Practice ProblemIII

1. From the formula of Newton's method:

$$x_{n+1} = x_n + \frac{f(x_n)}{f'(x_x)}$$

and $f(x) = x^3 - 2$, $f'(x) = 3x^2$.
$x_1 = 1$, $f(x_1) = -1$, $f'(x_1) = 3$,

$$x_2 = 1 - \frac{-1}{3} = 1 + \frac{1}{3} = \frac{4}{3}$$

$f(\frac{4}{3}) = (\frac{4}{3})^3 - 2 = \frac{10}{27}$, $f'(\frac{4}{3}) = 3 \cdot (\frac{4}{3})^2 = \frac{16}{3}$,

$$x_3 = \frac{4}{3} - \frac{\frac{10}{27}}{\frac{16}{3}} = \frac{4}{3} - \frac{5}{72} = \frac{91}{72}$$

2. $f(x) = x^3 - 3x^2 + 3$
$f'(x) = 3x^2 - 6x = 3x(x-2)$, $f''(x) = 6x - 6 = 6(x-1)$
$\Rightarrow$ $f$ has critical points when $f'(x) = 0$ or $x = 0, 2$.
$f$ has point of inflection when $f'' = 0$ at $x = 1$.
$f''(0) = -6 < 0 \Rightarrow f$ has local maximum at $x = 0$.
$f''(2) = 6 > 0 \Rightarrow f$ has local minimum at $x = 2$.
$f' > 0$ on $(-\infty, 0)$, $(2, \infty) \Rightarrow$ increasing.
$f' < 0$ on $(0, 2) \Rightarrow$ decreasing.
$f'' > 0$ on $(-\infty, 1) \Rightarrow$ concave down.
$f'' > 0$ on $(1, \infty) \Rightarrow$ concave up.

| | | 0 | 1 | 2 | |
|---|---|---|---|---|---|
| $f'(x)$ | $+$ | | $-$ | $-$ | $+$ |
| $f''(x)$ | $-$ | | $-$ | $+$ | $+$ |

3. Since $\lim_{x\to 1} x^a - 1 = \lim_{x\to 1} x^b - 1 = 0$, use l'Hospital's rule

$$\lim_{x\to 1} \frac{x^a - 1}{x^b - 1} = \lim_{x\to 1} \frac{ax^{a-1}}{bx^{b-1}} = \frac{a}{b}$$

4. Let $x$ be the length of the square corners and $y$ be the length of remainder. Then the volume of the box is $V = x^2 y$, which is the function we want to maximize. Use the condition $2x + y = 24$ , or $y = 24 - 2x$, we can eliminate $y$ in $V$:

$$V = x^2(24 - 2x) = 24x^2 - 2x^3$$

To find the critical number,

$$V' = 48x - 6x^2 = 6x(8 - x) = 0 \Rightarrow x = 0 \textit{ or } 8$$

It's clear that $x = 8$ is what we want.

5.

$$\begin{aligned}\int_2^4 \frac{1+x-x^2}{x^2}dx &= \int_2^4 (\frac{1}{x^2}+\frac{1}{x}-1)dx \\ &= -\frac{1}{x}+\ln x - x \mid_2^4 \\ &= (-\frac{1}{4}+\ln 4-4)-(-\frac{1}{2}+\ln 2-2) \\ &= \ln 2 - \frac{7}{4}\end{aligned}$$

6. Notice that

$$\int_{2x}^{3x+1} \sin(t^2)dt = \int_a^{3x+1} \sin(t^2)dt - \int_a^{2x} \sin(t^2)dt$$

Let $u = 3x+1$, then

$$\frac{d}{dx}\int_a^{3x+1} \sin(t^2)dt = \frac{d}{du}\int_a^{u} \sin(t^2)dt \cdot \frac{du}{dx} = \sin(u^2)\cdot 3 = 3\sin((3x+1)^2)$$

Similarly, $\frac{d}{dx}\int_a^{2x} \sin(t^2)dt = 2\sin((2x)^2)$. So

$$\begin{aligned}\frac{d}{dx}\int_{2x}^{3x+1} \sin(t^2)dt &= \frac{d}{dx}\int_a^{3x+1} \sin(t^2)dt - \frac{d}{dx}\int_a^{2x} \sin(t^2)dt \\ &= 3\sin((3x+1)^2) - 2\sin((2x)^2) \\ &= 3\sin(9x^2+6x+1) - 2\sin(4x^2)\end{aligned}$$

7. The linear approximation of $f(x)$ is

$$L(x) = f(a) + f'(a)(x-a)$$

Let $f(x) = x^{\frac{1}{3}}$ and set $a = 8$. Then

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}},\ f(8) = 8^{\frac{1}{3}} = 2,\ f'(8) = \frac{1}{3}\cdot 8^{-\frac{2}{3}} = \frac{1}{12}$$

$$8.3^{\frac{1}{3}} = f(8.3) \approx L(8.3) = 2 + \frac{1}{12}(8.3-8) = 2 + \frac{0.3}{12} = 2.025$$

8.

$$f'(c) = \frac{f(2)-f(0)}{2-0} = \frac{4^3\cdot 2^2 - 2^3\cdot 2^0}{2} = \frac{256-8}{2} = \frac{248}{8} = 124$$