Review Problems - Exam 1

1. $\int \frac{4x+3}{x^4+6x^2}\,dx = \int \frac{4x+3}{x^2(x^2+6)}\,dx$

*Use partial fractions

$\frac{4x+3}{x^2(x^2+6)} = \frac{Ax+B}{x^2} + \frac{Cx+D}{x^2+6}$

$4x+3 = (Ax+B)(x^2+6) + (Cx+D)x^2$

system of equations:
$$0 = Ax^3 + Cx^3 \Rightarrow A = -C$$
$$0 = Bx^2 + Dx^2 \Rightarrow B = -D$$
$$4x = 6Ax \Rightarrow A = \tfrac{2}{3} \Rightarrow C = -\tfrac{2}{3}$$
$$3 = 6B \Rightarrow B = \tfrac{1}{2} \Rightarrow D = -\tfrac{1}{2}$$

So $\int \frac{4x+3}{x^2(x^2+6)}\,dx = \int \frac{\frac{2}{3}x+\frac{1}{2}}{x^2} + \frac{-\frac{2}{3}x-\frac{1}{2}}{x^2+6}\,dx$

$$= \frac{2}{3}\int \frac{1}{x}\,dx + \frac{1}{2}\int \frac{1}{x^2}\,dx - \frac{2}{3}\int \frac{x}{x^2+6}\,dx - \frac{1}{2}\int \frac{1}{x^2+6}\,dx$$

$$= \frac{2}{3}\ln|x| - \frac{1}{2}\frac{1}{x} - \frac{1}{3}\ln|x^2+6| - \frac{1}{2}\frac{1}{\sqrt{6}}\arctan\left(\frac{x}{\sqrt{6}}\right) + C$$

2. $\int e^{\sqrt{x}}\,dx$

$*$w-sub $\quad w = \sqrt{x} \Rightarrow w^2 = x$
$\qquad\qquad\qquad\qquad 2w\,dw = dx$

$= \int e^{w} \cdot 2w\,dw = 2\int w e^{w}\,dw$

$*$int. by parts

$u = w \qquad dv = e^{w}dw$

$du = dw \qquad v = e^{w}$

$= 2\left[w \cdot e^{w} - \int e^{w}dw\right]$

$= 2\left[we^{w} - e^{w}\right] + C$

$= 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$

3. $\int \cos^4 x \sin^3 x\,dx$

$*$leave a $\sin x$, replace the rest with $\sin^2 x = 1 - \cos^2 x$

$= \int \cos^4 x \sin^2 x \cdot \sin x\,dx$

$= \int \cos^4 x\,(1 - \cos^2 x)\,\sin x\,dx$

let $u = \cos x$, $du = -\sin x\,dx$

$= -\int u^4 - u^6\,du$

$= -\dfrac{u^5}{5} + \dfrac{u^7}{7} + C$

$= -\dfrac{\cos^5 x}{5} + \dfrac{\cos^7 x}{7} + C$

4. $\displaystyle\int_0^\infty \frac{x\,dx}{(5+2x^2)^2}$ * improper because of infinite interval $[0,\infty)$

$$= \lim_{t\to\infty} \int_0^t \frac{x\,dx}{(5+2x^2)^2}$$

* u-sub $u = 5+2x^2$ $\quad x=t \Rightarrow u = 5+2t^2$

$du = 4x\,dx \quad x=0 \Rightarrow u=5$

$$= \lim_{t\to\infty} \int_5^{5+2t^2} \frac{1}{u^2}\,\frac{1}{4}\,du$$

$$= \lim_{t\to\infty} \frac{1}{4} \cdot -\frac{1}{u} \Bigg|_5^{5+2t^2}$$

$$= \frac{1}{4}\left[\underbrace{\lim_{t\to\infty} -\frac{1}{5+2t^2}}_{\to 0} + \frac{1}{5}\right]$$

$$= \frac{1}{4}\cdot\frac{1}{5} = \frac{1}{20}$$

5. $\int_0^{\pi} \sin(e^x)\,dx$, TRAP Rule with $n=6$

$$\approx \frac{\Delta x}{2}\left[f(x_0)+2f(x_1)+\cdots+2f(x_5)+f(x_6)\right]$$

$$\Delta x = \frac{\pi-0}{6} = \frac{\pi}{6}$$

$x_0 = 0$
$x_1 = \frac{\pi}{6}$
$x_2 = \frac{\pi}{3}$
$x_3 = \frac{\pi}{2}$
$x_4 = \frac{2\pi}{3}$
$x_5 = \frac{5\pi}{6}$
$x_6 = \pi$

$$\approx \frac{\pi}{12}\left[\sin(e^0)+2\sin(e^{\pi/6})+2\sin(e^{\pi/3})+2\sin(e^{\pi/2}) + 2\sin(e^{2\pi/3})+2\sin(e^{5\pi/6})+\sin(e^{\pi})\right]$$

$$\approx \frac{\pi}{12}\left[.8415+1.9863+.5756+-1.9904 + 1.9294+1.8188+-.9126\right]$$

$$\approx \frac{\pi}{12}\left[4.2485\right]$$

$$\approx 1.1123$$

6. $\displaystyle\int_{1/2}^{1} \frac{\sqrt{4x^2-1}}{x}\,dx$

* Trig sub

$2x = \sec\theta$

$2dy = \sec\theta\tan\theta\, d\theta$

change endpts $x=1 \Rightarrow 2 = \sec\theta \Rightarrow \theta = \frac{\pi}{3}$

$x = \frac{1}{2} \Rightarrow 1 = \sec\theta \Rightarrow \theta = 0$

$$= \int_0^{\pi/3} \frac{\sqrt{\sec^2\theta - 1}}{\cancel{\tfrac{1}{2}\sec\theta}}\, \cancel{\tfrac{1}{2}\sec\theta}\tan\theta\, d\theta$$

$$= \int_0^{\pi/3} \sqrt{\tan^2\theta}\, \tan\theta\, d\theta$$

$$= \int_0^{\pi/3} \tan^2\theta\, d\theta$$

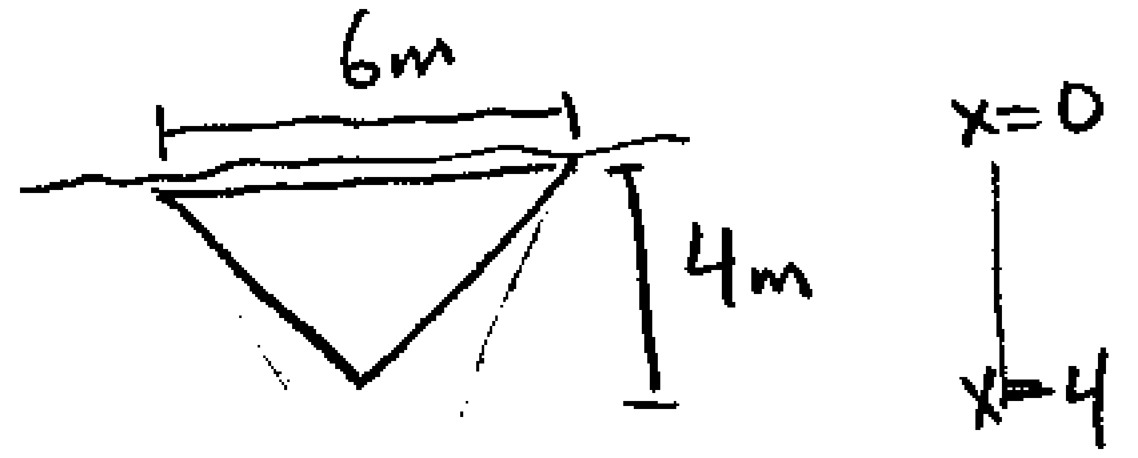
$$= \int_0^{\pi/3} \sec^2\theta - 1\, d\theta$$

$$= \tan\theta - \theta \Big|_0^{\pi/3}$$

$$= \sqrt{3} - \frac{\pi}{3} - 0 + 0$$

$$= \sqrt{3} - \frac{\pi}{3}$$

7.



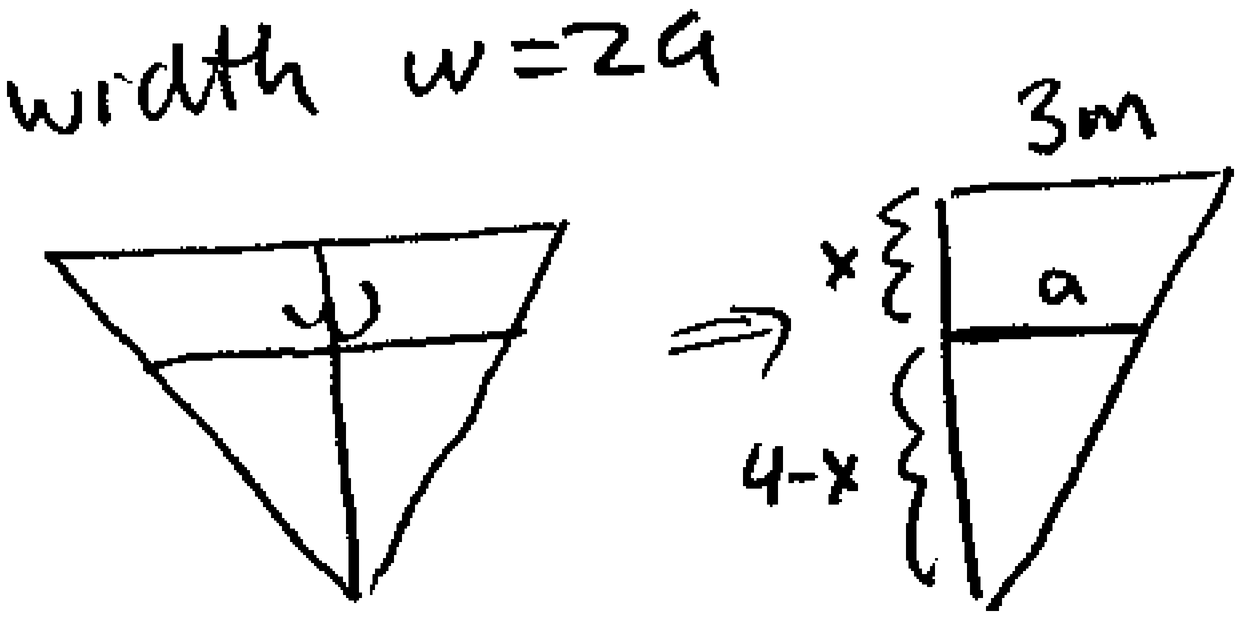
* put $x=0$ at water level

limits of integration will be $x=0$ to $x=4$

$$F=\int_0^4 \delta\, d\, w\, dx$$

depth $d=x$ because $x=0$ at water level

width $w=2a$



similar triangles

$$\frac{4-x}{4}=\frac{a}{3}$$

$$a=3\left(1-\frac{x}{4}\right)$$

$$w=6\left(1-\frac{x}{4}\right)$$

$$F=\int_0^4 \delta x\cdot 6\left(1-\frac{x}{4}\right)dx$$

$$=6\delta\int_0^4 x-\frac{x^2}{4}\,dx$$

$$=6\delta\left[\frac{x^2}{2}-\frac{x^3}{12}\right]_0^4$$

$$=6\delta\left[\frac{16}{2}-\frac{64}{12}-0\right]$$ $*\delta=9800$

$$=6\delta\left[8-\frac{16}{3}\right]=156800\text{ N}$$

8. Arc length of $y = \frac{x^4}{16} + \frac{1}{2x^2}$ on $1 \le x \le 2$

*use x as variable so

$$L = \int_a^b \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx$$

$$= \int_1^2 \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx$$

$$\frac{dy}{dx} = \frac{4x^3}{16} - \frac{2}{2x^3} = \frac{x^3}{4} - \frac{1}{x^3}$$

$$\left[\frac{dy}{dx}\right]^2 = \left(\frac{x^3}{4} - \frac{1}{x^3}\right)^2 = \frac{x^6}{16} + 2\,\frac{\cancel{x^3}}{\cancel{4}_2} \cdot \frac{-1}{\cancel{x^3}} + \frac{1}{x^6}$$

$$= \frac{x^6}{16} - \frac{1}{2} + \frac{1}{x^2}$$

$$\left[\frac{dy}{dx}\right]^2 + 1 = \frac{x^6}{16} + \frac{1}{2} + \frac{1}{x^6} = \left(\frac{x^3}{4} + \frac{1}{x^3}\right)^2$$

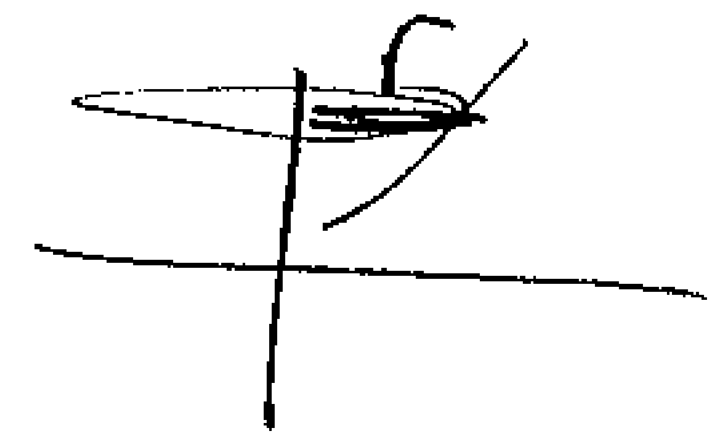
so $$L = \int_1^2 \sqrt{\left(\frac{x^3}{4} + \frac{1}{x^3}\right)^2}\, dx$$

$$= \int_1^2 \frac{x^3}{4} + \frac{1}{x^3}\, dx$$

$$= \frac{x^4}{16} - \frac{1}{2x^2} \Big|_1^2 = \frac{16}{16} - \frac{1}{8} - \frac{1}{16} + \frac{1}{2}$$

$$= \frac{21}{16}$$

9. Surface Area of last problem around y-axis



$A = \int 2\pi r \, ds$

* x is variable

$ds = \sqrt{\left(\frac{x^3}{4} + \frac{1}{x^3}\right)^2} \, dx$

$r = x$

$$A = \int_1^2 2\pi x \sqrt{\left(\frac{x^3}{4} + \frac{1}{x^3}\right)^2} \, dx$$

$$= \int_1^2 2\pi x \left(\frac{x^3}{4} + \frac{1}{x^3}\right) dx$$

$$= \int_1^2 2\pi \left(\frac{x^4}{4} + \frac{1}{x^2}\right) dx$$

$$= 2\pi \left[\frac{x^5}{20} - \frac{1}{x}\right]_1^2$$

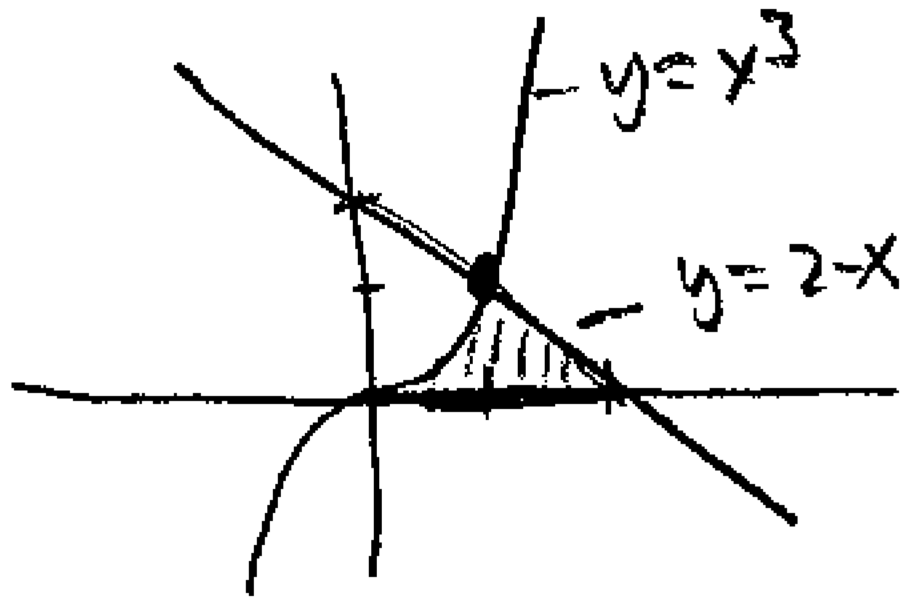
$$= 2\pi \left[\frac{2^5}{20} - \frac{1}{2} - \frac{1}{20} + 1\right]$$

$$= 2\pi \left[\frac{41}{20}\right] = \frac{41\pi}{10}$$

10. Find centroid of region bounded by $y=x^3$

$x+y=2 \Rightarrow y=2-x$

$y=0$



intersections

$x^3 = 2-x$

$x^3+x-2=0$

$(x-1)(x^2+x+2)=0$

$\Rightarrow x=1$

$$M_y = \int_a^b \rho x f(x)dx = \int_0^1 \rho x \cdot x^3 dx + \int_1^2 \rho x \cdot (2-x)\, dx$$

$$= \rho \frac{x^5}{5}\Big|_0^1 + \rho\left[x^2 - \frac{x^3}{3}\right]_1^2$$

$$= \frac{\rho}{5} + \rho\left[4 - \frac{8}{3} - 1 + \frac{1}{3}\right] = \frac{13\rho}{15}$$

$$M_x = \int_a^b \frac{1}{2}\rho [f(x)]^2 dx = \int_0^1 \frac{\rho}{2} x^6 dx + \int_1^2 \frac{\rho}{2}(2-x)^2 dx$$

$$= \frac{\rho}{2}\frac{x^7}{7}\Big|_0^1 + \frac{\rho}{2}\cdot\frac{-1}{3}(2-x)^3\Big|_1^2$$

$$= \frac{\rho}{14} - \cancel{\frac{\rho}{6}(0)} + \frac{\rho}{6} = \frac{5\rho}{21}$$

$$\text{mass} = \rho\int_a^b f(x)dx = \rho\int_0^1 x^3 dx\ \rho\int_1^2 2-x\, dx$$

$$= \rho\left[\frac{x^4}{4}\Big|_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2\right]$$

$$= \rho\left[\frac{1}{4} + 4 - 2 - 2 + \frac{1}{2}\right] = \frac{3\rho}{4}$$

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{\text{mass}}, \frac{M_x}{\text{mass}}\right) = \left(\frac{\frac{13\rho}{15}}{\frac{3\rho}{4}}, \frac{\frac{5\rho}{21}}{\frac{3\rho}{4}}\right) = \left(\frac{52}{45}, \frac{20}{63}\right)$$