Power Series Summary Answers

1. Power series for $f(x) = \dfrac{x^2}{x-4}$ centered $a=6$

Since we're look for a power series centered at $x=6$, we need to make sure that the powers of x in the numerator are in the form $(x-6)$ so

$$\frac{x^2}{x-4} = \frac{x^2-12x+36}{x-4} + \frac{12x-36}{x-4} = \frac{(x-6)^2}{x-4} + \frac{12x-72}{x-4} + \frac{36}{x-4}$$

$$= \frac{(x-6)^2}{x-4} + \frac{12(x-6)}{x-4} + \frac{36}{x-4}$$

Now for each term, we'll find the power series.

- $$\frac{(x-6)^2}{x-4} = \frac{(x-6)^2}{x-6+6-4} = \frac{(x-6)^2}{x-6+2} = \frac{(x-6)^2}{2}\left(\frac{1}{\frac{x-6}{2}+1}\right)$$

$$= \frac{(x-6)^2}{2}\left(\frac{1}{1-\left(\frac{-(x-6)}{2}\right)}\right)$$

$u = \dfrac{-(x-6)}{2}$, so sub into $\dfrac{1}{1-u} = \displaystyle\sum_{n=0}^{\infty} u^n$

$$\frac{(x-6)^2}{x-4} = \frac{(x-6)^2}{2}\sum_{n=0}^{\infty}\left(\frac{-(x-6)}{2}\right)^n = \frac{(x-6)^2}{2}\sum_{n=0}^{\infty}\frac{(-1)^n(x-6)^n}{2^n}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n(x-6)^{n+2}}{2^{n+1}}$$

- $$\frac{12(x-6)}{x-4} = \frac{12(x-6)}{2}\left(\frac{1}{1-\left(\frac{-(x-6)}{2}\right)}\right) = 6(x-6)\sum_{n=0}^{\infty}\frac{(-1)^n(x-6)^n}{2^n}$$

$$= 6\sum_{n=0}^{\infty}\frac{(-1)^n(x-6)^{n+1}}{2^n}$$

$\cdot \dfrac{36}{x-4} = \dfrac{36}{2}\left(\dfrac{1}{1-\left(\frac{-(x-6)}{2}\right)}\right) = 18\sum_{n=0}^{\infty}\dfrac{(-1)^n(x-6)^n}{2^n}$

Thus $f(x) = \dfrac{x^2}{x-6} = \sum_{n=0}^{\infty}\dfrac{(-1)^n(x-6)^{n+2}}{2^{n+1}} + 6\sum_{n=0}^{\infty}\dfrac{(-1)^n(x-6)^{n+1}}{2^n} + 18\sum_{n=0}^{\infty}\dfrac{(-1)^n(x-6)^n}{2^n}$

These series converge for $|u|<1$, so for $\left|-\frac{(x-6)}{2}\right|<1$

$\Rightarrow |x-6|<2 \Rightarrow -2<x-6<2 \Rightarrow 4<x<8$

$R=2$

2. $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{(n+1)^2 5^n}$

Ratio Test

$\lim_{n\to\infty}\left|\dfrac{x^{n+1}}{(n+2)^2 5^{n+1}}\cdot\dfrac{(n+1)^2 5^n}{x^n}\right| = \lim_{n\to\infty}\dfrac{|x|}{5}\cdot\dfrac{(n+1)^2}{(n+2)^2} = \dfrac{|x|}{5}$

Converges when $\frac{|x|}{5}<1 \Rightarrow |x|<5 \Rightarrow R=5$

So the (open) interval of convergence is $-5<x<5$

Test endpoints:

When $x=-5$, get $\displaystyle\sum_{n=0}^{\infty}\frac{(-5^n)}{(n+1)^2 5^n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2}$ which converges by the Alt. Series Test.

When $x=5$, get $\displaystyle\sum_{n=0}^{\infty}\frac{5^n}{(n+1)^2 5^n} = \sum_{n=0}^{\infty}\frac{1}{(n+1)^2}$ which converges by comparison to $\sum\frac{1}{n^2}$. $\Rightarrow I: [-5,5]$

3. $f(x) = \frac{1}{5+x^2} = \frac{1}{5}\left(\frac{1}{1-\left(-\frac{x^2}{5}\right)}\right)$

Let $u = \frac{-x^2}{5}$ & sub into $\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n$ for $|u|<1$

$$\frac{1}{5+x^2} = \frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{-x^2}{5}\right)^n = \frac{1}{5}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{5^n} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{5^{n+1}}$$

converges for $\left|\frac{-x^2}{5}\right|<1 \Rightarrow |x|^2<5 \Rightarrow |x|<\sqrt{5}$

$\Rightarrow R=\sqrt{5}$

4. Note that $\frac{d}{dx}\left(\frac{1}{5+x^2}\right) = \frac{-2x}{(5+x^2)^2}$

Thus $-\frac{1}{2x}\frac{d}{dx}\left(\frac{1}{5+x^2}\right) = \frac{1}{(5+x^2)^2}$

Since $\frac{1}{5+x^2} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{5^{n+1}} = \frac{1}{5} - \frac{x^2}{5^2} + \frac{x^4}{5^3} - \frac{x^6}{5^4} + \cdots$

$$\frac{d}{dx}\left(\frac{1}{5+x^2}\right) = \sum_{n=1}^{\infty}\frac{(-1)^n(2n)x^{2n-1}}{5^{n+1}} = \frac{-2x}{5^2} + \frac{4x^3}{5^3} - \frac{6x^5}{5^4} + \cdots$$

So $-\frac{1}{2x}\frac{d}{dx}\left(\frac{1}{5+x^2}\right) = \frac{-1}{2x}\sum_{n=1}^{\infty}\frac{(-1)^n(2n)x^{2n-1}}{5^{n+1}} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\, n x^{2n-2}}{5^{n+1}}$

So $\frac{1}{(5+x^2)^2} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1} n x^{2n-2}}{5^{n+1}}$

Since for problem 3 $R=\sqrt{5}$, it is the same here.

5. Since $\int \frac{2x}{5+x^2}\,dx = \ln|5+x^2| + C$, we need to find the power series for $\frac{2x}{5+x^2} = 2x\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{5^{n+1}}$

$$= 2\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{5^{n+1}}$$

Now we can integrate

$$\int \frac{2x}{5+x^2}\,dx = \int 2\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{5^{n+1}}\,dx = \int 2\left[\frac{x}{5} - \frac{x^3}{5^2} + \frac{x^5}{5^3} - \frac{x^7}{5^4} + \dots\right]dx$$

$$\ln|5+x^2| = C + 2\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{5^{n+1}(2n+2)} = C + 2\left[\frac{x^2}{10} - \frac{x^4}{4\cdot 5^2} + \frac{x^6}{6\cdot 5^3} - \frac{x^8}{8\cdot 5^4} + \dots\right]$$

So $\ln|5+x^2| = C + \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{5^{n+1}(n+1)}$

To find C, Let $x=0$, then

$$\ln|5+0| = C + \sum_{n=0}^{\infty}\frac{(-1)^n 0}{5^{n+1}(n+1)}$$

$$\Rightarrow C = \ln|5|$$

Thus $\ln|5+x^2| = \ln|5| + \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{5^{n+1}(n+1)}$

Since the radius of convergence in problem 3 is $R=\sqrt{5}$, it is the same here.

6. First we need to find the power series for $\frac{x}{1-x^8}$

$$\frac{x}{1-x^8} = x \cdot \frac{1}{1-x^8}$$

so let $u = x^8$ & sub into

$$\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n \quad \text{for } |u|<1$$

$$\frac{x}{1-x^8} = x\sum_{n=0}^{\infty} (x^8)^n = \sum_{n=0}^{\infty} x^{8n+1}$$

converges when $|x^8|<1$

$\Rightarrow |x|<1$

$\Rightarrow R=1$

Then $$\int \frac{x}{1-x^8}dx = \int \sum_{n=0}^{\infty} x^{8n+1} \, dx = \int x + x^9 + x^{17} + x^{25} + \cdots dx$$

$$\int \frac{x}{1-x^8}dx = C + \sum_{n=0}^{\infty} \frac{x^{8n+2}}{8n+2} = C + \frac{x^2}{2} + \frac{x^{10}}{10} + \frac{x^{18}}{18} + \frac{x^{26}}{26} + \cdots$$

Since we integrated to find this power series, the radius of convergence is the same as the original,

so $R=1$