RESEARCH STATEMENT

JOHN HALL

The representation theory of the symmetric group plays a large role in many areas of combinatorics, for example in the study of symmetric functions. The full transformation semigroup is perhaps the most natural generalization of the symmetric group, and yet its representation theory is not well understood.

The most immediate objective of my current research is to describe the structure of the full transformation semigroup algebra. This task is greatly complicated by the algebra not being semisimple. One plan of attack is to introduce parameters in such a way that the algebra becomes generically semisimple. The deformed algebra can then be analyzed, perhaps giving insight into the original problem. In my research I have come up with several interesting deformations. One makes the algebra "as semisimple as possible", while another leads to an eigenvalue result involving Schur functions.

The Full Transformation Semigroup

The *Full Transformation Semigroup on n letters*, denoted T_n, is the semigroup of all set maps $w : [n] \to [n]$, where the multiplication is the usual composition. In what follows, $\pi = \{\pi_1, \dots, \pi_k\}$ will always denote a set partition of $[n]$ with blocks ordered by increasing smallest element, $P = \{p_1, \dots, p_k\}$ a subset of $[n]$ with $p_1 < \dots < p_k$, and σ an element of the symmetric group S_k. Let $w_{\pi,P,\sigma} \in T_n$ denote the map taking $x \in \pi_i$ to $p_{\sigma(i)}$.

The subgroup consisting of the invertible elements of T_n, i.e., the bijective maps, is precisely the symmetric group S_n. So the elements of T_n can be thought of as generalized permutations, and we can ask which of the many combinatorial aspects of the symmetric group can be extended in a meaningful way to the full transformation semigroup.

Let $\mathbb{C}T_n$ denote the full transformation semigroup algebra. $\mathbb{C}T_n$ has a chain of two-sided ideals

$$\mathbb{C}T_n = I_n \supseteq I_{n-1} \supseteq \dots \supseteq I_1 \supseteq I_0 = 0,$$

where for $1 \leq k \leq n$, I_k as a vector space is the complex span of the maps of rank less than or equal to k (the *rank* of a map is the cardinality of its image). For $1 \leq k \leq n$ define the algebras $A_{n,k} = I_k / I_{k-1}$. We can think of $A_{n,k}$ as being the algebra spanned by the maps of rank k, where two maps multiply to zero if their composition has rank less than k. The top quotient $A_{n,n}$ is isomorphic to $\mathbb{C}S_n$, the group algebra of the symmetric group, and is therefore semisimple. However the remaining algebras $A_{n,k}$ are not semisimple.

Problem 1. *Give an explicit combinatorial construction of the irreducible modules for $A_{n,k}$.*

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Problem 2. *Find a basis for the radical* $\sqrt{A_{n,k}}$.

These two problems motivated my early research on T_n, and remain long-term goals. It is known that the irreducible modules for $A_{n,k}$ are indexed by partitions $\lambda \vdash k$. In fact Hewitt and Zuckerman give a calculation in [9] that generates all irreducible matrix representations for $A_{n,k}$. However, their methods are difficult to apply in practice and do not even determine the dimensions of the representations. These dimensions are now also known, thanks to a more recent character result of Putcha [13]. Regardless of the approach, it is clear that the non-semisimplicity of $A_{n,k}$ causes great difficulties. This has led me to define several deformations of the algebra.

The first deformation. Let $w_1 = w_{\pi,P,\phi}$ and $w_2 = w_{\rho,R,\psi}$ be two maps of rank k. Notice that in order for the product $w_1 w_2$ to be nonzero in $A_{n,k}$ each element of $R = \{r_1, \dots, r_k\}$ must lie in a different block of π. In this situation we can associate to the maps w_1 and w_2 the permutation $\tau \in S_k$ defined by the condition $r_i \in \pi_{\tau(i)}$.

Now define a new associative multiplication in $A_{n,k}$ by

$$w_1 * w_2 := x^{\mathrm{inv}(\tau)} w_1 w_2,$$

where $\mathrm{inv}(\tau)$ is the number of inversions of τ. Let $A_{n,k}(x)$ denote the algebra with the multiplication $*$. As we shall see, there is a sense in which $A_{n,k}(x)$ is "as semisimple as possible" for generic x.

Let $\mathfrak{A}$ be an associative algebra, and Π an $n \times m$ matrix over $\mathfrak{A}$. The *Munn matrix algebra* $\mathcal{A} = \mathcal{M}(\mathfrak{A}; m, n; \Pi)$ is the set of all $m \times n$ matrices with entries in $\mathfrak{A}$, where multiplication is defined by $X \cdot Y := X \Pi Y$. Π is called a *sandwich matrix.*

$A_{n,k}$ is isomorphic to the Munn matrix algebra $\mathcal{M}(\mathbb{C}S_k; \binom{n}{k}, S(n,k); \Pi)$, where Π is the $S(n,k) \times \binom{n}{k}$ sandwich matrix

$$(\Pi)_{\pi,P} = \begin{cases} \tau & \text{if } p_i \in \pi_{\tau(i)}, 1 \le i \le k, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Note that the non-zero entries of Π are precisely the permutations τ that arise in the $*$ multiplication. Hence the first deformation preserves the Munn matrix algebra structure.

Proposition 1. *$A_{n,k}(x)$ is isomorphic to the Munn matrix algebra $\mathcal{M}(\mathbb{C}S_k; \binom{n}{k}, S(n,k); \Pi(x))$, where $\Pi(x)$ is the $S(n,k) \times \binom{n}{k}$ sandwich matrix defined by*

$$(\Pi(x))_{\pi,P} = \begin{cases} x^{\mathrm{inv}(\tau)}\tau & \text{if } p_i \in \pi_{\tau(i)}, 1 \le i \le k, \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

It is natural to ask in what way the semisimplicity of $\mathcal{A}$ depends on the sandwich matrix Π. Clifford and Preston [4] have shown that a Munn matrix algebra of the form $\mathcal{A} = \mathcal{M}(\mathbb{C}G; m, n; \Pi)$ is semisimple if and only if Π is non-singular, in the sense that Π has an inverse in the ring of $n \times m$ matrices over $\mathbb{C}G$. Note that in particular for semisimplicity we need $m = n$. However the sandwich matrix for $A_{n,k}(x)$ is $S(n,k) \times \binom{n}{k}$. What can we do?

Define the *rank* of Π to be the size of the largest non-singular minor of Π. (So, in particular, $\mathrm{rank}(\Pi) \le \min(m,n)$.) A result of McAlister [11] states that if $\mathcal{A} = \mathcal{M}(\mathfrak{A}; m, n; \Pi)$ is a Munn matrix algebra, and Π is *suitable* of rank r, then $\mathcal{A}/\sqrt{\mathcal{A}} \cong \left(\mathfrak{A}/\sqrt{\mathfrak{A}}\right)_r$, the algebra of all $r \times r$ matrices with entries in $\mathfrak{A}/\sqrt{\mathfrak{A}}$.

The additional condition that Π be suitable is a bit stronger than having an invertible submatrix of size r, but it is trivially satisfied if $r = \min(m, n)$. To show that the rank of $\Pi(x)$ is $\binom{n}{k}$ we define a special set of partitions of $[n]$.

Definition 1. *A set partition* π *is* cyclically contiguous *if the blocks of* π *are intervals, with the possible exception of the first block, which may be of the form* $\{1, 2, \ldots, i\} \cup \{j, j+1, \ldots, n\}$*, i.e., the union of an initial segment and a terminal segment. For example,* $\pi = 1256|3|4$ *is cyclically contiguous.*

There is an obvious bijection between cyclically contiguous partitions of $[n]$ into k blocks and k-subsets of $[n]$. Let $\Pi^c(x)$ be the $\binom{n}{k} \times \binom{n}{k}$ submatrix of $\Pi(x)$ consisting of the rows corresponding to maps w such that $\pi(w)$ is cyclically contiguous. I have shown that $\Pi^c(x)$ is nonsingular for generic x. Thus $\Pi(x)$ is suitable of rank $\binom{n}{k}$, and we have

Theorem 1. *For generic* x, $A_{n,k}(x)/\sqrt{A_{n,k}(x)} \cong (\mathbb{C}S_k)_{\binom{n}{k}}$.

Note that the rank of Π cannot be any larger than its width $\binom{n}{k}$. So no deformation of $A_{n,k}$ that preserves the Munn matrix algebra structure can be any more semisimple than $A_{n,k}(x)$.

Let $A^c_{n,k}(x)$ be the subalgebra of $A_{n,k}(x)$ spanned by the maps whose partitions are cyclically contiguous. $A^c_{n,k}(x)$ is also a Munn matrix algebra, with sandwich matrix $\Pi^c(x)$.

Grood [7] has generalized the classical Specht-module construction for the symmetric group (see [10]) to describe the irreducible modules of the *rook monoid* R_n, another semigroup containing the symmetric group. As it turns out, $\mathbb{C}R_n$ has a similar tower of ideals

$$\mathbb{C}R_n = J_n \supseteq J_{n-1} \supseteq \ldots \supseteq J_1 \supseteq J_0 = 0,$$

and defining $B_{n,k} = J_k/J_{k-1}$ we have $B_{n,k} \cong (\mathbb{C}S_k)_{\binom{n}{k}} \cong A^c_{n,k}(x)$. I have extended my first deformation so that it interpolates between $A^c_{n,k}(x)$ and $B_{n,k}$.

Problem 3. *Determine the irreducible modules and irreducible characters for* $A^c_{n,k}(x)$.

Problem 4. *How do the characters of* $A^c_{n,k}(x)$ *depend on* x*, and what happens when* $x = 1$*?*

The second deformation. There is another associative multiplication we can define on $A_{n,k}$. The symmetric group S_k acts on the maps of rank k by

$$\sigma w_{\pi,P,\phi} := w_{\pi,P,\sigma\phi}.$$

Now define

$$w_1 \circ w_2 := \sum_{\sigma \in S_k} p_{\rho(\sigma)} \sigma w_1 w_2,$$

where $\rho(\sigma) \vdash k$ is the cycle type of σ, and $p_{\rho(\sigma)}$ is the corresponding power-sum symmetric function in the variables $x_1, \ldots, x_k$. Let $A_{n,k}(\mathbf{x})$ denote the algebra with the multiplication $\circ$. Explicit calculations for small values of n and k suggest that $A_{n,k}(\mathbf{x})$ is no more semisimple than $A_{n,k}$, i.e., that even for generic values of the x_i we have $\dim\sqrt{A_{n,k}(\mathbf{x})} = \dim\sqrt{A_{n,k}}$. However something interesting does come out of this multiplication.

The following fact gives a useful characterization of the radical of an algebra.

Fact 1. *Let $\mathfrak{A}$ be a finite-dimensional associative algebra and $\{v_1, \ldots, v_N\}$ a basis of $\mathfrak{A}$. Define the $N \times N$ Gram matrix M for $\mathfrak{A}$ by $(M)_{i,j} = tr(v_i v_j)$. Then the nullspace of M is $\sqrt{\mathfrak{A}}$.*

If we define $M_{n,k}(\mathbf{x})$ to be the Gram matrix for $A_{n,k}(\mathbf{x})$ we have

Proposition 2.

$$(M_{n,k}(\mathbf{x}))_{i,j} = \begin{cases} S(n,k)k! \sum_{\lambda \vdash k} \frac{k!}{f^\lambda} \chi^\lambda(\mu) s_\lambda^2 & w_i w_j \text{ induces a permutation of cycle} \\ & \text{type } \mu \text{ on the image of } w_i \\ 0 & \text{otherwise.} \end{cases}$$

Here for λ a partition of k, χ^λ is the corresponding irreducible character of S_k, $f^\lambda = \chi^\lambda(1)$ its dimension, and s_λ the associated Schur function.

In the sequel we will always normalize the Gram matrix $M_{n,k}(\mathbf{x})$ by dividing by the constant $S(n,k)k!$.

In the semisimple case $k = n$ one can use Frobenius' factorization of the group determinant (see [2]) to derive the following result about the normalized matrix $M_{n,n}(\mathbf{x})$.

Theorem 2. *The eigenvalues of $M_{n,n}(\mathbf{x})$ are $\pm\left(\frac{n!}{f^\lambda} s_\lambda\right)^2$, $\lambda \vdash n$, where the positive values appear with multiplicity $\binom{f^\lambda+1}{2}$ and the negative values appear with multiplicity $\binom{f^\lambda}{2}$.*

Corollary 1. *The algebra $A_{n,n}(\mathbf{x})$ is semisimple if and only if the values of the parameters x_i avoid the zeros of the Schur functions s_λ.*

An analogous result for $k < n$ is so far only conjectural.

Conjecture 1. *The non-zero eigenvalues of $M_{n,k}(\mathbf{x})$, $k < n$, are of the form cs_λ^2 for $\lambda \vdash k$, where c is an integer or $\pm$ the square-root of an integer. The "multiplicity" of s_λ^2, i.e., the sum of the multiplicities of the cs_λ^2, is $(\binom{n}{k} f^\lambda)^2$ for $\lambda \vdash k$, $\lambda \neq 1^k$, and $\binom{n-1}{k-1}^2$ for $\lambda = 1^k$.*

The conjectured multiplicities are the dimensions of the irreducible modules for $A_{n,k}$.

Problem 5. *Compute the eigenvalues of $M_{n,k}(\mathbf{x})$.*

The radical. One of our original questions about the non-deformed algebras $A_{n,k}$ is whether we can provide some sort of combinatorial description for the radical $\sqrt{A_{n,k}}$. A basis might be too much to hope for, but even a spanning set would be very useful. The following two propositions describe families of elements in $\sqrt{A_{n,k}}$.

Proposition 3. *Fix a subset $P = \{p_1, \ldots, p_k\} \subseteq [n]$ and an integer partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of n into distinct parts, $k > 1$. Given a set partition $\pi = \{\pi_1, \ldots, \pi_k\}$ of type λ we associate a permutation ν_π by $|\pi_i| = \lambda_{\nu_\pi(i)}$. Then*

$$x_{\lambda,P} := \sum_\pi sgn(\nu_\pi) \sum_{\sigma \in S_k} sgn(\sigma) w_{\pi,P,\sigma} \in \sqrt{A_{n,k}},$$

where the outer sum is over all set partitions π of type λ.

Proposition 4. *Fix a subset $P = \{p_1, \ldots, p_{k+1}\} \subseteq [n]$ and a set partition $\pi = \{\pi_1, \ldots, \pi_k\}$ of $[n]$, $k < n$. Then*

$$x_{\pi,P} := \sum_{i=1}^{k+1} (-1)^i \sum_{\sigma \in S_k} sgn(\sigma) w_{\pi, P\setminus\{i\}, \sigma} \in \sqrt{A_{n,k}}.$$

The elements $\{x_{\pi,P}\}$ span $\sqrt{A_{n,1}}$ for all n, and $\{x_{\lambda,P}\}$ and $\{x_{\pi,P}\}$ together span $\sqrt{A_{4,2}}$, but in general their span is some proper subideal of the radical.

Problem 6. *Find larger classes of elements in $\sqrt{A_{n,k}}$.*

Problem 7. *Find a recursive formula, allowing us to use known elements in $\sqrt{A_{n,k}}$ to derive elements in $\sqrt{A_{n+1,k}}$ or $\sqrt{A_{n+1,k+1}}$.*

Other Areas of Interest

The Partition Algebra. The full transformation semigroup algebra $\mathbb{C}T_n$ is a subalgebra of the partition algebra $P_n(m)$. This larger algebra was developed independently by P. Martin and V.F.R. Jones, and is discussed extensively in [8]. For V a complex vector space of dimension m there is an action of $P_n(m)$ on the n-fold tensor product $V^{\otimes n}$. The symmetric group S_m, thought of as a subgroup of $GL_m(\mathbb{C})$, acts diagonally on $V^{\otimes n}$. The importance of the partition algebra can be seen in the fact that these two actions generate full centralizers of each other, mirroring the Schur-Weyl duality of $\mathbb{C}S_n$ and $GL_m(\mathbb{C})$.

$\mathbb{C}T_n \subseteq P_n(m)$ so we trivially have $\mathrm{End}_{\mathbb{C}T_n}(V^{\otimes n}) \supseteq \mathrm{End}_{P_n(m)}(V^{\otimes n})$. Since $P_n(m)$ is a much larger algebra than $\mathbb{C}T_n$ and also involves the parameter m it is a bit surprising that we in fact have equality.

Proposition 5. *$End_{\mathbb{C}T_n}(V^{\otimes n}) = End_{P_n(m)}(V^{\otimes n})$.*

Because $\mathbb{C}T_n$ is not semisimple, the usual Schur-Weyl duality results do not hold.

Problem 8. *Find a well-defined action of the semisimple quotient of $\mathbb{C}T_n$ on $V^{\otimes n}$.*

Alternating Sign Matrices. An *alternating sign matrix (ASM) of order* n is an $n \times n$ matrix of 0's, 1's and -1's such that every row and column sums to 1 and the nonzero entries alternate in sign across rows and down columns. Alternating sign matrices can be thought of as generalized permutation matrices; the smallest ASM that is *not* a permutation matrix is given below, along with a more general example.

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Two examples of alternating sign matrices.

These matrices were first studied by Robbins and Rumsey in the early 1980's. They arose out of a study of Dodgson's condensation method for evaluating determinants.

The number A_n of order-n ASM's is given by the product formula

$$A_n = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}.$$

The search for a proof of this formula was a long-standing open problem (see [3]), finally solved independently by Zeilberger and Kuperberg in 1995. Curiously, while the number A_n is also known to count totally symmetric self-complementary plane partitions inside a $2n \times 2n \times 2n$ box as well as order-n descending plane partitions, no explicit bijection is known between any of these sets of objects.

More recently, ASM's have shown up in several surprising ways. In particular, Razumov and Stroganov [14] have formulated a number of conjectures relating ASM's to the ground-state eigenvector of a certain Hamiltonian operator. I have found a way to state one of their main conjectures in terms of the Laplacian and adjacency matrices of a graph.

Consider the graph $G(N,k)$ whose vertices correspond to the k-subsets of an N-set, represented by words of k 1's and $(N-k)$ -1's, where two vertices are connected by an edge if the corresponding words differ by a single transposition of (cyclically) adjacent entries. Let

$$H_{N,k} = \frac{N}{2} I - L_{N,k} - A_{N,k},$$

where $L_{N,k}$ and $A_{N,k}$ are, respectively, the Laplacian matrix and the adjacency matrix for $G(N,k)$. Let H_N be the diagonal block matrix with blocks $H_{N,k}$, $k = 0, 1, \ldots, N$.

Our restatement of Razumov and Stroganov's conjecture is as follows.

Conjecture 2. *Let $N = 2n+1$, and let $\vec{v}_n$ be the eigenvector of H_N corresponding to the minimum eigenvalue. Then the ratio of the largest non-zero component of $\vec{v}_n$ to the smallest is A_n, the number of $n \times n$ alternating sign matrices.*

Not much is known about the structure of the graphs $G(N,k)$. Enns [6] has shown that these graphs are Hamiltonian (with a few small exceptions) if and only N and k not both even, $k \neq 2$ or $N-2$. It is not clear how the restatement presented in Conjecture 2 helps in deriving the eigenvectors and eigenvalues of H_N.

Problem 9. *Describe the structure of the graph $G(N,k)$.*

Fully-Packed Loops and Linking Patterns. A *fully-packed loop model (FPL) of order n* is a graph drawn on an $(n+2) \times (n+2)$ array of vertices, minus the corners, such that every interior vertex has degree two, and alternate exterior vertices, numbered $1, \ldots, 2n$ as shown in the example below, have degree 1 (the remaining exterior vertices have degree 0).

An order-5 fully-packed loop model.

Notice that if we start from a numbered vertex, there will be a unique path to follow using edges of the graph; this path will lead to another numbered vertex. In such a way we have open loops connecting the $2n$ numbered vertices in some non-crossing fashion. There may also be closed loops within the graph. Order-n fully-packed loop models are in bijection with $n \times n$ alternating sign matrices.

An order-n FPL, via its *linking pattern*, naturally determines a non-crossing pairing of $2n$ points around a circle. For example, the FPL shown above determines the pairing $\{1 \leftrightarrow 2, 3 \leftrightarrow 8, 4 \leftrightarrow 5, 6 \leftrightarrow 7, 9 \leftrightarrow 10\}$. Wieland [17] has shown that for any two non-crossing pairings π and π' that are conjugate via a rotation or a reflection the number of FPL states with linking pattern π equals the number of FPL states with linking pattern π'.

This result has led Razumov and Stroganov [15] to formulate another conjecture. For $i = 1, \ldots, 2n$ define h_i to be the operator which acts on a non-crossing pairing π as follows. If $i \leftrightarrow (i+1) \in \pi$ then $h_i(\pi) = \pi$. On the other hand, if $i \leftrightarrow j, (i+1) \leftrightarrow k \in \pi$ then $h_i(\pi) = \pi'$, where π' is the non-crossing pairing identical to π in all respects except that $i \leftrightarrow (i+1), j \leftrightarrow k \in \pi'$. In other words, h_i forces the pairing $i \leftrightarrow i+1$ (h_{2n} is defined cyclically, so that it forces the pairing $2n \leftrightarrow 1$).

The action of h_i on non-crossing pairings.

For π a non-crossing pairing of $2n$ points around a circle, define $A_n(\pi)$ to be the number of order-n fully-packed loop models with linking pattern π. Let $H = \sum_{i=1}^{2n} h_i$ act on the complex span of all non-crossing pairings. Razumov and Stroganov have conjectured that the vector $\vec{v} = \sum_{\pi} A_n(\pi)\pi$ is an eigenvector for H, with eigenvalue $2n$.

Wieland's result is obtained by defining operators acting on the FPL's that simultaneously flip pairs of parallel edges.

Problem 10. *Define local operators acting on a single pair of parallel edges, which somehow generalizes the action of the h_i on the non-crossing pairings.*

Zuber [18] has recently formulated additional conjectures on FPL's, concerning explicit and recursive formulas for $A_n(\pi)$ for several classes of linking patters π.

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School of Mathematics, University of Minnesota, Minneapolis, MN 55455
E-mail address: `jhall@math.umn.edu`