Strategy for Testing Series

1. Look at the form of the series. Is it a p-series or a geometric series? If the series is a p-series, it is convergent for $p > 1$ and divergent for $p \le 1$. If it is geometric, then if $|r| < 1$, the series is convergent and divergent if $|r| \ge 1$.
   Example. $\sum \frac{1}{n^{5/4}}$ converges since $p = \frac{5}{4} > 1$
   Example. $\sum_{n=1}^{\infty} 3 \left(\frac{10}{9}\right)^{n-1}$ diverges since $|r| = \frac{10}{9} > 1$.

2. Does the series look similar to a p-series or geometric series? Then use a comparison test. Remember that the Comparison Test works for series with only positive terms, so if the series $\sum a_n$ has some negative terms, consider using the comparison test to $\sum |a_n|$ and test for absolute convergence.
   Example. $\sum_{n=1}^{\infty} \frac{n \cos(n)}{(n+2)^3}$ converges since it is absolutely convergent when compared to $\sum \frac{1}{n^2}$.

3. Does the $\lim_{n\to\infty} a_n \ne 0$? Then apply the test for divergence.
   Example. $\sum_{n=0}^{\infty} \frac{4n^2 + 6n}{(n+1)^2}$ diverges since $\frac{4n^2+6n}{(n+1)^2} \to 4 \ne 0$

4. Is the series of the from $\sum (-1)^{n-1} b_n$ or $\sum (-1)^n b_n$? Then the Alternating Series Test might apply.
   Example. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{n^2+1}$ converges since i. $\frac{n}{n^2+1} < \frac{n+1}{(n+1)^2+1}$ (How can we show this? There are 2 ways.) and ii. $\frac{n}{n^2+1} \to 0$ as $n \to \infty$.

5. Does the series involve factorials, products or constants to the nth power? If so the Ratio Test is helpful. Keep in mind that the Ratio Test is not helpful for algebraic functions of n.
   Example. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

6. If the series is of the form $\sum (b_n)^n$, then the Root Test is useful.
   Example. $\sum_{n=1}^{\infty} \frac{(3n^2+1)^n}{(5n)^{2n}}$

7. If $a_n = f(n)$ and it looks like you could integrate $f(n)$, then the Integral Test is useful, given that the hypotheses hold.
   Example. $\sum_{n=1}^{\infty} n e^{n^2}$

8. Try again! Don't give up if your first try doesn't work!

9. Do a lot of problems! Experience with different kinds of series is the best strategy for knowing what test to use.

Practice Problems (taken from section 11.7):

Test each series for convergence or divergence:

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n+1)^n}{n^{2n}}$

9. $\displaystyle\sum_{k=1}^{\infty} k^2 e^{-k}$

15. $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n+2)}$

18. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}-1}$

20. $\displaystyle\sum_{k=1}^{\infty} \frac{k+5}{5^k}$

31. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^2-1}}{n^3+2n^2+5}$