Answers to Strategies for Integration Worksheet

5. $\int_0^2 \frac{2t}{(t-3)^2}\, dt$

Strategy 1. Integration by parts

Let $u = 2t$, $dv = (t-3)^{-2}dt$. Then $du = 2dt$ and $v = -(t-3)^{-1}$, and

$$\begin{aligned}\int_0^2 \frac{2t}{(t-3)^2}\, dt &= 2t \cdot -\frac{1}{t-3}\Big|_0^2 - \int_0^2 \frac{-1}{t-3} \cdot 2\, dt \\ &= -\frac{2t}{t-3}\Big|_0^2 + 2\ln|t-3|\Big|_0^2 \\ &= \frac{4}{1} + 0 + 2\ln|-1| - 2\ln|-3| \\ &= 4 - 2\ln(3)\end{aligned}$$

Strategy 2. Partial Fractions

$$\begin{aligned}\frac{2t}{(t-3)^2} &= \frac{A}{t-3} + \frac{B}{(t-3)^2} \\ 2t &= A(t-3) + B \\ &\Rightarrow A = 2 \text{ and } B = 6 \\ \int_0^2 \frac{2t}{(t-3)^2}\, dt &= \int_0^2 \frac{2}{t-3} + \frac{6}{(t-3)^2}\, dt \\ &= \left[2\ln|t-3| - \frac{6}{t-3}\right]_0^2 \\ &= 2\ln|-1| - 2\ln|-3| - \frac{6}{-1} + \frac{6}{-3} \\ &= 4 - 2\ln(3)\end{aligned}$$

Strategy 3. U-Substitution. Let $u = t-3$. Then $t = u+3$, $du = dt$, and when $t = 0 \Rightarrow u = -3$, when $t = 2 \Rightarrow u = -1$

$$\begin{aligned}\int_0^2 \frac{2t}{(t-3)^2}\, dt &= \int_{-3}^{-1} \frac{2(u+3)}{u^2}\, du \\ &= 2\int_{-3}^{-1} \frac{1}{u} + \frac{3}{u^2}\, du \\ &= 2\left[\ln|u| - \frac{3}{u}\right]_{-3}^{-1} \\ &= 2\left[\ln|-1| - \ln|-3| - frac3{-1} + \frac{3}{-3}\right] \\ &= -2\ln(3) + 4\end{aligned}$$

6. $\displaystyle\int \frac{x}{\sqrt{3-x^4}}\,dx$

Strategy 1. U-sub, then trig sub. Let $u = x^2$, $du = 2x\,dx$.

$$\int \frac{x}{\sqrt{3-x^4}}\,dx = \int \frac{1}{\sqrt{3-u^2}} \cdot \frac{1}{2}\,du$$

Let $u = \sqrt{3}\sin\theta$, $du = \sqrt{3}\cos\theta\,d\theta$

$$\begin{aligned}
\int \frac{1}{\sqrt{3-u^2}} \cdot \frac{1}{2}\,du &= \frac{1}{2}\int \frac{1}{\sqrt{3-3\sin^2\theta}} \cdot \sqrt{3}\cos\theta\,d\theta \\
&= \frac{\sqrt{3}}{2}\int \frac{\cos\theta}{\sqrt{3}\,\sqrt{1-\sin^2\theta}}\,d\theta \\
&= \frac{1}{2}\int \frac{\cos\theta}{\sqrt{\cos^2\theta}}\,d\theta \\
&= \frac{1}{2}\int d\theta \\
&= \frac{1}{2}\theta + c \\
&= \frac{1}{2}\arcsin\left(\frac{u}{\sqrt{3}}\right) + c \\
&= \frac{1}{2}\arcsin\left(\frac{x^2}{\sqrt{3}}\right) + c
\end{aligned}$$

Strategy 2. Trig sub $x^2 = \sqrt{3}\sin\theta$, $2x\,dx = \sqrt{3}\cos\theta\,d\theta$.

$$\begin{aligned}
\int \frac{x}{\sqrt{3-x^4}} &= \int \frac{1}{\sqrt{3-3\sin^2\theta}} \cdot \frac{\sqrt{3}}{2}\cos\theta\,d\theta \\
&= \frac{1}{2}\int \frac{\cos\theta}{\sqrt{\cos^2\theta}}\,d\theta \\
&= \frac{1}{2}\int d\theta \\
&= \frac{1}{2}\theta + c \\
&= \frac{1}{2}\arcsin\left(\frac{u}{\sqrt{3}}\right) + c \\
&= \frac{1}{2}\arcsin\left(\frac{x^2}{\sqrt{3}}\right) + c
\end{aligned}$$

13. $\int \sin^3\theta\,\cos^5\theta\,d\theta$

Strategy 1. Leave 1 $\sin x$, replace the others using $\sin^2 x = 1-\cos^2 x$.

$$\int \sin^3\theta\,\cos^5\theta\,d\theta = \int (1-\cos^2\theta)\cos^5\theta\cdot\sin\theta\,d\theta$$
$$= \int (\cos^5\theta - \cos^7\theta)\cdot\sin\theta\,d\theta$$

Let $u=\cos\theta,\ du = -\sin\theta\,d\theta$

$$= -\int u^5 - u^7\,du$$
$$= -\frac{u^6}{6} + \frac{u^8}{8} + c$$
$$= -\frac{\cos^6\theta}{6} + \frac{\cos^8\theta}{8} + c$$

Strategy 2. Leave 1 $\cos x$, replace the others using $\cos^2 x = 1-\sin^2 x$.

$$\int \sin^3\theta\,\cos^5\theta\,d\theta = \int \sin^3\theta(1-\sin^2\theta)^2\cdot\cos\theta\,d\theta$$
$$= \int \sin^3\theta(1-2\sin^2\theta+\sin^4\theta)\cdot\cos\theta\,d\theta$$

Let $u=\sin\theta,\ du=\cos\theta\,d\theta$

$$= \int u^3(1-2u^2+u^4)du$$
$$= \int u^3 - 2u^5 + u^7\,du$$
$$= \frac{u^4}{4} - \frac{u^6}{3} + \frac{u^8}{8} + c$$
$$= \frac{\sin^4\theta}{4} - \frac{\sin^6\theta}{3} + \frac{\sin^8\theta}{8} + c$$

27. $\displaystyle\int \frac{dx}{1+e^x}$

Strategy 1. U-substitution. Let $u = e^x$, then $du = e^x\,dx \Rightarrow \frac{du}{u} = dx$

$$\int \frac{dx}{1+e^x} = \int \frac{1}{1+u} \cdot \frac{1}{u}\,du$$

Then partial fractions:

$$\begin{aligned}
\frac{1}{u(u+1)} &= \frac{A}{u} + \frac{B}{u+1} \\
1 &= A(u+1) + Bu \\
\Rightarrow A &= 1 \text{ and } B = -1 \\
\int \frac{1}{u(u+1)}\,du &= \int \frac{1}{u} - \frac{1}{u+1}\,du \\
&= \ln|u| - \ln|u+1| + c \\
&= \ln|e^x| - \ln|e^x+1| + c
\end{aligned}$$

Strategy 2. A different u-substitution. Let $u = 1 + e^x$, then $du = e^x\,dx \Rightarrow \frac{du}{u-1} = dx$

$$\int \frac{dx}{1+e^x} = \int \frac{1}{u} \cdot \frac{1}{u-1}\,du$$

Then partial fractions:

$$\begin{aligned}
\frac{1}{u(u-1)} &= \frac{A}{u} + \frac{B}{u-1} \\
1 &= A(u-1) + Bu \\
\Rightarrow A &= -1 \text{ and } B = 1 \\
\int \frac{1}{u(u+1)}\,du &= \int \frac{-1}{u} + \frac{1}{u-1}\,du \\
&= -\ln|u| + \ln|u+1| + c \\
&= -\ln|e^x+1| + \ln|e^x| + c
\end{aligned}$$

29. $\int_0^5 \frac{3w-1}{w+2}\, dw$

Strategy. Long division or 'creative' factoring to get

$$\begin{aligned}\int_0^5 \frac{3w-1}{w+2}\, dw &= \int_0^5 3 - \frac{7}{w+2}\, dw \\ &= \left[3w - 7\ln|w+2|\right]_0^5 \\ &= 15 - 0 - 7\ln|7| + 7\ln|2| = 15 + 7\ln\left|\frac{2}{7}\right|\end{aligned}$$

** could use u-sub ($u = w + 2$) as well, but the problem ends up being very similar.

45. $\int x^5 e^{-x^3}\,dx$

Strategy. W-substitution, then integration by parts. Let $w = -x^3$, $dw = -3x^2\,dx$.

$$\int x^5 e^{-x^3}\,dx = \int -we^w \cdot \frac{-1}{3}\,dw$$

Choose $u = w$ and $dv = e^w\,dw$. Then $du = dw$ and $v = e^w$.

$$\begin{aligned}
\frac{1}{3}\int we^w\,dw &= \frac{1}{3}\left[w \cdot e^w - \int e^w dw\right] \\
&= \frac{1}{3}\left[w\,e^w - e^w\right] + c \\
&= \frac{1}{3}\left[-x^3\,e^{-x^3} - e^{-x^3}\right] + c
\end{aligned}$$

63. $\displaystyle\int \frac{\sin 2x}{1+\cos^4 x}\,dx$

Strategy 1. Rewrite $\sin 2x$ using the double angle formula, then u-sub $u = \cos x$.

$$\int \frac{\sin 2x}{1+\cos^4 x}\,dx = \int \frac{2\sin x \cos x}{1+\cos^4 x}\,dx$$
$$= \int \frac{-2u}{1+u^4}\,du$$

Let $w = u^2,\ dw = 2u\,du$

$$= \int -\frac{1}{1+w^2}\,dw$$
$$= -\arctan(w) + c$$
$$= -\arctan(u^2) + c$$
$$= -\arctan(\cos^2 x) + c$$

Strategy 2. Rewrite $cos^2 x$ using a half angle identity, then u-sub $u = \frac{1}{2}(1+\cos 2x)$, $du = -\sin 2x dx$.

$$\int \frac{\sin 2x}{1+\cos^4 x}\,dx = \int \frac{\sin 2x}{1+\frac{1}{4}(1+\cos 2x)^2}\,dx$$
$$= \int \frac{-1}{1+u^2}\,du$$
$$= -\arctan(u) + c$$
$$= -\arctan\left(\frac{1}{2}(1+\cos 2x)\right) + c$$
$$= -\arctan(\cos^2 x) + c$$