Extra Credit Homework

This assignment is both an option to receive extra homework points and an opportunity to learn about some interesting and important mathematical ideas that you can already understand. There are 3 problems on this assignment. Submit one of the three problems for up to 10 points of extra credit. Practice writing neatly and organized (rewrite the solutions after solving the problems if you have to). Of the 10 points for a problem, 1 point may be taken off for poor organization, and another for the quality of your writing.

1. In most of Chapter 11, we cared more about whether a given infinite series converged or diverged. In most cases, this is all we are able to say about a series anyways. However, recall that we did initially learn about a couple situations where we could actually compute the exact value of the infinite sum. In particular, one example of this was a telescoping sum wehere we were able to write the partial sum s_n of the series in a simplified formula because many of the terms cancelled. For example, in order to compute

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

we used partial fractions and wrote $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$, so that

$$s_n = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}.$$

So by definition, the value of the infinite sum is

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1.$$

The other type of infinite sum for which we could actually compute the value were geometric series. For example, we can compute

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1/2}{1-1/2} = 1.$$

An interesting feature of this particular geometric series is that we can *also* write it as a telescoping series since

$$\frac{1}{2^n} = \frac{1}{2^{n-1}} - \frac{1}{2^n}$$

(check this equality is true). This means

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2^{n-1}} - \frac{1}{2^n}\right)$$

so that we can simplify the n^{th} partial sum as

$$s_n = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{n-1}} - \frac{1}{2^n}\right) = 1 - \frac{1}{2^n}.$$

Then, again by the definition,

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{2^n}\right) = 1.$$

(a) Show $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges using the ratio test.

(b) It may seem that is all we can say about this series. In fact, we can actually compute the value of this sum exactly by rewriting it as a telescoping sum as well. Find this value. (Hint: Let $p(n) = An + B$ be a polynomial of degree 1, and suppose

$$\frac{n}{2^n} = \frac{p(n)}{2^{n-1}} - \frac{p(n+1)}{2^n}.$$

Use this to solve for the constants A and B. Next, simplify the partial sum s_n as above. Finally, use this to compute the infinite sum as the $\lim_{n\to\infty} s_n$.)

(c) Now compute the value of the sum

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n}.$$

(Hint: Suppose

$$\frac{n^2}{2^n} = \frac{p(n)}{2^{n-1}} - \frac{p(n+1)}{2^n},$$

where now $p(n) = An^2 + Bn + C$ is a general polynomial of degree 2. Proceed as in the previous problem.)

(d) Generalize what you did in parts (b) in (c) to compute the value of $\sum_{n=1}^{\infty} \frac{n^3}{2^n}$.

(e) You can continue in this manner, computing the value of $\sum_{n=1}^{\infty} \frac{n^k}{2^n}$ for all positive integers k. If we let a_k be the value of this sum, we get an infinite sequence $\{a_0, a_1, a_2, \ldots\}$. So far you have obtained the values of a_0, a_1, a_2, and a_3. The next two terms in the sequence are $a_4 = 150$ and $a_5 = 1084$. Go to oeis.org (the Online Encyclopedia of Integer Sequences) and search for the first 6 terms of the sequence: $a_0, a_1, \ldots, a_5$. There is exactly one matching sequence if you do everything correctly. Give the other way way of describing this sequence in bold at the top, as well as the next 4 terms a_6, a_7, a_8, a_9.

2. This problem will require a bit of reading from the textbook. Throughout chapter 11, we learned about different series tests, but we could not use the tests to evaluate the exact value of the sum. We skipped over some material for these tests that do allow us to obtain approximations to the infinite series by taking a sufficiently large finite sum. In this problem, you will learn about these remainder estimates that allow you to do this. You will also learn how to *prove* a function is equal to its Taylor Series.

   (a) Read through the section "Estimating the Sum of a Series" in pages 700-702 on the Integral Test. Make sure you understand the examples on these pages. Then do Exercise 32 on page 704.

(b) Read through the section "Estimating Sums" in pages 712-713 on the Alternating Series Test. Then do Exercise 30 on page 714.

(c) Read through Example 8 on page 732. Then do Exercise 27 on page 733.

(d) Read about the Taylor Remainder Theorem in Section 11.10. This starts halfway down page 736 and goes to the top of page 739, and proves that e^x is equal to its Maclaurin Series. The top of page 740 contains another example, this time proving $\sin x$ equals its Maclaurin Series. Follow these examples to do Exercise 21 on page 746 (you will have to do Exercise 7 as well).

3. In this problem, you will learn a little bit about fractal geometry. You will first learn about the Koch snowflake, and then about some other interesting fractals, as well as fractal dimension.

   (a) Do problem 5 in the Problems Plus section for Chapter 11, on page 761.

   (b) Do an internet search for "fractal dimension". Find the fractal dimension of the Koch snowflake (or Koch curve). You will probably want to use the Wikipedia page which gives a list of fractals by Hausdorff dimension. Use this page to find the fractal dimension of the Sierpinski triangle, the Sierpinski carpet, and the boundary of the Mandelbrot set. A fractal is a shape, or curve, which is self-similar. That is, as you zoom in on a given region, it looks "similar" to regions you have already seen. The Hausdorff dimension measures the amount a fractal looks like 1-D or 2-D space.