J. of Mech. and Phys. of Solids 45 (1997) pp. 1275=-1280

ON THE NONEXISTENCE OF CERTAIN IDEAL FORMING OPERATIONS FOR EXTRUSION AND DRAWING DIES

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ABSTRACT

It is shown that an ideal forming operation for an extrusion or drawing die shrinks not only the cross sectional area but also the perimeter. It follows that for any billet shape there are product cross sections which can be produced by extrusion or drawing dies but not by ideal forming operations.

Keywords. B. ideally plastic material, C. optimization

1. INTRODUCTION

This work concerns the design of an extrusion or drawing die with well-lubricated walls which is to reshape an initial material strand with a prescribed upstream cross section $\mathcal{C}_u$ (the billet) into an emerging strand with a prescribed downstream cross section $\mathcal{C}_d$ (the product). It is assumed that the material is perfectly rigid-plastic and incompressible, that it obeys a Tresca yield condition, and that there is no strain hardening. The problem is to produce a shape for the die which minimizes the rate of energy dissipation per unit length of product, or, equivalently, the force required to push or pull the material through the die at a given constant rate.

For the case where the upstream and downstream profiles are rectangular strips of different thicknesses, it was shown by Richmond and Devenpeck (1962) that one could find a divergence-free two-dimensional velocity field $\mathbf{v}$ with the following properties: (i) $\mathbf{v}$ is an eigenvector of the rate-of-strain matrix

$$D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \tag{1.1}$$

which corresponds to the only positive eigenvalue of this matrix; (ii) $\mathbf{v}$ reduces to constant slug flow in the upstream and the downstream regions. They called such a flow a **streamlined flow**, and showed that if the die boundary lies on the two streamlines which connect the upstream boundary planes with the downstream boundary planes, the resulting die is optimal in the above sense.

The analogous result was obtained by Richmond (1965) (see also Richmond and Morrison (1967)) for the case of wire drawing, in which the upstream and downstream cross sections are discs of prescribed radii. The optimal die is again obtained as the stream surface of a streamlined flow which connects the upstream and downstream wires.

Hill (1967) extended the concept of streamlined flow to general three-dimensional problems, and gave it the name **ideal forming operation**. He presented an argument to show that a die which produces an ideal forming operation is optimal. However, Hill did not address the important question of whether there is an ideal forming operation which reshapes an initial material strand with a given cross section into an emerging strand with a given cross section. This question been considered by a number of authors (see, e. g., Kuusela (1986), Wienecki and Richmond (1992)), Chung and Richmond (1994), Weinberger (1995, 1996)), but it has not been settled.

In the present work we shall show that there are many pairs of cross sections with the property that the downstream cross section can be produced from the upstream one by means of an extrusion die but for which there is no ideal forming operation. In other words, the production of a streamlined flow is a property which cannot, in general, be expected of an optimal die. This means that the search for an optimal die needs to be conducted by other means.

Our result is based on a simple geometric property of an ideal forming operation. We shall assume that each of the cross sections is simply connected, and that it is the closure of its interior. The perimeter of such a cross section is defined to be the limit as $\epsilon \to 0$ of the infimum of the lengths of interior curves which lie within distance ϵ of the boundary.

THEOREM *Suppose there is an ideal forming operation which takes a prescribed simply connected upstream cross section $\mathcal{C}_u$ into a prescribed simply connected downstream cross section $\mathcal{C}_d$. Let p_u the perimeter of $\mathcal{C}_u$, and let p_d denote the perimeter of $\mathcal{C}_d$. Then*

$$p_d \leq p_u. \tag{1.2}$$

This result gives a variety of pairs of cross sections for which there is no ideal forming operation. For example, if one wishes to deform a circular cylinder of radius 1 into a bar whose cross section is a square of radius a, the fact that an extrusion die decreases the cross sectional area requires that $a \leq \pi^{1/2}$, but the inequality (1.2) states that an ideal forming operation cannot exist unless $a \leq \pi/2$.

Theorem 1 can also be used in the following argument to show that the set of product cross sections which can be obtained from a given billet by ideal forming operations is nowhere dense. The **Hausdorff distance** between two regions is the infimum of numbers d such that each point of each of the regions is within distance d of the other region.

Suppose the cross section $\mathcal{C}_u$ can be changed into $\mathcal{C}_d$ by an ideal forming operation. By making the boundary of the downstream cross section rougher, one can obtain a new cross section whose Hausdorff distance from the original one is arbitrarily small while its perimeter is so large that the inequality (2.2) is violated. That is, there is a cross section arbitrarily close to $\mathcal{C}_d$ which cannot be produced with an ideal forming operation.

2. PROOF OF THE THEOREM

Let $\mathbf{v}$ be a divergence-free velocity field in a domain which is topologically a cylinder. Suppose that the flow is tangential on the lateral boundaries, that on the upstream part $x_3 \leq 0$ one has $v_1 = v_2 = 0$, $v_3 = V_u$, and that on the downstream part $x_3 \geq L$ one has $v_1 = v_2 = 0$, $v_3 = V_d$. The prescribed speeds V_u and $V_d > V_u$ are constants. By the divergence theorem $V_u A_u = V_d A_d$ where A_u and A_d are the areas of the upstream and downstream cross sections.

When the Tresca yield condition is valid, the rate of energy dissipation of such a flow is given by the formula

$$\dot{E} = 2k \int |\lambda| dx dy dz$$

where $|\lambda|$ is the largest of the absolute values of the eigenvalues of the rate-of-strain matrix D, and k is the yield constant. (See Weinberger (1996).) The integral is over the region where $D \neq 0$. When the die boundary is prescribed, the equilibrium flow is that velocity field $\mathbf{v}$ which minimizes this integral. An optimal die is one in which the resulting minimum is minimized with respect to choices of the die boundary.

We see from Rayleigh's principle that

$$|\lambda| \geq \mathbf{v} \cdot D\mathbf{v}/|\mathbf{v}|^2 = \nabla \cdot (\ln|\mathbf{v}|\ \mathbf{v}). \tag{2.1}$$

Therefore the divergence theorem implies the lower bound

$$\dot{E} \geq 2kV_d A_d \ln(A_u/A_d),$$

which depends only on the prescribed data, and not on the die boundary.

A flow is an ideal forming operation when this lower bound is attained. This means that equality must hold in the inequality (2.1). This is the case if and only if at each point the characteristic value $|\lambda|$ is an eigenvalue of D and $\mathbf{v}$ is a corresponding eigenvector. Because the trace of D is zero, the other two eigenvalues must be nonpositive.

The streamlines of the velocity field $\mathbf{v}$ define a one-to-one mapping S from the upstream cross section $\mathcal{C}_u$ to the downstream cross section $\mathcal{C}_d$. We shall denote by $\ell(\gamma)$ the length of a curve γ. Our result is based on the following Lemma.

Lemma. *If $\mathbf{v}$ is the velocity field of an ideal forming operation, so that at every point $\mathbf{v}$ is an eigenvector of the matrix D which corresponds to its unique positive eigenvalue λ, then the image $S[\gamma]$ of any curve γ in the interior of $\mathcal{C}_u$ has the property*

$$\ell(S[\gamma]) \leq \ell(\gamma). \tag{2.2}$$

Proof. Define the two stream functions h_1 and h_2 by the equation

$$\begin{aligned} &\mathbf{v} \cdot \nabla h_\alpha = 0 \\ &h_\alpha(x_1, x_2, L) = x_\alpha \qquad \alpha = 1, 2. \end{aligned}$$

That is, the point $(h_1(\mathbf{x}), h_2(\mathbf{x}), L)$ on $\mathcal{C}_d$ lies on the streamline through (x_1, x_2, x_3). In particular, the mapping $(x_1, x_2) \to (h_1(x_1, x_2, 0), h_2(x_1, x_2, 0))$ is the streamline mapping S.

Because $\mathbf{v}$ is orthogonal to the gradients of h_1 and h_2, it must have the form $\mathbf{v} = a(\mathbf{x})\nabla h_1 \times \nabla h_2$. The condition $\nabla \cdot \mathbf{v} = 0$ implies that $\mathbf{v} \cdot \nabla a = 0$. Because at $x_3 = L$ we have $a = V_d$, it follows that*

$$\mathbf{v} = V_d \nabla h_1 \times \nabla h_2.$$

The conditions on $\mathbf{v}$ at $x_3 = 0$ become

$$\begin{aligned} &\frac{\partial h_1}{\partial x_3}(x_1, x_2, 0) = \frac{\partial h_2}{\partial x_3}(x_1, x_2, 0) = 0 \\ &\det\left(\frac{\partial(h_1, h_2)}{\partial(x_1, x_2)}(x_1, x_2, 0)\right) = \frac{V_u}{V_d} = \frac{A_d}{A_u}. \end{aligned} \tag{2.3}$$

* This representation was found independently by R. Roadman (private conversation).

Thus the mapping $(x_1, x_2) \to (\sqrt{A_u/A_d}h_1(x_1, x_2, 0), \sqrt{A_u/A_d}h_2(x_1, x_2, 0))$, which is a dilation of S, is area preserving. In particular, S is an invertible map from $\mathcal{C}_u$ onto $\mathcal{C}_d$.

We shall use the summation convention and the notation $_{,i}$ for the partial derivative with respect to x_i with $i = 1, 2, 3$.

For any constant vector (a_1, a_2), let $h = a_1h_1 + a_2h_2$. Then by the definitions of h_1 and h_2, we have $v_ih_{,i} = 0$. By differentiating this equation, we see that $v_ih_{,ij} = -v_{i,j}h_{,j}$. Thus

$$\begin{aligned} v_i(|\nabla h|^2)_{,i} &= 2v_ih_{,ij}h_{,j} \\ &= -2v_{i,j}h_{,i}h_{,j} \\ &= -2D_{ij}h_{,i}h_{,j}. \end{aligned}$$

Because ∇h is orthogonal to $\mathbf{v}$, which is the eigenvector corresponding to the only positive eigenvalue of D_{ij}, the quadratic form on the right is nonnegative. Because of the minus sign, it follows that $|\nabla h|^2$ is nondecreasing along streamlines. Therefore the upstream value of $|\nabla h|^2$ is bounded above by its downstream value. Since $h = a_1h_1 + a_2h_2$, since $h_\alpha(x_1, x_2, L) = x_\alpha$, and since $h_{\alpha,3}(x_3, x_2, 0) = 0$, we find that

$$h_{\alpha,\delta}(x_1, x_2, 0)h_{\beta,\delta}(x_1, x_2, 0)a_\alpha a_\beta \leq a_\alpha a_\alpha$$

for every vector (a_1, a_2). (Summation over Greek letters is over the values 1 and 2.) That is, both eigenvalues of the matrix $h_{\alpha,\delta}(x_1, x_2, 0)h_{\beta,\delta}(x_1, x_2, 0)$ are at most 1.

If we define the matrix $J_{\alpha\delta} = h_{\alpha,\delta}(x_1, x_2, 0)$, we see that the above matrix has the form JJ^*. It is well known that the matrix J^*J has the same eigenvalues as JJ^*. Therefore the eigenvalues of the matrix

$$g_{\alpha\beta}(x_1, x_2) = h_{\delta,\alpha}(x_1, x_2, 0)h_{\delta,\beta}(x_1, x_2, 0)$$

are also bounded above by 1.

A curve γ in the upstream profile which is given in the parametric form $x_\alpha = \phi_\alpha(\tau)$ is deformed into the curve $S[\gamma]: \; x_\alpha = h_\alpha(\phi_1(\tau), \phi_2(\tau), 0)$ in the downstream profile. The length of γ is the integral of $\sqrt{\phi'_\alpha\phi'_\alpha}$. The length of $S[\gamma]$ is the integral of $\sqrt{g_{\alpha\beta}\phi'_\alpha\phi'_\beta}$. Therefore the fact that the eigenvalues of $g_{\alpha\beta}$ are at most 1 implies the statement (2.2) of the Lemma.

Proof of the Theorem. By the Lemma, the streamline map is a continuous and continuously invertible map from the interior of $\mathcal{C}_u$ onto the interior of $\mathcal{C}_c$. The Lemma also shows that the continuous extension of this map to the boundary of $\mathcal{C}_u$ is a continuous continuously invertible map from this boundary onto the boundary of $\mathcal{C}_d$.

For any positive ϵ and δ, choose a curve γ in the interior of $\mathcal{C}_u$ which lies within distance ϵ of the boundary and whose length is less than $p_u + \delta$. By the inequality (2.2) the curve $S[\gamma]$ lies within ϵ of the boundary of $\mathcal{C}_d$ and has length less than $p_u + \delta$. Since ϵ and δ are arbitrary, the inequality (1.2) follows immediately. Thus Theorem 1 is established.

REMARKS. 1. If one confines one's attention to two-dimensional flows, all cross sections are strips and the inequality (1.2) is meaningless. For axially symmetric flows all cross sections are discs, and the the inequality (1.2) is automatic. Thus the Theorem says nothing about these cases.

2. Similar results can be obtained for cross sections which are multiply connected and/or disconnected.

3. The necessary condition (1.2) for the existence of an ideal forming process is but one consequence of the Lemma. For example, if $\mathcal{C}_u$ is convex, one can show that every even moment of the boundary of $\mathcal{C}_d$ with respect to the point which minimizes this moment is bounded above by the corresponding moment of the boundary of $\mathcal{C}_u$. A similar result holds for any even area moment divided by the cross sectional area.

4. By using the fact that the transformation $(x_1, x_2) \to (\sqrt{A_d/A_u}h_1(x_1, x_2, L),$ $(\sqrt{A_d/A_u}h_2(x_1, x_2, L))$ is area preserving, one can also obtain the inequality $\ell(S[\gamma])/A_d \geq \ell(\gamma)/A_u$, which leads to additional necessary conditions.

ACKNOWLEDGMENT

This work was partly supported by a grant from the ALCOA Foundation.

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