitrary it follows that

$$u(x, t) \leqslant v(|x| - ct, t) \text{ in } \mathbb{R}^n \times \mathbb{R}^+.$$

Fix an arbitrary $y \in \mathbb{R}^n$. Since $v(\xi, t)$ is a nonincreasing function of ξ and $|\zeta - y| \geqslant ct$ implies $|\zeta| - ct \geqslant -|y|$ it follows that

$$\max_{|\zeta - y| \geqslant ct} u(\zeta, t) \leqslant \max_{|\zeta - y| \geqslant ct} v(|\zeta| - ct, t) \leqslant v(-|y|, t).$$

Thus the assertion of the theorem is a consequence of (5.3).

Remark. The condition $u(x, 0) \equiv 0$ for $|x| > \rho$ is used in the proof of Theorem 5.1 only to obtain the inequality $u(x, 0) \leqslant w(x, 0)$ in $\mathbb{R}^n$. However, since $\omega_c(\xi)$ satisfies (5.2) it suffices to assume that $e^{s|x|}u(x, 0)$ is bounded for some s such that $s > -s_c$.

If we impose more restrictive conditions on $f(u)$ and $u(x, 0)$ we can obtain a result which is stronger than Theorem 5.1. Specifically, we can estimate u in terms of the plane wave $q^*(\xi)$ constructed in Section 4. In stating this result we use the notation $s^* = s_{c^*}$. We again let α be 1 if $f(u)$ satisfies (1.4), (1.6), or (1.6′) and $\alpha = a$ if f satisfies (1.5).

THEOREM 5.2. *Let $u \in [0, 1]$ denote a solution of (1.1) in $\mathbb{R}^n \times \mathbb{R}^+$, where $f(u)$ satisfies (1.3) and one of the conditions (1.4), (1.5), (1.6), or (1.6′). If $u(x, 0) \in [0, \alpha]$ and $e^{s|x|}u(x, 0)$ is bounded in $\mathbb{R}^n$ for some $s > -s^*$ then for each $h \in \mathbb{R}^+$ there exists a constant θ such that*

$$u(x, t) \leqslant q^*(|x| - c^*t + \theta)$$

in $\mathbb{R}^n \times [h, +\infty)$.

Proof. By hypothesis there exists a $k \in \mathbb{R}^+$ such that

$$u(x, 0) \leqslant ke^{-s|x|}$$

in $\mathbb{R}^n$. Let ν be an arbitrary unit vector in $\mathbb{R}^n$ and define

$$z(x, t) \equiv k \exp\{(\sigma + s^2)t - s(x \cdot \nu)\}$$

where

$$\sigma = \sup\{f(u)/u : 0 < u \leqslant 1\}.$$

Since $x \cdot \nu \leqslant |x|$,

$$z(x, 0) = ke^{-s(x \cdot \nu)} \geqslant ke^{-s|x|} \geqslant u(x, 0).$$

Moreover

$$u_t - \Delta u - \sigma u \leqslant u_t - \Delta u - f(u) = 0 = z_t - \Delta z - \sigma z.$$

Therefore, by the remark following Proposition 2.1, $z(x, t) \geqslant u(x, t)$ in $\mathbb{R}^n \times \mathbb{R}^+$. In particular, for arbitrary fixed $h \in \mathbb{R}^+$

$$u(x, h) \leqslant \tilde{k} e^{-s(x \cdot \nu)} \text{ in } \mathbb{R}^n,$$

where $\tilde{k} = k \exp(\sigma + s^2)h$. Since the unit vector ν is arbitrary it follows that

$$u(x, h) \leqslant \tilde{k} e^{-s|x|} \text{ in } \mathbb{R}^n. \tag{5.4}$$

The Corollary to Theorem 4.1 states that for any $\epsilon \in (0, 1)$ there is an $r = r(\epsilon) > 0$ such that $\xi > r$ implies

$$q^*(\xi) \geqslant (1 - \epsilon)^\xi e^{s^* \xi}. \tag{5.5}$$

Set $\delta = s^* + s$ and $\epsilon = (e^\delta - 1)/(e^\delta + 1)$. Then it follows from (5.4) and (5.5) that

$$q^*(|x|) \geqslant u(x, h) \tag{5.6}$$

provided that

$$|x| > r^* = \max\{r, \log \tilde{k}/\log(1 + \epsilon)\}.$$

Define

$$\mu \equiv \max_{|x| \leqslant r^*} u(x, h).$$

By Proposition 2.1, $0 < u(x, h) < \alpha$ so that $\mu \in (0, \alpha)$. If $q^*(r^*) \geqslant \mu$ then, since $q^*(\xi)$ is a decreasing function of ξ, $q^*(|x|) > \mu$ for all $|x| < r^*$. Therefore, in this case, (5.6) holds for all $x \in \mathbb{R}^n$. If $q^*(r^*) < \mu$ there exists $\theta' \in \mathbb{R}^+$ such that $q^*(r^* - \theta') = \mu$. Moreover, $|x| < r^*$ implies $q^*(|x| - \theta') > \mu \geqslant u(x, h)$ while, on the other hand, $u(x, h) \leqslant q^*(|x|) < q^*(|x| - \theta')$ for all x such that $|x| > r^*$. Thus there exists a constant $\theta' \geqslant 0$ such that

$$u(x, h) \leqslant q^*(|x| - \theta') \text{ in } \mathbb{R}^n.$$

If ν is any unit vector in $\mathbb{R}^n$ then $q^*(|x| - \theta') \leqslant q^*(x \cdot \nu - \theta')$ and it follows from Proposition 2.1 that

$$u(x, t) \leqslant q^*(x \cdot \nu - c^* t + \theta)$$

in $\mathbb{R}^n \times [h, +\infty)$ where $\theta = hc^* - \theta'$. Finally, since ν is arbitrary

$$u(x, t) \leqslant q^*(|x| - c^* t + \theta) \text{ in } \mathbb{R}^n \times [h, +\infty).$$

Remark. Fix $y \in \mathbb{R}^n$. Then in view of the monotonicity of q^*

$$\max_{|\zeta - y| \geqslant ct} u(\zeta, t) \leqslant q^*((c - c^*)t - |y| + \theta).$$

In particular, as in Theorem 5.1

$$\lim_{t\to+\infty} \max_{|\zeta-y|\geqslant ct} u(\zeta, t) = 0$$

provided that $c > c^*$. A more precise statement is obtained by noting that the Corollary to Theorem 4.1 implies

$$q^*(\xi) = o(e^{-\sigma\xi})$$

for any σ such that $\sigma < -s^*$. Therefore for $c > c^*$ and $\sigma < -s^*$

$$\max_{|\zeta-y|\geqslant ct} u(\zeta, t) = o(e^{-\sigma(c-c^*)t})$$

as $t \to +\infty$.

As we have seen, a disturbance with bounded support cannot be propagated with a speed larger than c^*. Because of the possibility of threshold effects such a disturbance may not be propagated at all; that is, it is possible that

$$\lim_{t\to+\infty} u(x, t) = 0.$$

However, our next result shows that if the disturbance is propagated with sufficient strength, then its speed of propagation is no smaller than c^*. Thus, in particular, c^* is the asymptotic speed of propagation.

THEOREM 5.3. *Let $u \in [0, 1]$ be a solution of* (1.1) *in $\mathbb{R}^n \times \mathbb{R}^+$ where $f(u)$ satisfies* (1.3) *and one of the conditions* (1.4), (1.5), (1.6), *or* (1.6′). *If*

$$\liminf_{t\to+\infty} u(x, t) \geqslant \alpha \tag{5.7}$$

uniformly on every compact subset of $\mathbb{R}^n$ then for any $c \in (0, c^*)$ and any $y \in \mathbb{R}^n$

$$\liminf_{t\to+\infty} \min_{|\zeta-y|\leqslant ct} u(\zeta, t) \geqslant \alpha.$$

The proof of Theorem 5.3 is based on the following lemma which establishes the existence of a special class of disturbances which travel with speeds arbitrarily close to c^*. This special class of disturbances provides the comparison functions which are used in the proof of the theorem.

Lemma 4.3 states that for each $c \in (0, c^*)$ and $\eta \in (\gamma_c, \alpha)$ the trajectory $U_{c,\eta}$ through $(\eta, 0)$ leaves the semistrip S at a point on the negative p-axis. The trajectory $U_{c,\eta}$ corresponds to the solution $q(\xi)$ of the initial value problem

$$q'' + cq' + f(q) = 0, \qquad q(0) = \eta, \qquad q'(0) = 0. \tag{5.8}$$

In view of the properties of $U_{c,\eta}$ there exists a number $b = b(c, \eta) > 0$ such that

$$q(b) = 0 \qquad \text{and} \qquad q' < 0 \text{ in } (0, b]. \tag{5.9}$$

Given the parameters $c \in (0, c^*)$, $\eta \in (\gamma_c, \alpha)$ and $\rho > (n-1)/c$ let $v(x, t)$ denote the solution of the initial value problem

$$
\begin{aligned}
v_t &= \Delta v + f(v) \text{ in } \mathbb{R}^n \times \mathbb{R}^+ \\
v(x, 0) = v_0(|x|) &\equiv \eta \qquad \text{for } |x| \leqslant \rho, \\
&\equiv q(|x| - \rho) \qquad \text{for } \rho < |x| \leqslant \rho + b, \\
&\equiv 0 \qquad \text{for } \rho + b < |x|,
\end{aligned}
\tag{5.10}
$$

where $q(\xi)$ denotes the solution of (5.8) satisfying (5.9). Strictly speaking we should indicate the dependence of b and q on c and η, and of v on c, η, and ρ. However, we shall omit this to avoid excessive notation.

LEMMA 5.1. *For given $c \in (0, c^*)$, $\eta \in (\gamma_c, \alpha)$ and $\rho \in ((n-1)/c, +\infty)$ let $v(x, t)$ denote the solution of the corresponding problem* (5.10). *Then*

$$\lim_{t \to +\infty} v(x, t) = \alpha$$

uniformly on compact subsets of $\mathbb{R}^n$ and

$$v(x, t) \geqslant \eta \qquad \text{for } |x| \leqslant \rho + (c - (n-1)/\rho)t \text{ and } t \in \mathbb{R}^+.$$

Proof. Let $\{\varphi_j(r)\}$ denote a sequence of $C_0^\infty[0, +\infty)$ functions such that $\varphi_j(|x|) \searrow v_0(|x|)$ as $j \to +\infty$ and $\varphi_j(|x|) > v_0(|x|)$ for $|x| \leqslant \rho + b$. If $v_j(x, t)$ denotes the solution of the initial value problem

$$
\begin{aligned}
v_t &= \Delta v + f(v) \text{ in } \mathbb{R}^n \times \mathbb{R}^+ \\
v(x, 0) &= \varphi_j(|x|) \text{ in } \mathbb{R}^n
\end{aligned}
$$

then, as is easily verified, $v_j(x, t) \searrow v(x, t)$ in $\mathbb{R}^n \times \mathbb{R}^+$ as $j \to +\infty$. By Proposition 2.1, $v_j > 0$ in $\mathbb{R}^n \times \mathbb{R}^+$.

Choose an arbitrary $c_1 \in (0, c - (n-1)/\rho)$ and define

$$W(x, t) \equiv v_0(|x| - c_1 t).$$

Then for every j, $W(x, 0) \leqslant \varphi_j(|x|)$ with the strict inequality for $|x| \leqslant \rho + b$. Moreover

$$
\begin{aligned}
W_t - \Delta W - f(W) &= -f(\eta) \qquad \text{for } |x| < \rho + c_1 t, \\
&= q'(|x| - c_1 t - \rho)(c - [(n-1)/|x|] - c_1) \\
&\qquad \text{for } \rho + c_1 t < |x| < \rho + b + c_1 t, \\
&= 0 \qquad \text{for } \rho + b + c_1 t < |x|.
\end{aligned}
$$

Since $c_1 < c - (n-1)/\rho$ and $q' < 0$ it follows that

$$W_t - \Delta W - f(W) \leqslant 0 \qquad \text{for } |x| \neq \rho + c_1 t \text{ or } \rho + b + c_1 t. \tag{5.11}$$

Also note that $q'(0) = 0$ implies that W is a continuously differentiable function of x for $|x| < \rho + b + c_1 t$.

Let $u_j \equiv v_j - W$. Then $u_j \in C(\mathbb{R}^n \times [0, +\infty))$ and $u_j(x, 0) = \varphi_j(|x|) - v_0(|x|) > 0$ for $|x| \leqslant \rho + b$. In view of (5.11)

$$(\partial u_j/\partial t) - \Delta u_j - f' u_j \geqslant 0 \quad \text{for } |x| < \rho + b + c_1 t \text{ with } |x| \neq \rho + c_1 t. \tag{5.12}$$

Here f' is evaluated somewhere between $v_j(x, t)$ and $W(x, t)$. We shall show that $u_j > 0$ in $\mathbb{R}^n \times \mathbb{R}^+$. Since $u_j(x, t) = v_j(x, t) > 0$ for $|x| \geqslant \rho + b + c_1 t$ and $t \in \mathbb{R}^+$, it suffices to consider only $(x, t) \in \mathbb{R}^n \times \mathbb{R}^+$ with $|x| < \rho + b + c_1 t$.

Suppose there exists a $t_0 \in \mathbb{R}^+$ and an $x_0 \in \mathbb{R}^n$ satisfying $|x_0| < \rho + b + c_1 t_0$ such that $u_j(x, t) > 0$ for all $(x, t) \in \mathbb{R}^n \times [0, t_0)$ with $|x| \leqslant \rho + b + c_1 t$ and $u_j(x_0, t_0) = 0$. By the strong maximum principle [4, p. 38] applied to the differential inequality (5.12) it follows that $|x_0| = \rho + c_1 t_0$. According to the boundary point lemma [4, p. 49] applied in the set $|x| \leqslant \rho + c_1 t$ for $t \in [0, t_0]$, the radial derivative $\partial u_j/\partial r$ is negative at (x_0, t_0). The same argument applied in the set $\rho + c_1 t \leqslant |x| \leqslant \rho + b + c_1 t$ for $t \in [0, t_0]$ yields $\partial u_j/\partial r > 0$ at (x_0, t_0). Since both v_j and W are continuously differentiable, this is a contradiction and we conclude that $u_j = v_j - W > 0$ in $\mathbb{R}^n \times \mathbb{R}^+$. Now let $j \to \infty$ to obtain

$$W(x, t) \leqslant v(x, t) \text{ in } \mathbb{R}^n \times \mathbb{R}^+. \tag{5.13}$$

It follows from the definition of W that $v \geqslant \eta$ for $|x| \leqslant \rho + c_1 t$, $t \in \mathbb{R}^+$. Since $c_1 \in (0, c - (n-1)/\rho)$ is arbitrary this proves the second assertion of Lemma 5.1.

In view of (5.13) and the definition of W, $v(x, h) \geqslant W(x, h) \geqslant W(x, 0) = v(x, 0)$ for any $h > 0$. By Proposition 2.1, $v(x, t + h) \geqslant v(x, t)$ so that v is an increasing function of t for each $x \in \mathbb{R}^n$. Since $v \in [0, 1]$ there exists a function $\tau(x)$ such that $v(x, t) \nearrow \tau(x)$ as $t \to +\infty$. By the same argument as that used in the proof of Proposition 2.2, it follows that $\Delta\tau + f(\tau) = 0$ in $\mathbb{R}^n$ and that the convergence of v to τ is uniform on compact subsets. Moreover,

$$W(x, t) \leqslant v(x, t) \leqslant \tau(x) \text{ in } \mathbb{R}^n \times \mathbb{R}^+.$$

The fact that $W(x, t) \nearrow \eta$ as $t \to +\infty$ implies that $\tau(x) \geqslant \eta$ in $\mathbb{R}^n$. It remains only to show that $\tau(x) \equiv \alpha$.

Let $z(x, t)$ denote the solution of the initial value problem

$$z_t = \Delta z + f(z) \text{ in } \mathbb{R}^n \times \mathbb{R}^+,$$
$$z(x, 0) = \eta \text{ in } \mathbb{R}^n.$$

By Proposition 2.1, $z(x, t) \leqslant \tau(x)$. On the other hand, z is independent of x so that $z(x, t) = \zeta(t)$ where $\zeta' = f(\zeta)$ and $\zeta(0) = \eta$. Therefore

$$t = \int_\eta^\zeta \frac{d\lambda}{f(\lambda)}.$$

Since $f > 0$ on $[\eta, \alpha)$ and $f(\alpha) = 0$ it follows that

$$\lim_{t\to\infty} \zeta(t) \geqslant \alpha.$$

This together with the fact that $\tau \leqslant \alpha$ completes the proof of the Lemma.

Proof of Theorem 5.3. Given $c \in (0, c^*)$ choose $\bar{c} \in (c, c^*)$ and $\bar{\rho} \in \mathbb{R}^+$ such that $c < \bar{c} - (n-1)/\bar{\rho}$. For an arbitrary $\bar{\eta} \in (\gamma_{\bar{c}}, \alpha)$ let $\bar{v}(x, t)$ denote the solution of the corresponding problem (5.10). By hypothesis, $\liminf_{t\to\infty} u(x, t) \geqslant \alpha$ uniformly on compact subsets of $\mathbb{R}^n$. Thus, in particular, there exists an $h > 0$ such that $u(x, h) \geqslant \bar{\eta}$ for $|x| \leqslant \bar{\rho} + \bar{b}$, where $\bar{b} = b(\bar{c}, \bar{\eta})$. Since $\bar{v}(x, 0) \leqslant \bar{\eta}$ for $|x| \leqslant \bar{\rho} + \bar{b}$ and $\bar{v}(x, 0) = 0$ for $|x| > \bar{\rho} + \bar{b}$ it follows that $u(x, h) \geqslant \bar{v}(x, 0)$ in $\mathbb{R}^n$. According to Proposition 2.1, $u(x, t + h) \geqslant \bar{v}(x, t)$ in $\mathbb{R}^n \times \mathbb{R}^+$. We therefore conclude from Lemma 5.1 that $u(x, t + h) \geqslant \bar{\eta}$ in $\mathbb{R}^n \times \mathbb{R}^+$ for $|x| \leqslant \bar{\rho} + \{\bar{c} - (n-1)/\bar{\rho}\}t$.

Fix $y \in \mathbb{R}^n$. Since $c < \bar{c} - (n-1)/\bar{\rho}$ the ball $\{\zeta : |y - \zeta| \leqslant ct\}$ is contained in the ball $\{\zeta : |\zeta| \leqslant \bar{\rho} + (\bar{c} - (n-1)/\bar{\rho})(t - h)\}$ for all sufficiently large $t \in \mathbb{R}^+$. Thus for each sufficiently large $t \in \mathbb{R}^+$

$$\min_{|\zeta - y| \leqslant ct} u(\zeta, t) \geqslant \bar{\eta}.$$

The result now follows because $\bar{\eta}$ can be chosen to be arbitrarily close to α.

The hair trigger results of Section 3 show that the hypothesis (5.7) of Theorem 5.3 is automatically satisfied by nontrivial solutions in the heterozygote intermediate and superior cases. Thus we have the following corollaries to Theorem 5.3.

COROLLARY 1. *Let $f(u)$ satisfy* (1.3) *and the conditions* (1.4) *of the heterozygote intermediate case. If $u \in [0, 1]$ is a solution of* (1.1) *in $\mathbb{R}^n \times \mathbb{R}^+$ such that $u \not\equiv 0$ then for any $y \in \mathbb{R}^n$ and $c \in [0, c^*)$*

$$\lim_{t\to+\infty} \min_{|\zeta - y| \leqslant ct} u(\zeta, t) = 1.$$

Remark. Theorem 5.1 together with Corollary 1 to Theorem 5.3 shows that c^* is an asymptotic speed of propagation in the following sense. Suppose that $u(x, 0)$ has bounded support. For any $\gamma \in (0, 1)$ define

$$r_\gamma(t) \equiv \inf\{|x| : u(x, t) = \gamma\}$$

and

$$R_\gamma(t) \equiv \sup\{|x| : u(x, t) = \gamma\}.$$

Then

$$\lim_{t\to+\infty} \frac{1}{t} r_\gamma(t) = \lim_{t\to+\infty} \frac{1}{t} R_\gamma(t) = c^*.$$

COROLLARY 2. *Let $f(u)$ satisfy* (1.3) *and the conditions* (1.5) *of the heterozygote superior case. If $u \in [0, 1]$ is a solution of* (1.1) *in $\mathbb{R}^n \times \mathbb{R}^+$ such that $u \not\equiv 0$ then for any $x \in \mathbb{R}^n$ and $c \in [0, c^*)$*

$$\liminf_{t\to+\infty} \min_{|\zeta-x|\leqslant ct} u(\zeta, t) \geqslant a.$$

Remark. By introducing the new dependent variable $w = 1 - u$ we can show that in the heterozygote superior case there is a $c^{**} > 0$ such that if $u \not\equiv 1$, then

$$\limsup_{t\to+\infty} \max_{|x-\zeta|\leqslant ct} u(\zeta, t) \leqslant a$$

for all $c \in [0, c^{**})$. If, moreover, $1 - u(x, 0)$ has bounded support, then by Theorem 5.1 applied to w,

$$\lim_{t\to+\infty} \min_{|x-\zeta|\geqslant ct} u(\zeta, t) = 1$$

for any $c > c^{**}$. There is no a priori information about the relative sizes of c^* and c^{**}. For example, if $f(u) = u(1 - u)(a - u)$ for some $a \in (0, 1)$ then $c^* = 2a^{1/2}$ and $c^{**} = 2(1 - a)^{1/2}$ so that the relative sizes of c^* and c^{**} depend upon the value of a.

6. THRESHOLD EFFECTS

In this section we investigate the stability of the rest state $u \equiv 0$ in the heterozygote inferior case (1.6) and the combustion case (1.6′). We shall show that the state $u \equiv 0$ is stable with respect to perturbations which are not too large on too large a set, but is unstable with respect to some perturbations with bounded support. Moreover, we shall show that if $u \to 1$ as $t \to \infty$ then the disturbance is propagated with asymptotic speed c^*.

We begin with a very simple stability result.

PROPOSITION 6.1. *Assume that $f(u) < 0$ for $u \in (0, a)$. Let $u \in [0, 1]$ be a solution of* (1.1) *in $\mathbb{R}^n \times \mathbb{R}^+$. If*

$$\mu \equiv \sup_{\mathbb{R}^n} u(x, 0) < a$$

then

$$\lim_{t\to+\infty} u(x, t) = 0$$

uniformly in $\mathbb{R}^n$.

Proof. Let $z(x, t)$ denote the solution of the initial value problem

$$\begin{aligned} z_t &= \Delta z + f(z) \text{ in } \mathbb{R}^n \times \mathbb{R}^+ \\ z(x, 0) &= \mu \text{ in } \mathbb{R}^n. \end{aligned}$$

Then z is independent of x; that is, $z(x, t) = \zeta(t)$ with $\zeta' = f(\zeta)$ and $\zeta(0) = \mu$. Therefore

$$t = \int_\mu^\zeta \frac{d\lambda}{f(\lambda)},$$

and it follows that $\zeta \to 0$ as $t \to +\infty$. On the other hand, by Proposition 2.1, $u(x, t) \leqslant \zeta(t)$, which proves the assertion.

We shall now show that the state $u \equiv 0$ is stable with respect to a class of perturbations which may be large on a small set. This theorem is an extension of a result of Kanel' [11]. In its present form it was first given in [1] for the one-dimensional case. We assume that

$$f(u) \leqslant 0 \text{ for } u \in [0, a] \text{ and } f(a') > 0 \text{ for some } a' \in (a, 1). \tag{6.1}$$

For any $\eta \in [0, a)$ define

$$\mu = \mu(\eta) \equiv \sup_{u \in (\eta, 1)} f(u)/(u - \eta).$$

Note that $\mu > 0$. In what follows we shall use the notation

$$[q]^+ = \max(q, 0).$$

THEOREM 6.1. *Assume that $f(u)$ satisfies (1.3) and (6.1). Let $u(x, t) \in [0, 1]$ be a solution of (1.1) in $\mathbb{R}^n \times \mathbb{R}^+$ such that for some $\eta \in [0, a)$.*

$$\int_{\mathbb{R}^n} [u(x, 0) - \eta]^+ \, dx \leqslant \left\{ \frac{2\pi n}{e\mu(\eta)} \right\}^{n/2} (a - \eta). \tag{6.2}$$

Then

$$\limsup_{t \to +\infty} u(x, t) \leqslant \eta \tag{6.3}$$

uniformly in $\mathbb{R}^n$. If, moreover, $f(u) < 0$ for $u \in (0, \eta]$ then

$$\lim_{t \to \infty} u(x, t) = 0$$

uniformly in $\mathbb{R}^n$.

Proof. In view of (6.1), $f(u) \leqslant \mu[u - \eta]^+$. Let $w(x, t)$ denote the solution of the initial value problem

$$\begin{aligned} w_t &= \Delta w + \mu w \text{ in } \mathbb{R}^n \times \mathbb{R}^+ \\ w(x, 0) &= [u(x, 0) - \eta]^+ \text{ in } \mathbb{R}^n. \end{aligned} \tag{6.4}$$

By Proposition 2.1, $w \geqslant 0$ in $\mathbb{R}^n \times \mathbb{R}^+$. Therefore the function $v(x, t) \equiv \eta + w(x, t)$ satisfies the differential inequality

$$v_t - \Delta v - f(v) \geqslant v_t - \Delta v - \mu[v - \eta]^+ = w_t - \Delta w - \mu w = 0$$

in $\mathbb{R}^n \times \mathbb{R}^+$ together with the initial condition $v(x, 0) \geqslant u(x, 0)$ in $\mathbb{R}^n$. It follows from Proposition 2.1 that $v(x, t) \geqslant u(x, t)$ in $\mathbb{R}^n \times \mathbb{R}^+$. From the standard formula for the solution of problem (6.4) we obtain

$$\begin{aligned} v(x, t) &= \eta + \frac{e^{\mu t}}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-|x-\xi|^2/4t}[u(\xi, 0) - \eta]^+ \, d\xi \\ &\leqslant \eta + \frac{e^{\mu t}}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} [u(\xi, 0) - \eta]^+ \, d\xi. \end{aligned}$$

In particular, if (6.2) holds then

$$u(x, n/2\mu) \leqslant v(x, n/2\mu) \leqslant a.$$

Applying Proposition 2.1 again, one finds that $u(x, t) \leqslant a$ in $\mathbb{R}^n \times [n/2\mu, +\infty)$. Hence $f(u) \leqslant 0$ for $t \geqslant n/2\mu$.

Let $z(x, t)$ denote the solution of the initial value problem

$$\begin{aligned} z_t &= \Delta z \text{ in } \mathbb{R}^n \times (n/2\mu, +\infty) \\ z(x, n/2\mu) &= v(x, n/2\mu) \text{ in } \mathbb{R}^n. \end{aligned}$$

Then since $f(u) \leqslant 0$ we can apply Proposition 2.1 to obtain $u(x, t) \leqslant z(x, t)$ in $\mathbb{R}^n \times [n/2\mu, +\infty)$. Note that

$$\begin{aligned} z(x, t) &= \int_{\mathbb{R}^n} G(x - \xi, t - n/2\mu)\, v(\xi, n/2\mu)\, d\xi \\ &= \eta + \int_{\mathbb{R}^n} G(x - \xi, t - n/2\mu)\, w(\xi, n/2\mu)\, d\xi \\ &= \eta + e^{n/2} \int_{\mathbb{R}^n} G(x - \xi, t)[u(\xi, 0) - \eta]^+ \, d\xi \\ &\leqslant \eta + \left(\frac{e}{4\pi t}\right)^{n/2} \int_{\mathbb{R}^n} [u(\xi, 0) - \eta]^+ \, d\xi \end{aligned}$$

where $G(x, t) = (4\pi t)^{-n/2} \exp\{-|x|^2/4t\}$. In particular, $z \to \eta$ as $t \to +\infty$ uniformly in $\mathbb{R}^n$, and (6.3) follows.

If $f(u) < 0$ in $(0, \eta]$ then, by continuity, the same holds on a slightly larger interval, say $(0, \eta + \epsilon)$ for some $\epsilon > 0$. Since $u(x, t) \leqslant z(x, t) < \eta + \epsilon$ for sufficiently large t, we can apply Proposition 6.1 to conclude that $u \to 0$ uniformly in $\mathbb{R}^n$ as $t \to +\infty$.

Remark. If $f(u)$ satisfies (1.3) and the condition (1.6) of heterozygote inferiority then the rest state $u \equiv 0$ is stable with respect to any perturbation which satisfies (6.2) for some $\eta \in [0, a)$. In the combustion case (1.6′) the rest state is stable if (6.2) holds with $\eta = 0$. This was proved by Kanel' [11].

We now exhibit a class of perturbations with respect to which the state $u \equiv 0$ is unstable. In particular, the class in question includes some perturbations of bounded support.

THEOREM 6.2. *Assume that $f(u)$ satisfies (1.3) and either (1.6) or (1.6′). Let $u(x, t) \in [0, 1]$ denote a solution of (1.1) in $\mathbb{R}^n \times \mathbb{R}^+$ and suppose that*

$$u(x, 0) \geqslant v_0(|x - x_0|) \tag{6.5}$$

for some member $v_0(r)$ of the three-parameter family of functions defined in (5.10) and some $x_0 \in \mathbb{R}^n$. Then for any $y \in \mathbb{R}^n$ and $c \in (0, c^*)$

$$\lim_{t\to\infty} \min_{|\zeta-y|\leqslant ct} u(\zeta, t) = 1.$$

Proof. In view of (6.5), it follows from Proposition 2.1 that

$$u(x, t) \geqslant v(x - x_0, t) \text{ in } \mathbb{R}^n \times \mathbb{R}^+,$$

where $v(x, t)$ denotes the solution of (5.10) with initial data $v_0(|x|)$. According to Lemma 5.1

$$\lim_{t\to+\infty} u(x, t) = 1$$

uniformly on compact subsets of $\mathbb{R}^n$. The assertion of the theorem now follows directly from Theorem 5.3.

Remark 1. Theorems 6.1 and 6.2 show that a threshold phenomenon occurs in the heterozygote inferior case. In particular, the advantageous allele A does not survive unless it is initially present with sufficient density in a sufficiently large territory.

Remark 2. Assume that $0 < f(u) \leqslant ku^\beta$ in $(0, \alpha)$ for some $k > 0$ and $\beta > 1 + 2/n$, and that $f(u) < 0$ in $(\alpha, 1)$ if $\alpha \in (0, 1)$. In view of Theorem 3.2 there are positive initial functions $u(x, 0)$ for which $u(x, t) \to 0$ as $t \to +\infty$. On the other hand, it is not difficult to verify that Lemma 4.3 holds also in this case and that $\gamma_c = q_c \searrow 0$ as $c \searrow 0$. In the proofs of Lemma 5.1, Theorem 5.3, and Theorem 6.2 the hypothesis on f is used only to assure the validity of Lemma 4.3. Thus all of these results continue to hold in the present case. In particular, if

$$u(x, 0) \geqslant v_0(|x - x_0|)$$

for some member $v_0(r)$ of the three-parameter family of functions defined by (5.10) and for some $x_0 \in \mathbb{R}^n$ then

$$\liminf_{t\to+\infty} \min_{|y|\leqslant ct} u(x, t) \geqslant \alpha$$

for any $c \in (0, c^*)$. Since $\gamma_c \searrow 0$ as $c \searrow 0$, there exist arbitrarily small initial data with compact support for which the solution of the initial value problem grows. Thus in this case there are also threshold effects.

Remark 3. The gap between the conditions of Theorems 6.1 and 6.2 is in the nature of the problem. The function $u \equiv a$ is an unstable steady state solution and an examination of the phase plane trajectories shows that there also exist many periodic steady state solutions as well as traveling wave solutions with values in (0, 1).

Remark 4. Theorem 6.2 can be regarded as a rather strong stability result for the state $u \equiv 1$. In particular, it states that $u \equiv 1$ is stable with respect to any perturbation with bounded support and, indeed, with respect to any perturbation bounded above by one of the functions $1 - v_0(|x - x_0|)$. A different stability condition for this case (with $n = 1$) is due to Chafee [2].

Remark 5. It follows from the definition of γ_c in Lemma 4.3 that as c approaches zero γ_c approaches the number $\kappa \geqslant a$ defined by

$$\int_0^\kappa f(u)\, du = 0.$$

Hence for any $\eta \in (\kappa, 1)$ there is a member v_0 of the family of functions defined (5.10) which satisfies $v_0 \leqslant \eta$ and has bounded support. Thus the conclusion of Theorem 6.2 holds if $u(x, 0) \geqslant \eta > \kappa$ on a sufficiently large ball.

Remark 6. For $n = 1$, Fife and McLeod [The approach of solutions of nonlinear diffusion equations to travelling front solutions, *Arch. Rat. Mech. Anal.* **65** (1977), 335–361.] have obtained the following considerably stronger result: If f satisfies (1.3) and (1.6) and if $u_0 \geqslant \eta > a$ on a sufficiently large interval there exist constants θ_1 and θ_2 and positive constants K and ν so that

$$|u(x, t) - q^*(x - c^*t - \theta_1) - q^*(-x - c^*t - \theta_2) + 1| < Ke^{-\nu t}$$

Note that $a < \kappa$ and that the solution approaches a sum of two traveling waves.

APPENDIX: PROOF OF LEMMA 3.1

Lemma 3.1 was first proved by Hayakawa [7], but only for $n = 1$ and 2. For arbitrary n it is a special case of a more general result proved by Kobayashi, Sirao, and Tanaka [12, 16]. Here we shall give a somewhat simplified version of Hayakawa's proof which works for all $n \geqslant 1$.

of p, one can use the method of successive approximations to solve (A.1) in $\mathbb{R}^n \times (0, T']$ for sufficiently small $T' \in \mathbb{R}^+$ subject to the initial condition

$$p(x, 0) = p_0(x). \tag{A.2}$$

The function $p(x, t)$ is continuous in $\mathbb{R}^n \times [0, T']$ and is the unique bounded solution of the initial value problem (A.1), (A.2) in $\mathbb{R}^n \times [0, T']$. Moreover, if $p(\xi, \tau)$ is continuous in $\mathbb{R}^n \times [0, t]$, then for any $t_0 \in [0, t)$, p satisfies the integral identity

$$p(x, t) = \int_{\mathbb{R}^n} G(x - \xi, t - t_0)\, p(\xi, t_0)\, d\xi$$

$$+ k \int_{t_0}^{t} d\tau \int_{\mathbb{R}^n} G(x - \xi, t - \tau)\{p(\xi, \tau)\}^{\beta}\, d\xi. \tag{A.3}$$

Here

$$G(x, t) = (4\pi t)^{-n/2}\, e^{-|x|^2/4t}$$

is the fundamental solution of the equation of heat conduction. If $p(x, T')$ is again bounded, we can extend this solution to a strip $\mathbb{R}^n \times (T', T'']$ so that the set of T' with the property that p is a bounded solution in $\mathbb{R}^n \times [0, T']$ is open.

We assume that

$$p_0(x) \geqslant 0 \qquad \text{and} \qquad p_0(x) \not\equiv 0 \text{ in } \mathbb{R}^n, \tag{A.4}$$

and we shall prove that there exists a $T \in \mathbb{R}^+$ such that $p(x, t)$ is the solution of problem (A.1), (A.2) in $\mathbb{R}^n \times [0, T)$ and

$$\limsup_{t \nearrow T} \sup_{x \in \mathbb{R}^n} p(x, t) = +\infty. \tag{A.5}$$

Suppose, in contradiction to (A.5), that $p(x, t)$ is bounded in $\mathbb{R}^n \times [0, T]$ for every $T \in \mathbb{R}^+$. In view of (A.4), it follows from Proposition 2.1 that $p > 0$ in $\mathbb{R}^n \times \mathbb{R}^+$. By (A.3) with $t_0 = 1$ and $t = 2$,

$$p(x, 2) > \int_{\mathbb{R}^n} G(x - \xi, 1)\, p(\xi, 1)\, d\xi.$$

Set $\nu = \min\{p(\xi, 1): |\xi| \leqslant 1\}$. Then $\nu > 0$ and

$$p(x, 2) > \nu \int_{|\xi|\leqslant 1} G(x - \xi, 1)\, d\xi \geqslant c\, e^{-|x|^2/2},$$

where

$$c = \nu(4\pi)^{-n/2} \int_{|\xi|\leqslant 1} e^{-|\xi|^2/2}\, d\xi.$$

Set $\theta = k^{-1/2}c^{-1/n}$ and define

$$\rho(x, t) \equiv c^{-1}p(\theta x, \theta^2 t + 2).$$

Then $\rho(x, t)$ is positive and bounded in $\mathbb{R}^n \times [0, T]$ for every $T \in \mathbb{R}^+$ and

$$\rho(x, 0) \geqslant e^{-\gamma|x|^2}, \tag{A.6}$$

where $\gamma = \theta^2/2$. Moreover, since $G(\theta x, \theta^2 t) = \theta^{-n}G(x, t)$ it follows from (A.3) that

$$\rho(x, t) = \int_{\mathbb{R}^n} G(x - \xi, t)\, \rho(\xi, 0)\, d\xi + \int_0^t d\tau \int_{\mathbb{R}^n} G(x - \xi, t - \tau)\{\rho(\xi, \tau)\}^\beta\, d\xi \tag{A.7}$$

in $\mathbb{R}^n \times [0, +\infty)$.

We shall prove that for each integer $N \geqslant 0$

$$\rho(x, t) \geqslant \sum_{j=0}^{N} \sigma_j(x, t) \text{ in } \mathbb{R}^n \times [0, +\infty), \tag{A.8}$$

where

$$\sigma_j(x, t) = \beta^{\pi_j}(1 + 4\gamma t)^{-n/2} \left\{\frac{\beta - 1}{4\gamma} \log(1 + 4\gamma t)\right\}^{w_j} \exp\left\{-\frac{\gamma\beta^j\, |x|^2}{1 + 4\gamma t}\right\}, \tag{A.9}$$

$w_j = (\beta^j - 1)(\beta - 1)$, and

$$\begin{aligned} \pi_j &= 0 && \text{for } j = 0 \\ &= -\left(\frac{n}{2} + 1\right) \sum_{l=0}^{j-1} (j - l)\beta^l && \text{for } j \geqslant 1. \end{aligned}$$

To prove (A.8) we first note the identity

$$\int_{\mathbb{R}^n} G(x - \xi, t)\, e^{-A|\xi|^2}\, d\xi = (1 + 4At)^{-n/2}\, e^{-A|x|^2/(1+4At)} \tag{A.10}$$

it follows from (A.6), (A.9), and (A.10) that

$$\rho(x, t) \geqslant \int_{\mathbb{R}^n} G(x - \xi, t)\, \rho(\xi, 0)\, d\xi \geqslant (1 + 4\gamma t)^{-n/2}\, e^{-\gamma|x|^2/(1+4\gamma t)} = \sigma_0(x, t). \tag{A.11}$$

Thus (A.8) holds with $N = 0$. On the other hand, if (A.8) holds with $N = m \geqslant 0$ then we conclude from (A.7), (A.11) and the elementary inequality

$$\left(\sum_{j=0}^{m} \sigma_j\right)^{\beta} \geqslant \sum_{j=0}^{m} \sigma_j{}^{\beta}$$

that

$$\rho(x, t) \geqslant \sigma_0(x, t) + \sum_{j=0}^{m} \int_0^t d\tau \int_{\mathbb{R}^n} G(x - \xi, t - \tau)\{\sigma_j(\xi, \tau)\}^{\beta}\, d\xi.$$

Therefore, (A.8) is valid for $N = m + 1$ provided that

$$\int_0^t d\tau \int_{\mathbb{R}^n} G(x - \xi, t - \tau)\{\sigma_j(\xi, \tau)\}^{\beta}\, d\xi \geqslant \sigma_{j+1}(x, t) \tag{A.12}$$

for all $j \geqslant 0$.

Let I_j denote the integral on the left in (A.12). In view of (A.9)

$$I_j = \beta^{\beta\pi_j} \left(\frac{\beta - 1}{4\gamma}\right)^{\beta w_j} \int_0^t (1 + 4\gamma\tau)^{-n\beta/2}\{\log(1 + 4\gamma\tau)\}^{\beta w_j}\, d\tau \int_{\mathbb{R}^n} G(x - \xi, t - \tau)$$
$$\cdot \exp\left\{-\frac{\gamma\beta^{j+1}\, |\, \xi\, |^2}{1 + 4\gamma\tau}\right\} d\xi.$$

Now apply (A.10) with $A = \gamma\beta^{j+1}/(1 + 4\gamma\tau)$ to obtain

$$I_j = \beta^{\beta\pi_j} \left(\frac{\beta - 1}{4\gamma}\right)^{\beta w_j} \int_0^t (1 + 4\gamma\tau)^{n(1-\beta)/2}$$
$$\cdot \{1 + 4\gamma\tau + 4\gamma\beta^{j+1}(t - \tau)\}^{-n/2}\{\log(1 + 4\gamma\tau)\}^{\beta w_j}$$
$$\cdot \exp\left\{-\frac{\gamma\beta^{j+1}\, |\, x\, |^2}{1 + 4\gamma\tau + 4\gamma\beta^{j+1}(t - \tau)}\right\} d\tau.$$

Recall that $\beta = 1 + 2/n > 1$. For $\tau \in [0, t]$

$$1 + 4\gamma\tau \leqslant 1 + 4\gamma\tau + 4\gamma\beta^{j+1}(t - \tau) \leqslant \beta^{j+1}(1 + 4\gamma t).$$

Thus

$$I_j \geqslant \beta^{\beta\pi_j - n(j+1)/2} \left(\frac{\beta - 1}{4\gamma}\right)^{\beta w_j} (1 + 4\gamma t)^{-n/2} \exp\left\{-\frac{\gamma\beta^{j+1} \mid x \mid^2}{1 + 4\gamma t}\right\}$$

$$\cdot \int_0^t (1 + 4\gamma\tau)^{-1} \{\log(1 + 4\gamma\tau)\}^{\beta w_j} \, d\tau$$

$$= \beta^{\beta\pi_j - n(j+1)/2} \left(\frac{\beta - 1}{4\gamma}\right)^{\beta w_j} (1 + 4\gamma t)^{-n/2}$$

$$\cdot \exp\left\{-\frac{\gamma\beta^{j+1} \mid x \mid^2}{1 + 4\gamma t}\right\} \{4\gamma(\beta\omega_j + 1)\}^{-1} \{\log(1 + 4\gamma t)\}^{\beta w_j + 1}.$$

It is easily verified that $\beta\omega_j + 1 = \omega_{j+1}$, $(\beta\omega_j + 1)^{-1} > (\beta - 1)\,\beta^{-(j+1)}$, and $\beta\pi_j - ((n/2) + 1)(j + 1) = \pi_{j+1}$. The estimate (A.12) follows immediately from these observations together with the last estimate for I_j.

Since $\pi_0 = 0$ and $1 + n/2 = \beta/(\beta - 1)$, the exponents in (A.9) are related by the formula

$$\pi_{l+1} - \pi_l = -(1 + n/2)\,\omega_{l+1} = -\kappa(\beta^{l+1} - 1)$$

for $l \geqslant 0$, where $\kappa = \beta/(\beta - 1)^2$. Summing on l from 0 to $j - 1$ yields

$$\pi_j = -\beta\kappa\omega_j + j\kappa$$

and (A.9) can be rewritten in the form

$$\sigma_j(x, t) = \beta^{j\kappa}(1 + 4\gamma t)^{-n/2} \left\{\beta^{-\beta\kappa} \frac{\beta - 1}{4\gamma} \log(1 + 4\gamma t)\right\}^{w_j} \exp\left\{-\frac{\gamma\beta^j \mid x \mid^2}{1 + 4\gamma t}\right\}.$$

There exists a $t^* \in \mathbb{R}^+$ such that

$$\log(1 + 4\gamma t^*) = [4\gamma/(\beta - 1)]\,\beta^{\beta\kappa}.$$

Then

$$\sigma_j(0, t^*) \geqslant (1 + 4\gamma t^*)^{-n/2}$$

and it follows from (A.8) that

$$\rho(0, t^*) \geqslant (N + 1)(1 + 4\gamma t^*)^{-n/2}$$

for every integer $N \geqslant 0$. This contradicts the boundedness of ρ and hence of ρ in $\mathbb{R}^n \times [0, T]$ for every $T \in \mathbb{R}^+$. Therefore (A.5) holds and the lemma is proved.

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