Answers to Final Exam Review

1. a. $f(x) = \cos(-2x)$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$\left.\begin{aligned} f'(x) &= 2\sin(-2x) \\ f''(x) &= -4\cos(-2x) \\ f'''(x) &= -8\sin(-2x) \\ f^{(4)}(x) &= 16\cos(-2x) \\ f^{(5)}(x) &= 32\sin(-2x) \\ f^{(6)}(x) &= -64\cos(-2x) \end{aligned}\right\} \Rightarrow \begin{aligned} f(0) &= 1 \\ f'(0) &= 0 \\ f''(0) &= -4 \\ f'''(0) &= 0 \\ f^{(4)}(0) &= 16 \\ f^{(5)}(0) &= 0 \\ f^{(6)}(0) &= -64 \end{aligned}$$

$$\Rightarrow \cos(-2x) = 1 + 0x - \frac{4}{2!}x^2 + 0x^3 + \frac{16}{4!}x^4 + 0x^5 + \frac{-64}{6!}x^6 + \cdots$$

$$\cos(-2x) = \underset{n=0}{1} - \underset{n=1}{\frac{4}{2!}x^2} + \underset{n=2}{\frac{16}{4!}x^4} - \underset{n=3}{\frac{64}{6!}x^6} + \cdots$$

$$\boxed{\cos(-2x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!}}$$

b. $f(x) = \cos(-2x)$

Since $\cos(u) = \sum_{n=0}^{\infty} \frac{(-1)^n u^{2n}}{(2n)!}$, we can substitute $u = -2x$

$$\text{so } \cos(-2x) = \sum_{n=0}^{\infty} \frac{(-1)^n(-2x)^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \overbrace{(-1)^{2n}}^{=1 \text{ for all } n} 2^{2n} x^{2n}}{(2n)!}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!}}$$

* notice the 2 answers are the same!

2. $(5+x)^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{5^{n+1}}$ for $|x|<5$

Since $\frac{d}{dx}\left((5+x)^{-1}\right) = \frac{-1}{(5+x)^2} = -(5+x)^{-2}$,

we need to differentiate.

$$\frac{d}{dx}\left((5+x)^{-1}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{5^{n+1}}\right)$$

$$-(5+x)^{-2} = \sum_{n=1}^{\infty} \frac{(-1)^n n x^{n-1}}{5^{n+1}}$$

* Note series starts at $n=1$ now since $\frac{d}{dx}(n=0 \text{ term}) = 0$.

$$(5+x)^{-2} = -\sum_{n=1}^{\infty} \frac{(-1)^n n x^{n-1}}{5^{n+1}}$$

$$\boxed{(5+x)^{-2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n x^{n-1}}{5^{n+1}}} \quad (\text{for } |x|<5)$$

3. Unit vector parallel to $6\vec{i}-4\vec{j}+12\vec{k}$

Any scalar multiple of $6\vec{i}-4\vec{j}+12\vec{k}$ is parallel to it.

To make a unit vector, we can multiply by the scalar $\frac{1}{\text{magnitude}}$

$$|6\vec{i}-4\vec{j}+12\vec{k}| = \sqrt{6^2+(-4)^2+12^2} = \sqrt{196} = 14$$

$$\frac{1}{14}\left(6\vec{i}-4\vec{j}+12\vec{k}\right) = \boxed{\frac{3}{7}\vec{i} - \frac{2}{7}\vec{j} + \frac{6}{7}\vec{k}}$$

4. Angle between $5\vec{i}+4\vec{j}-8\vec{k}$ and $6\vec{i}-3\vec{j}+7\vec{k}$

$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$ where θ is the angle between $\vec{a}$ & $\vec{b}$

$$\cos\theta = \frac{(5\vec{i}+4\vec{j}-8\vec{k})\cdot(6\vec{i}-3\vec{j}+7\vec{k})}{|5\vec{i}+4\vec{j}-8\vec{k}|\,|6\vec{i}-3\vec{j}+7\vec{k}|}$$

$$\cos\theta = \frac{30-12-56}{\sqrt{25+16+64}\,\sqrt{36+9+49}} = \frac{-38}{\sqrt{105}\,\sqrt{94}}$$

$$\arccos\left(\frac{-38}{\sqrt{9870}}\right) = \theta$$

$$\boxed{\theta = 1.963 \text{ radians}}$$

5. $\vec{v} = 3\vec{i}-2\vec{j}+5\vec{k}$, $\vec{w} = 4\vec{i}+6\vec{j}+7\vec{k}$

$$\text{Proj}_{\vec{w}}\vec{v} = \frac{\vec{v}\cdot\vec{w}}{|\vec{w}|^2}\vec{w} = \frac{12-12+35}{(16+36+49)}(4\vec{i}+6\vec{j}+7\vec{k})$$

$$= \boxed{\frac{35}{101}(4\vec{i}+6\vec{j}+7\vec{k})}$$

$$\text{Proj}_{\vec{v}}\vec{w} = \frac{\vec{v}\cdot\vec{w}}{|\vec{v}|^2}\vec{v} = \frac{35}{(9+4+25)}(3\vec{i}-2\vec{j}+5\vec{k})$$

$$= \boxed{\frac{35}{38}(3\vec{i}-2\vec{j}+5\vec{k})}$$

$$\text{Comp}_{\vec{w}}\vec{v} = \frac{\vec{v}\cdot\vec{w}}{|\vec{w}|} = \boxed{\frac{35}{\sqrt{101}}}$$

$$|\vec{v}|\cos\theta = \frac{\vec{v}\cdot\vec{w}}{|\vec{w}|} = \text{comp}_{\vec{w}}\vec{v} = \boxed{\frac{35}{\sqrt{101}}}$$

6. equation of plane determined by 3 pts.

- need a pt on plane & the normal vector $\vec{n}$
- Find $\vec{n}$ by taking cross product of ~~2 pts on~~ 2 vectors in the plane

1st vector: displacement vector for $(3,0,-2)$ to $(11,-5,2)$

$$\langle 8,-5,4\rangle$$

2nd vector: displacement vector for $(3,0,-2)$ to $(3,7,4)$

$$\langle 0,7,6\rangle$$

$$\langle 8,-5,4\rangle \times \langle 0,7,6\rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 8 & -5 & 4 \\ 0 & 7 & 6 \end{vmatrix} = -58\vec{i} - 48\vec{j} + 56\vec{k}$$

$\langle -58,-48,56\rangle$ is parallel to $\langle -29,-24,28\rangle$

so we can use $\vec{n} = \langle -29,-24,28\rangle$

Then eqn of plane is $\vec{n}\cdot\langle x-3, y-0, z+2\rangle = 0$

$$\langle -29,-24,28\rangle \cdot \langle x-3, y, z+2\rangle = 0$$

$$-29x + 87 - 24y + 28z + 56 = 0$$

$$\boxed{-29x - 24y + 28z + 143 = 0}$$

* This is not a unique answer. If you used different displacement vectors or a different point your answer could look different.

7. $\vec{r}_1(t) = \langle 5t+3, -2t+7, 4t \rangle \Rightarrow$ direction $\langle 5,-2,4 \rangle$

$\vec{r}_2(s) = \langle 8, 6s+1, 3s+2 \rangle \Rightarrow$ direction $\langle 0,6,3 \rangle$

* not parallel

Check intersecting:

$5t+3 = 8 \Rightarrow 5t = 5 \Rightarrow t = 1$

$-2t+7 = 6s+1$

$4t = 3s+2 \Rightarrow 4(1) = 3s+2 \Rightarrow s = \frac{2}{3}$

check in 2nd eqn $-2(1)+7 \stackrel{?}{=} 6(\frac{2}{3})+1$

$5 \stackrel{?}{=} 5$ ✓

$\Rightarrow$ intersect when $t=1$ & $s = \frac{2}{3}$, so at the point $(8,5,4)$

8. $3x-2y+z=1 \Rightarrow$ normal vector $\langle 3,-2,1 \rangle$

$2x+y-3z=3 \Rightarrow$ normal vector $\langle 2,1,-3 \rangle$

* not parallel

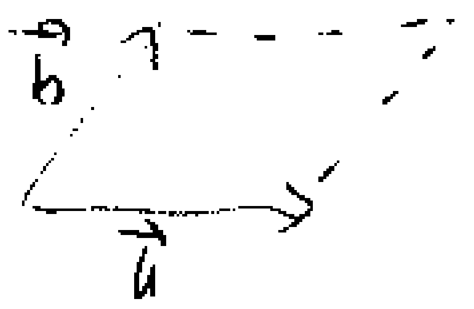
Need to find angle between normal vectors

$$\cos\theta = \frac{\langle 3,-2,1 \rangle \cdot \langle 2,1,-3 \rangle}{|\langle 3,-2,1 \rangle| \, |\langle 2,1,-3 \rangle|}$$

$$= \frac{6-2-3}{\sqrt{9+4+1}\,\sqrt{4+1+9}} = \frac{1}{13}$$

$\arccos\left(\frac{1}{13}\right) = \theta$

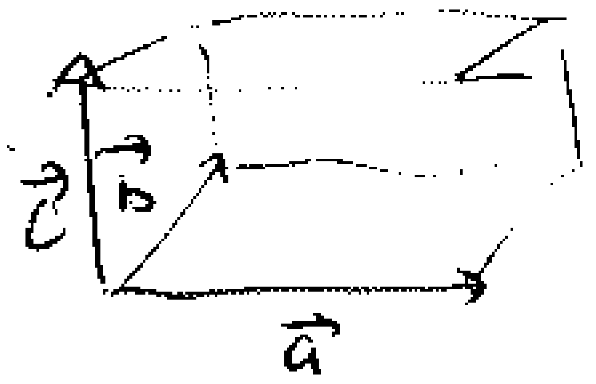
$\boxed{\theta = 1.494 \text{ radians}}$

9. a. 

Area is $|\vec{a}\times\vec{b}|$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 6 & 3 & -1 \\ 0 & 1 & 2 \end{vmatrix} = 7\vec{i} - 12\vec{j} + 6\vec{k} = \langle 7, -12, 6\rangle$$

$$|\vec{a}\times\vec{b}| = \sqrt{7^2 + (-12)^2 + 6^2} = \sqrt{49+144+36} = \boxed{\sqrt{229}}$$

b. 

Volume is $|\vec{c}\cdot(\vec{a}\times\vec{b})|$

$$|\langle 4, -2, 5\rangle \cdot \langle 7, -12, 6\rangle| = |28 + 24 + 30| = |82| = \boxed{82}$$

10. equation of plane. - need a pt & the normal vector.

- parallel to $2x+4y+8z=17$ $\Rightarrow$ parallel normal vectors
  $\Rightarrow$ we can use $\langle 2, 4, 8\rangle = \vec{n}$

- still need a pt. We can use any point on the line so let $t=0$, and get $(3, 0, 8)$

$$\vec{n}\cdot\langle x-3, y-0, z-8\rangle = 0$$

$$\langle 2,4,8\rangle \cdot \langle x-3, y-0, z-8\rangle = 0$$

$$2x - 6 + 4y + 8z - 64 = 0$$

$$\boxed{2x + 4y + 8z - 70 = 0}$$