Math 1272, Power Series Summary (11.8 and 11.9)

- A **power series centered at** $x = a$ has the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + ...$$

where x is a variable, a is a number and c_n is a coefficient.
Example. $1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} + \frac{x^4}{16} + ... = \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$ is a power series centered at $x = 0$.
$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$ is not a power series, it's just a normal series.

- We use the ratio or root test to determine **where a power series converges**.
Example. For what values of x does $\sum_{n=1}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$ converge? [This is a power series centered at $x = -2$.]

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)(x+2)^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{n(x+2)^n}\right| = \lim_{n\to\infty} \frac{|x+2|}{3}\frac{n+1}{n}$$
$$= \frac{|x+2|}{3} \lim_{n\to\infty} 1 + \frac{1}{n} = \frac{|x+2|}{3}$$

So the series converges when $\frac{|x+2|}{3} < 1$, so when $|x+2| < 3$. Then the radius of convergence is $R = 3$ and the (open) interval of convergence is $-3 < x + 2 < 3$, or rather, $-5 < x < 1$. Additionally, we need to check the endpoints of the the interval. So when $x = -5$, the series is

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{3}$$

which diverges by the test for divergence. When $x = 1$, the series is

$$\sum_{n=1}^{\infty} \frac{n}{3}$$

which also diverges by the test for divergence. Thus the interval of convergence is $-5 < x < 1$.

- **Power series represent functions** (where they converge).
Example. $f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$.

- **Substitution** is one technique we can use to find new power series from ones we already know.
Example. Power series at $a = 0$ for $\frac{3x^2}{2+x}$.

$$\frac{3x^2}{2+x} = \frac{3x^2}{2}\left(\frac{1}{1+x/2}\right) = \frac{3x^2}{2}\left(\frac{1}{1-(-x/2)}\right) = \frac{3x^2}{2}\sum_{n=0}^{\infty}\left(\frac{-x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{3(-1)^n}{2^{n+1}}x^{n+2}$$

So $\dfrac{3x^2}{2+x} = \displaystyle\sum_{n=0}^{\infty} \frac{3(-1)^n}{2^{n+1}} x^{n+2}$ for $|\frac{-x}{2}| < 1$, so for $|x| < 2$.
Example. Power series at $a = 3$ for $\frac{1}{4-x}$.

$$\frac{1}{4-x+3-3} = \frac{1}{(4-3)-(x-3)} = \frac{1}{1-(x-3)} = \sum_{n=0}^{\infty} (x-3)^n$$

for $|x-3| < 1$, i.e. $2 < x < 4$.

- **Term-by-Term differentiation** - keeps the same R radius of convergence.
  Example. Find the power series for $\dfrac{1}{(1-x)^2}$.
  Note that $\dfrac{d}{dx}\left(\dfrac{1}{1-x}\right) = \dfrac{1}{(1-x)^2}$
  Since $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + ... = \displaystyle\sum_{n=0}^{\infty} x^n$ for $|x| < 1$
  We can take the derivative of both sides to find the wanted power series:
  $\dfrac{1}{(1-x)^2} = 1 + 2x + 3x^2 + ... = \displaystyle\sum_{n=1}^{\infty} nx^{n-1}$ for $|x| < 1$.
- **Term-by-Term integration** - keeps the same R radius of convergence, don't forget to find C, the constant of integration.
  Example. Find the power series for $\ln|x+4|$.
  Given that $\dfrac{1}{4+x} = \dfrac{1}{4}\left(\dfrac{1}{1-(-x/4)}\right) = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{4^{n+1}}$ for $|x| < 4$, we can find

$$\ln|4+x| = \int \frac{1}{4+x} dx = \int \left(\frac{1}{4} - \frac{x}{4^2} + \frac{x^2}{4^3} - \frac{x^3}{4^4} + ...\right) dx$$

$$= C + \frac{x}{4} - \frac{x^2}{2 \cdot 4^2} + \frac{x^3}{3 \cdot 4^3} - \frac{x^4}{4 \cdot 4^4} + ... = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(n+1)4^{n+1}} = C + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n4^n}$$

When we plug in $x = 0$ in the above equation, we get $\ln 4 = C$, so

$$\ln|4+x| = \ln 4 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n4^n} \quad \text{for } |x| < 4$$

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Practice Problems:

1. Find a power series representation at $a = 6$ for $f(x) = \frac{x^2}{x-4}$. What is the radius of convergence?

2. Find the radius and interval of convergence for $\sum_{n=0}^{\infty} \frac{x^n}{(n+1)^2 5^n}$.

3. Find a power series representation for $f(x) = \frac{1}{5+x^2}$. What is the radius of convergence?

4. Using your answer to 3, find a power series representation for $g(x) = \frac{1}{(5+x^2)^2}$. What is the radius of convergence?

5. Using your answer to 3, find a power series representation for $h(x) = \ln(5+x^2)$. What is the radius of convergence. (Hint: To evaluate the integral, use your answer to 3, to find the power series for $\frac{x}{5+x^2}$, then find the power series for $h(x)$.)

6. Find a power series representation of $\int \frac{x}{1-x^8}\,dx$. What is the radius of convergence?