Answers to Strategy for Test Series Problems

2. $\sum_{n=1}^{\infty} \frac{(2n+1)^n}{n^{2n}}$

Since both numerator and denominator are raised to the nth power, the Root Test is useful.

$$\lim_{n\to\infty} \left|\frac{(2n+1)^n}{n^{2n}}\right|^{1/n} = \lim_{n\to\infty} \left(\frac{|2n+1|^n}{|n^2|^n}\right)^{1/n} = \lim_{n\to\infty} \frac{|2n+1|}{|n^2|}$$

$$= \lim_{n\to\infty} \frac{2n+1}{n^2} = 0$$

Since $0<1$, by the Root test, the series converges.

9. $\sum_{k=1}^{\infty} k^2 e^{-k}$

Since this series has rational function of k times a constant (e) raised to a power of k, this is a good canidate for the Ratio test

$$a_k = \frac{k^2}{e^k} \quad \text{and} \quad a_{k+1} = \frac{(k+1)^2}{e^{k+1}}$$

$$\lim_{k\to\infty} \left|\frac{(k+1)^2}{e^{k+1}} \cdot \frac{e^k}{k^2}\right| = \lim_{k\to\infty} \frac{1}{e} \frac{(k+1)^2}{k^2} = \frac{1}{e}$$

Since $\frac{1}{e}<1$, by the Ratio Test the series converges.

15. $\sum_{n=0}^{\infty} \frac{n!}{2\cdot5\cdot8\cdots(3n+2)}$

The factorial means we should use the Ratio test

$a_n = \frac{n!}{2\cdot5\cdot8\cdots(3n+2)}$ and $a_{n+1} = \frac{(n+1)!}{2\cdot5\cdot8\cdots(3n+2)(3n+5)}$

$$\lim_{n\to\infty}\left|\frac{(n+1)!}{2\cdot5\cdot8\cdots(3n+2)(3n+5)}\cdot\frac{2\cdot5\cdot8\cdots(3n+2)}{n!}\right| = \lim_{n\to\infty}\left|\frac{n+1}{3n+5}\right| = \frac{1}{3}$$

Since $\frac{1}{3}<1$, by the Ratio Test this series converges.

18. $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}-1}$

This is an alternating series, so try the Alt. Series Test.

1. $b_{n+1} \le b_n$ $\qquad b_n = \frac{1}{\sqrt{n}-1}$ & $b_{n+1} = \frac{1}{\sqrt{n+1}-1}$

since $n+1 > n$

$\sqrt{n+1} > \sqrt{n}$

$\sqrt{n+1}-1 > \sqrt{n}-1$

so $\frac{1}{\sqrt{n+1}-1} < \frac{1}{\sqrt{n}-1}$ ✓

2. $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{\sqrt{n}-1} = \lim_{n\to\infty}\frac{\overbrace{1/\sqrt{n}}^{\to 0}}{1-\underbrace{1/\sqrt{n}}_{\to 0}} = \frac{0}{1} = 0$ ✓

Therefore the series converges.

* Note: This series is conditionally convergent since $\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}-1}$ diverges. (How could you show this??)

20. $\sum_{k=1}^{\infty} \frac{k+5}{5^k}$

Ratio Test again:

$a_k = \frac{k+5}{5^k}, \quad a_{k+1} = \frac{k+6}{5^{k+1}}$

$$\lim_{k\to\infty} \left| \frac{k+6}{5^{k+1}} \cdot \frac{k+5}{5^k} \right| = \frac{1}{5}$$

Since $\frac{1}{5} < 1$, by the Ratio Test the series converges.

31. $\sum_{n=1}^{\infty} \frac{\sqrt{n^2-1}}{n^3+2n^2+5}$

Since this series has algebraic expressions of n, so a comparison test usually works, & since these are more complicated expressions, use the limit comparison test.

$$\frac{\sqrt{n^2-1}}{n^3+2n^2+5} \sim \frac{\sqrt{n^2}}{n^3} = \frac{1}{n^2}$$

$$\lim_{n\to\infty} \left| \frac{\sqrt{n^2-1}}{n^3+2n^2+5} \cdot \frac{n^2}{1} \right| = \lim_{n\to\infty} \frac{\sqrt{n^6-n^4}}{n^3+2n^2+5}$$

$$= \lim_{n\to\infty} \frac{\sqrt{1-1/n^2}}{1+2/n+5/n^3} = 1$$

Since $0 < 1 < \infty$, and since $\sum \frac{1}{n^2}$ converges, by the limit comp. test, the series converges.