Review problems for Exam 2

1. Solve $y' = \frac{3x^2}{y}$

$$\frac{dy}{dx} = \frac{3x^2}{y}$$

$$y\,dy = 3x^2 dx$$

$$\int y\,dy = \int 3x^2 dx$$

$$\frac{y^2}{2} = x^3 + c$$

$$\boxed{y^2 = 2x^3 + 2c}$$

2. Solve the IVP $y' = 3x^2 y$, $y(2) = e^9$

$$\frac{dy}{dx} = 3x^2 y$$

$$\frac{dy}{y} = 3x^2 dx$$

$$\ln|y| = x^3 + c$$

$$y = e^{x^3+c}$$

gen. solution: $y = Ce^{x^3}$

I.V.P

$y(2) = e^9 \Rightarrow e^9 = Ce^{2^3} = Ce^8$
$= e^{8+c}$
$\Rightarrow c = 1$

$$\boxed{y(x) = e^{x^3+1}}$$

3. Solve $xy' - y = x\ln x$

$$\frac{dy}{dx} - \frac{1}{x}y = \ln x$$

$$I(x) = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = x^{-1}$$

$$x^{-1}\frac{dy}{dx} - x^{-2}y = x^{-1}\ln x$$

$$(x^{-1}y)' = x^{-1}\ln x$$

$$\int (x^{-1}y)'\,dx = \int x^{-1}\ln x\,dx$$

$u = \ln x$
$du = x^{-1}dx$

$$x^{-1}y = (\ln x)^2 + C$$

$$\boxed{y = x(\ln x)^2 + cx}$$

4. Solve $y' = \frac{-y}{x} + \sin x \Rightarrow \frac{dy}{dx} + \frac{1}{x}y = \sin x$

$$I(x) = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

$$x\frac{dy}{dx} + y = x\sin x$$

$$(xy)' = x\sin x$$

$$\int (xy)'\,dx = \int x\sin x\,dx$$

Integration by parts
$u = x \qquad dv = \sin x\,dx$
$du = dx \qquad v = -\cos x$

$$xy = -x\cos x + \int \cos x\,dx$$

$$xy = -x\cos x + \sin x + c$$

$$\boxed{y = -\cos x + \frac{1}{x}\sin x + \frac{c}{x}}$$

5. $(x^3-1)^3 y' = x^2 y, \quad y(0)=2$

$$\frac{dy}{y} = \frac{x^2}{(x^3-1)^3}\,dx$$

u-sub
$u = x^3-1$
$du = 3x^2\,dx$

$$\int \frac{dy}{y} = \int \frac{1}{3}\,\frac{1}{u^3}\,du$$

$$\ln|y| = \frac{1}{3}\cdot\frac{u^{-2}}{-2} + C$$

$$\ln|y| = -\frac{1}{6}(x^3-1)^{-2} + C$$

$$y = Ce^{-1/6(x^3-1)^{-2}}$$

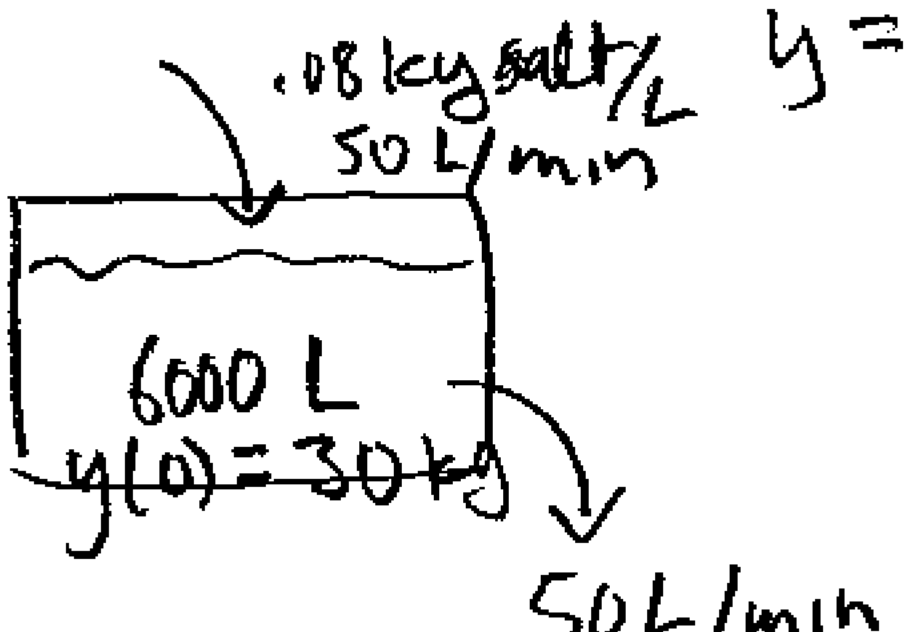
$$y(0)=2 \Rightarrow 2 = Ce^{-1/6(-1)^{-2}}$$

$$2 = Ce^{-1/6}$$

$$C = 2e^{1/6}$$

$$\boxed{y = 2e^{1/6 - 1/6(x^3-1)^{-2}}}$$

6.



$y = $ kg of salt

$$\frac{dy}{dt} = \text{rate in} - \text{rate out}$$

$$= .08\,\frac{kg}{L}\cdot\frac{50L}{min} - \frac{y\,kg}{6000L}\cdot\frac{50L}{min}$$

$$\frac{dy}{dt} = 4 - \frac{y}{120} = \frac{480-y}{120}$$

$$\frac{dy}{480-y} = \frac{dt}{120}$$

$$\frac{-dy}{480-y} = \frac{-dt}{120}$$

$$\frac{dy}{y-480} = \frac{-dt}{120}$$

$$\int \frac{dy}{y-480} = \int -\frac{dt}{120}$$

6. continued

$$\ln|y-480| = \frac{-t}{120} + C$$

$$y-480 = Ce^{-t/120}$$

$$y = Ce^{-t/120} + 480$$

IVP: $y(0)=30 \Rightarrow 30 = Ce^{0} + 480$

$$C = -450$$

$$\boxed{y = 480 - 450e^{-t/120}}$$

7. $x = 3t^2,\ y = 2t^3$

a. $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{6t^2}{6t} = t$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}} = \frac{\frac{d}{dt}[t]}{6t} = \frac{1}{6t}$$

b. $L = \int_0^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$$= \int_0^2 \sqrt{(6t)^2 + (6t^2)^2}\, dt$$

$$= \int_0^2 \sqrt{36t^2 + 36t^4}\, dt$$

$$= \int_0^2 \sqrt{36t^2}\sqrt{1+t^2}\, dt$$

$$= \int_0^2 6t\sqrt{1+t^2}\,dt \qquad \left[\begin{array}{l} u = t^2+1 \\ du = 2t\,dt \end{array}\right. \begin{cases} t=2 \Rightarrow u=5 \\ t=0 \Rightarrow u=1 \end{cases}$$

$$= \int_1^5 3\sqrt{u}\,du$$

$$= 3\cdot\frac{2}{3}u^{3/2}\Big|_1^5 = 2\left[(5)^{3/2}-1\right]$$

7c. $$SA = \int_0^2 2\pi\, x(t)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_0^2 2\pi(3t^2)\cdot 6t\sqrt{1+t^2}\,dt \qquad \left[\begin{array}{l} u = t^2+1 \\ du = 2t\,dt \\ t^2 = u-1 \end{array}\right. \begin{cases} t=2 \Rightarrow u=5 \\ t=0 \Rightarrow u=1 \end{cases}$$

$$= \int_1^5 18\pi(u-1)\sqrt{u}\,du$$

$$= 18\pi\int_1^5 u^{3/2}-u^{1/2}\,du$$

$$= 18\pi\left[\frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}\right]_1^5$$

$$= 18\pi\left[\frac{2}{5}(5)^{5/2}-\frac{2}{3}(5)^{3/2}-\frac{2}{5}+\frac{2}{3}\right]$$

8. rectangular $(-5\sqrt{3}, 5)$ to polar

$r^2 = (-5\sqrt{3})^2 + 5^2$

$r^2 = 75 + 25 = 100$

$r = \pm 10$

$$\boxed{\begin{array}{l} \left(10, \frac{5\pi}{6}\right), \left(10, \frac{-7\pi}{6}\right) \\ \left(-10, \frac{11\pi}{6}\right), \left(10, -\frac{\pi}{6}\right) \end{array}}$$

$\tan(\theta) = \frac{5}{-5\sqrt{3}} = -\frac{1}{\sqrt{3}}$

$\Rightarrow \theta = \frac{5\pi}{6}, \frac{-7\pi}{6}, \ldots$

* in quad II

For negative r,

$\theta = \frac{5\pi}{6} + \pi = \frac{11\pi}{6}$

$\theta = \frac{5\pi}{6} - \pi = -\frac{\pi}{6}$

9. outside $r=1$ & inside $r = 2\cos(4\theta)$

$r = 2\cos 4\theta$ is a rose with 8 pedals and max $r = \pm 2$

$r = \pm 2 \Rightarrow \cos(4\theta) = \pm 1$

$4\theta = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi, 7\pi, 8\pi$

$\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, 2\pi \rightarrow$ correspond to tips of pedals

$r = 0 \Rightarrow \cos(4\theta) = 0$

$4\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots$

$\theta = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \ldots$

$r = 1$ is a circle of radius 1 centered at the origin

Intersection pts: $\cos(4\theta) = \pm\frac{1}{2}$

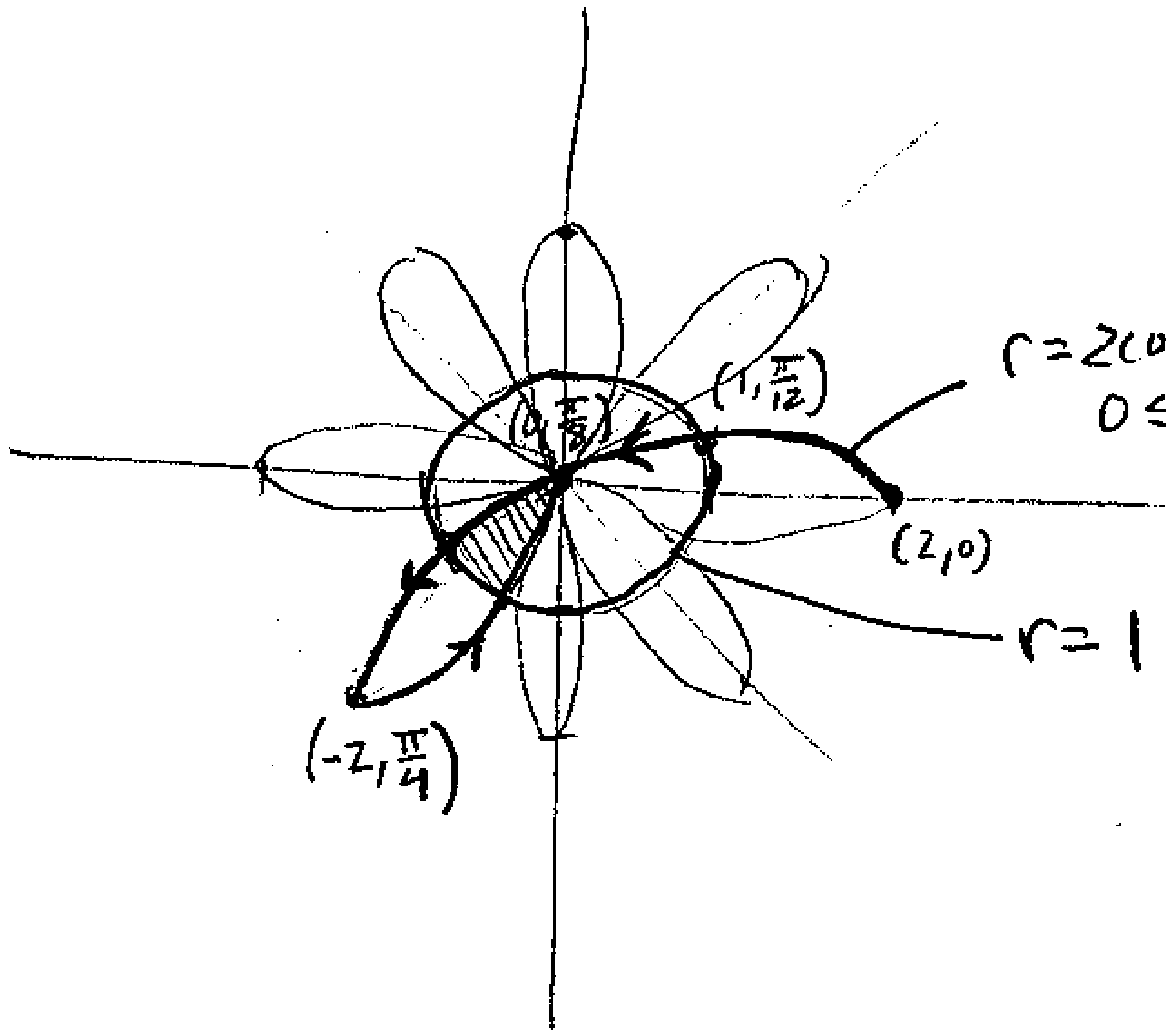
$4\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \ldots$

$\theta = \frac{\pi}{12}, \frac{\pi}{6}, \frac{\pi}{3}, \frac{5\pi}{12}, \ldots$

9. cont.

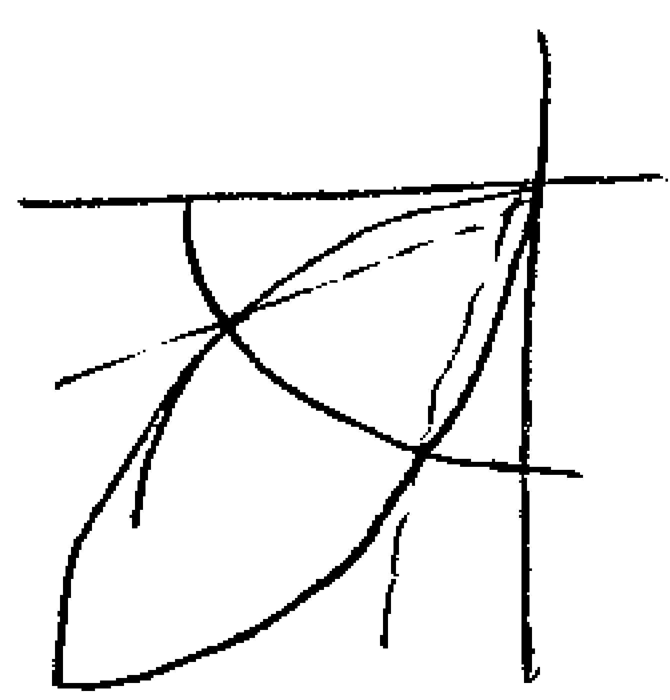
points to plot

θ	r
0	2
$\frac{\pi}{12}$	1
$\frac{\pi}{8}$	0
$\frac{\pi}{6}$	-1
$\frac{\pi}{4}$	-2
$\frac{\pi}{3}$	-1
$\frac{3\pi}{8}$	0



By symmetry, I know that finding the shaded area is $\frac{1}{8}$th of the area we need to find.

Close up of area:



See how from $r=0$ to $r=1$ the rose bounds the region then from intersection to intersec. the circle bounds Then from intersection to $r=0$ the rose bounds again

So Area =

$$8\left[\underbrace{\int_{\frac{\pi}{8}}^{\pi/6} \frac{1}{2}\left(2\cos(4\theta)\right)^2 d\theta}_{} + \int_{\pi/6}^{\pi/3} \frac{1}{2}(1)^2 d\theta + \underbrace{\int_{\pi/3}^{3\pi/8} \frac{1}{2}\left(2\cos(4\theta)\right)^2 d\theta}_{}\right]$$

The same by symmetry

9 cont.

$$A = 16\int_{\frac{\pi}{8}}^{\pi/6} \frac{1}{2}\cdot 4\cos^2(4\theta)\,d\theta + 8\int_{\frac{\pi}{6}}^{\pi/3} \frac{1}{2}\,d\theta$$

$$= 32\int_{\frac{\pi}{8}}^{\pi/6} \frac{1}{2}\left(1+\cos(8\theta)\right)d\theta + 4\theta\Big|_{\frac{\pi}{6}}^{\pi/3}$$

$$= 16\left[\theta + \frac{\sin(8\theta)}{8}\right]_{\frac{\pi}{8}}^{\pi/6} + 4\left[\frac{\pi}{3} - \frac{\pi}{6}\right]$$

$$= 16\left[\frac{\pi}{6} - \frac{\pi}{8} + \frac{\sin\left(\frac{4\pi}{3}\right)}{8} - \frac{\sin\pi}{8}\right] + 4\left[\frac{\pi}{6}\right]$$

$$= 16\left[\frac{\pi}{24} - \frac{\sqrt{3}}{16} - 0\right] + \frac{2\pi}{3}$$

$$= \frac{4\pi}{3} - \sqrt{3}$$

7. Alternative

$x = 2+3t, \quad y = \cosh(3t)$

a. $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{3\sinh(3t)}{3} = \sinh(3t)$

$\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}} = \dfrac{\frac{d}{dt}\left[\sinh(3t)\right]}{3} = \dfrac{3\cosh(3t)}{3} = \cosh(3t)$

b. $L = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$= \int_0^1 \sqrt{(3)^2 + (3\sinh(3t))^2}\, dt$

$= \int_0^1 \sqrt{9}\sqrt{1+\sinh^2(3t)}\, dt$

$= \int_0^1 3\sqrt{\cosh^2(3t)}\, dt$

$= \int_0^1 3\cosh(3t)\,dt = 3\left[\dfrac{\sinh(3t)}{3}\right]_0^1 = \sinh(3) - \sinh(0)$

c. $SA = \int_0^1 2\pi y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$= \int_0^1 2\pi\cosh(3t) \cdot 3\cosh(3t)\,dt$

$= 6\pi\int_0^1 \cosh^2(3t)\,dt$

just stop here

or use $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$