Answers to Review Problems (set 2)

1. $\left\{ \frac{\arctan(n^2)}{6^n+2} \right\}$

$$\lim_{n\to\infty} \frac{\arctan(n^2)}{6^n+2} = \lim_{x\to\infty} \frac{\arctan(x^2)}{6^x+2} = \frac{\pi/2}{\infty} = 0.$$

2. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{3/2}}$

Use Integral Test on $f(x) = \frac{1}{x(\ln x)^{3/2}}$

- Because dedominator $=0$ when $x=0$ & $x=1$, & we have $n \geq 2$, then $f(x)$ is continuous on $[2,\infty)$.
- Since $n \geq 2$, $x>0$ & $\ln x > 0$ for $x>1$, we have $f(x)$ is positive on $[2,\infty)$
- Since x & $\ln x$ are increasing functions, $x \ln x$ is an increasing function, so $f(x) = \frac{1}{x\ln x}$ is decreasing on $[2,\infty)$

or verify that $f'(x) = \dfrac{x(\ln x)^{3/2}\cdot 0 - \left[(\ln x)^{3/2} + \frac{3}{2}(\ln x)^{1/2}\right]}{x^2(\ln x)^3}$

$$f'(x) = \frac{-\left[(\ln x)^{3/2} + \frac{3}{2}(\ln x)^{1/2}\right]}{x^2(\ln x)^3} < 0$$

for x in $[2,\infty)$.

Now we can apply the test:

$$\int_2^{\infty} \frac{1}{x(\ln x)^{3/2}} = \lim_{t\to\infty} \int_2^{t} \frac{1}{x(\ln x)^{3/2}} = \lim_{t\to\infty} \left.\frac{-2}{(\ln x)^{1/2}}\right|_2^t$$

$$= \lim_{t\to\infty} \frac{-2}{(\ln t)^{1/2}} + \frac{2}{(\ln 2)^{1/2}} = \frac{2}{(\ln 2)^{1/2}} < \infty \Rightarrow \text{integral converges.}$$

$\Rightarrow$ Series converges.

3. $\sum_{n=1}^{\infty} \frac{\sin(n)}{(2n+1)(2n-1)}$

Since the sequence $\{\sin(n)\}$ has both positive and negative terms, but is not alternating, so try looking for absolute convergence.

The series of absolute values is $\sum_{n=1}^{\infty} \frac{|\sin(n)|}{4n^2-1}$

We can show this converges using a comparison Test.

$$\frac{|\sin(n)|}{4n^2-1} \leq \frac{1}{4n^2-1}$$

Also $\lim_{n\to\infty} \frac{4n^2-1}{4n^2} = 1$, and since $\sum \frac{1}{4n^2}$ converges (p-series with $p>1$), by the limit comparison test $\sum \frac{1}{4n^2-1}$ converges, thus, by direct comparison $\sum \frac{|\sin(n)|}{4n^2-1}$ also converges,

Therefore $\sum_{n=1}^{\infty} \frac{\sin(n)}{(2n+1)(2n-1)}$ is absolutely convergent & hence convergent.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}3^n}{2n^2+4}$

This is an Alternating series, but the Ratio Test will work better.

$$\lim_{n\to\infty} \left| \frac{(-1)^n 3^{n+1}}{2(n+1)^2+4} \cdot \frac{2n^2+4}{(-1)^{n-1}3^n} \right| = \lim_{n\to\infty} \left| \frac{3(2n^2+4)}{2(n+1)^2+4} \right| = 3$$

Since $3>1$, by the Ratio Test the series diverges.

6. $\displaystyle\sum_{k=1}^{\infty} (\cos(\pi k)) \frac{5k+3}{k^2+4}$

Since $\cos(\pi n) = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$

and $(-1)^n = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$

The series is the same as $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{5k+3}{k^2+4}$

This is an alternating series, so use that test.

- $b_{n+1} \le b_n$ (decreasing)

  Look at the related function $f(x) = \frac{5x+3}{x^2+4}$

  $$f'(x) = \frac{-(5x^2+6x-20)}{(x^2+4)^2} < 0$$

  when $(5x^2+6x-20) > 0$

  get $5x^2+6x-20=0$ when $x = .7448 \; -1.344$

  so $\quad + \quad -1.344 \quad - \quad .744 \quad +$ $\Rightarrow f'(x)<0$ when $x > .744$

$\Rightarrow$ sequence is decreasing.

$\bullet \lim_{k\to\infty} \frac{5k+3}{k^2+4} = 0$ ✓

Thus by the Alt. series test $\sum_{k=1}^{\infty} \cos(\pi k) \frac{5k+3}{k^2+4}$ converges.

7. $\sum_{n=1}^{\infty} \frac{10^n (x+2)^n}{(n+1)4^{2n+1}}$

Ratio Test:

$$\lim_{n\to\infty} \left| \frac{10^{n+1}(x+2)^{n+1}}{(n+2)4^{2n+3}} \cdot \frac{(n+1)4^{2n+1}}{10^n (x+2)^n} \right| = \lim_{n\to\infty} \frac{10|x+2|}{4^2} \frac{n+1}{n+2} = \frac{10}{16}|x+2|$$

$\Rightarrow$ converges if $\frac{5}{8}|x+2| < 1 \Rightarrow |x+2| < \frac{8}{5}$ $\quad R = \frac{8}{5}$

$\Rightarrow -\frac{8}{5} < x+2 < \frac{8}{5} \Rightarrow -\frac{18}{5} < x < -\frac{2}{5}$

Test endpoints

$x = -\frac{18}{5} \Rightarrow \sum_{n=1}^{\infty} \frac{10^n(-\frac{8}{5})^n}{(n+1)4^{2n+1}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{4(n+1)}$

which converges by Alt. Series Test

- $b_{n+1} = \frac{1}{4(n+2)} < \frac{1}{4(n+1)} = b_n$
- $\lim_{n\to\infty} \frac{1}{4(n+1)} = 0$

$x = -\frac{2}{5} \Rightarrow \sum_{n=1}^{\infty} \frac{10^n(-\frac{2}{5})^n}{(n+1)4^{2n+1}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)4^{n+1}}$

which converges by Alt. Series Test.

- $b_{n+1} = \frac{1}{(n+2)4^{n+2}} < \frac{1}{(n+1)4^{n+1}} = b_n$
- $\lim_{n\to\infty} \frac{1}{(n+1)4^{n+1}} = 0$

$\Rightarrow I = [-\frac{18}{5}, \frac{-2}{5}]$

8a. power series for $f(x) = \ln|2+x|$

First note $\ln|2+x| = \int \frac{1}{2+x}\,dx + c$

and $\frac{1}{2+x} = \frac{1}{2}\frac{1}{1+\frac{x}{2}} = \frac{1}{2}\cdot\frac{1}{1-(-\frac{x}{2})} = \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{2^n} = \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{2^{n+1}}$

for $|-\frac{x}{2}|<1 \Rightarrow |x|<2$ $(R=2)$

Then $\int \frac{1}{2+x}\,dx = \int \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{2^{n+1}}\,dx$

$\ln|2+x| = C + \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{(n+1)2^{n+1}}$

$x=0 \Rightarrow C = \ln|2|$

Thus $\ln|2+x| = \ln|2| + \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{(n+1)2^{n+1}}$ with $R=2$.

b. To find power series for $\int \frac{x}{1+x^3}\,dx$, need to first find power series for $\frac{x}{1+x^3}$

$\frac{x}{1+x^3} = x\cdot\frac{1}{1-(-x^3)} = x\sum_{n=0}^{\infty}(-1)^n x^{3n} = \sum_{n=0}^{\infty}(-1)^n x^{3n+1}$ for $|x|<1$

$\int \frac{x}{1+x^3}\,dx = \int \sum_{n=0}^{\infty}(-1)^n x^{3n+1}\,dx$

$\int \frac{x}{1+x^3}\,dx = C + \sum_{n=0}^{\infty}\frac{(-1)^n x^{3n+2}}{3n+2}$ for $|x|<1$.