Γ-Convergence Approximation to Piecewise Constant Mumford-Shah Segmentation

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Abstract. Piecewise constant Mumford-Shah segmentation [17] has been rediscovered by Chan and Vese [6] in the context of region based active contours. The work of Chan and Vese demonstrated many practical applications thanks to their clever numerical implementation using the level-set technology of Osher and Sethian [18]. The current work proposes a Γ-convergence formulation to the piecewise constant Mumford-Shah model, and demonstrates its simple implementation by the iterated integration of a linear Poisson equation. The new formulation makes unnecessary some intermediate tasks like normal data extension and level-set reinitialization, and thus lowers the computational complexity.

1 Introduction: The Mumford-Shah Segmentation Model

The Mumford-Shah segmentation model [17] is built upon a generic image model into which the edge feature is explicitly incorporated as in [10]. Consider the following image generation model:

$$\Gamma \longrightarrow u \xrightarrow{\oplus n} u_0,$$

where in the reverse order, u_0 denotes an observed image, n an additive Gaussian noise field, and u piecewise smooth (or cartoonish) image patches consistent with a given edge layout Γ.

From Bayesian point of view [10,16], segmentation is to estimate the posterior probability

$$p(\Gamma, u \mid u_0), \quad \text{or equivalently,} \quad p(u_0 \mid u, \Gamma)p(u, \Gamma)/p(u_0).$$

In the Markovian setting [7], the joint prior can be expressed by

$$p(u, \Gamma) = p(u \mid \Gamma)p(\Gamma).$$

Thus by putting aside the constant $p(u_0)$ and working with the energy function (or the *logarithmic likelihood function*) $E = -\log p$, one obtains the structure of the Mumford-Shah model up to an ineffectual constant:

$$E[\Gamma, u \mid u] = E[\Gamma] + E[u \mid \Gamma] + E[u_0 \mid u, \Gamma].$$

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The *full* Mumford-Shah model [15,17] is in fact explicitly expressed by:

$$E[\Gamma, u \mid u] = \sigma \text{length}(\Gamma) + \beta \int_{\Omega \setminus \Gamma} |\nabla u|^2 dx + \lambda \int_{\Omega} (u - u_0)^2 dx,$$

where $dx = dx_1 dx_2$ denotes the area element of a 2-D domain Ω. The model has become a classic and been studied by numerous researchers (e.g., the monograph [15]).

For images made of piecewise *homogeneous* stochastic patches, only their constant averages can be identified as the cartoonish pieces, i.e.,

$$u(x) \equiv C_i, \qquad x \in \Omega_i, \quad \text{and} \quad \Omega_i \in \pi(\Omega \mid \Gamma).$$

Here the notation $\pi(\Omega \mid \Gamma)$ denotes the partitioning of the entire image domain Ω given an edge layout Γ, or the collection of *connected components* of $\Omega \setminus \Gamma$ topologically speaking. The original Mumford-Shah model is then reduced to the piecewise constant model, or simply, the *reduced* Mumford-Shah model:

$$E[(C_i's), \Gamma \mid u_0] = \sigma \text{length}(\Gamma) + \lambda \sum_{\Omega_i \in \pi(\Omega \mid \Gamma)} \int_{\Omega_i} (u(x) - C_i)^2 dx.$$

Mathematically this reduced model is a proper asymptotic limit of the full Mumford-Shah model, as discussed in the original paper of Mumford and Shah [17], or Chan and Shen [5]. Recently in the award-winning paper [6,21] (2003 Best Paper of IEEE Signal Processing Society), Chan and Vese rediscovered this model in the context of region based active contours. As in [6], we shall mainly focus on the 2-phase model to illustrate our primary contributions:

$$E[C_+, C_-, \Gamma \mid u_0] = \sigma \text{length}(\Gamma) + \lambda \int_{\Omega_+} (u(x) - C_+)^2 dx + \lambda \int_{\Omega_-} (u(x) - C_-)^2 dx, \tag{1}$$

where Γ partitions Ω into the interior Ω_+ and exterior Ω_-. As remarkably demonstrated by Chan and Vese [6,21], such a 2-phase model has already witnessed numerous intriguing applications in astronomy and medicine.

Chan and Vese have successfully implemented the above model using the level-set computing technology invented and continuously advanced by Stan Osher and James Sethian [18]. Multiphase frameworks have also been developed by Chan and Vese [21], and lately by Lie, Lysaker, and Tai [13].

The current work is complementary to the above level-set approach. Inspired by the Γ-convergence approximation to the full Mumford-Shah model developed by Ambrosio and Tortorelli [1], we propose a new Γ-convergence formulation of the reduced Mumford-Shah model, and its robust and fast computational implementation. As in [1], the new formulation overcomes the fundamental theoretical and computational difficulties resulting from the free-boundary nature of the Mumford-Shah model (both the full and the reduced). The computation is reduced to the iterated integration of a linear Poisson equation, which can be easily and efficiently implemented in Matlab in a uniform code, without extra

intermediate processing steps (e.g., normal extension and reinitialization in the level-set approach) [21].

The organization of the paper goes as follows. Section 2 briefly reviews the essence of the Γ-convergence approximation to the full Mumford-Shah model. In Section 3, we introduce our new Γ-convergence approximate model to the reduced (i.e., piecewise constant) Mumford-Shah model. Efficient computational schemes and examples of generic test images are presented in Section 4.

2 Γ-Convergence Approximation to the Full M.-S. Model

Γ-convergence has its rigorous mathematical definition in metric spaces [1]. The intuition in the current context could be easily revealed by phase-field modelling in superconductors, as in the the works of Ginzburg and Landau [11]. We now briefly explain the core idea in terms of approximation theory, which will then naturally bring out the new model.

In the Γ-convergence setting [1,8,14], a curve Γ (in 2-D) is instead represented by a 2-D function $z = z_\epsilon(x_1, x_2) \in [0, 1]$, depending upon a small scale parameter ϵ. The energy associated with such a *phase field* z is defined as

$$L_\epsilon[z] = \int_\Omega \epsilon |\nabla z|^2 dx + \int_\Omega \frac{(1-z)^2}{4\epsilon} dx.$$

Since $\epsilon \ll 1$, under any finite energy bound, the second term demands the phase field $z = z_\epsilon(x_1, x_2)$ to be as close to 1 as possible almost everywhere on the image domain Ω.

Suppose in addition that along some narrow bands (intended to be the ϵ-neighborhoods of a curve Γ) the field z sharply drops down to zero. The graph of z then looks like a canyon along its valley line Γ. The entire Γ-convergence machinery is built upon the following remarkable approximation result:

$$L_\epsilon[z] \simeq \text{length}(\Gamma). \tag{2}$$

Rigorous mathematical analysis is more involved but a qualitative glimpse is not too far beyond the level of Advanced Calculus as presented below.

Applying the generic inequality $2AB \leq A^2 + B^2$, one has

$$L_\epsilon[z] \geq \int_\Omega |\nabla z||z-1| dx = \frac{1}{2} \int_\Omega |\nabla w|, \qquad w = (1-z)^2,$$

where the graph of $w = (1-z)^2$ looks like a set of walls. Most contributions to the integral come from a narrow band along Γ since w is flat away from it. With smooth Γ, the narrow tubular neighborhood can then be parameterized by the tangential (arc length) and normal coordinates s and n. Since w remains almost constant along the tangential direction, we have $|\nabla w(s, n)| \simeq |\partial w / \partial n|$, and

$$\frac{1}{2} \int_\Omega |\nabla w| \simeq \int_\Gamma \int_{-\epsilon}^{\epsilon} \frac{1}{2} \left| \frac{\partial w}{\partial n} \right| dn ds = \int_\Gamma \frac{1}{2} \text{TV}(w(s, \cdot)) ds.$$

For any fixed s, the total variation $\mathrm{TV}(w(s,\cdot)$ along the normal direction is ideally 2, since each shoulder of the wall contributes 1 (by ascending from 0 to 1 and then descending from 1 to 0). Hence we have shown qualitatively that

$$L_\epsilon[z] \geq \mathrm{length}(\Gamma).$$

Assisted with a suitable ordinary differential equation [1], one can further show that the lower bound can indeed be approached.

Notice that the above analysis crucially relies upon the assumption that z does touch down to the zero along Γ. But the energy form $L_\epsilon[z]$ alone does not guarantee it. In Ambrosio and Tortorelli's approximation [1], it is explicitly enforced through the second term of the Mumford-Shah model:

$$E_\epsilon[z] = \sigma\left(\int_\Omega \epsilon|\nabla z|^2 dx + \int_\Omega \frac{(1-z)^2}{4\epsilon} dx\right) + \beta\int_\Omega z^2|\nabla u|^2 dx + \lambda\int_\Omega (u-u_0)^2 dx.$$

Along the jump (edge) set Γ, ∇u is not classically defined, or remains very large (or expensive) even after discrete sampling or continuous blurring. Thus the second term forces z to touch down to zero along Γ to bound the total energy.

3 Γ-Convergence Form of the Reduced M.-S. Model

For the reduced (piecewise constant) Mumford-Shah model, the lack of the gradient term loses the control factor that forces the field z to drop near edges. In the current paper, therefore, we propose a proper variation of Ambrosio and Tortorelli's original formulation for the full Mumford-Shah model [1]. As in Chan and Vese [6], we shall primarily focus on the 2-phase model, and multiphase extensions can be similarly accomplished as in Vese and Chan [21], and in particular, in the recent work of Lie, Lysaker, and Tai [13].

To explicitly enforce two-phase separation without turning to the gradient information ∇u, we propose to replace the original phase field energy by

$$L_\epsilon[z] = \int_\Omega \left(9\epsilon|\nabla z|^2 + \frac{(1-z^2)^2}{64\epsilon}\right) dx.$$

The range of z is restricted within $[-1,1]$. Since $\epsilon \ll 1$, a bounded energy will force $z=1$ or $z=-1$ almost everywhere. Following the similar inequality in the preceding section, one has

$$L_\epsilon[z] \geq \frac{3}{4}\int_\Omega |\nabla z||1-z^2| dx = \frac{3}{4}\int_\Omega \left|\nabla\left(z - \frac{z^3}{3}\right)\right| dx \simeq \frac{3}{4}\int_\Gamma\int_{-\epsilon}^{\epsilon} \mathrm{TV}(w) dn ds,$$

where $w = w(z) = z(1-z^2/3)$ is a monotone function on $z \in [-1,1]$, and the local curvilinear coordinates have been applied along the transition medial line (where $z=0$), as in the preceding section. Since $w(-1) = -2/3$ and $w(1) = 2/3$, one has $\mathrm{TV}(w(z(s,\cdot))) = 4/3$ locally along each s-normal line. Thus we have qualitatively established the lower bound:

$$L_\epsilon[z] \geq \mathrm{length}(\Gamma).$$

Further elaborate study shows that the hyperbolic tangent transition:

$$z(s,n) = \tanh\left(\frac{n}{24\epsilon}\right)$$

can approach the lower bound as $\epsilon \to 0$. Thus $L_\epsilon[z]$ well approximates the length of Γ.

In the ideal scenario of two *pure* phases, one then defines their associated regions separately:

$$\Omega_\pm = \{x \in \Omega \mid z = \pm 1\}.$$

The associated indicator functions are ideally given by

$$1_+(x) = \left(\frac{1+z}{2}\right)^2, \qquad 1_-(x) = \left(\frac{1-z}{2}\right)^2.$$

(The square is mainly for computational stability in case that z strays away from $[-1,1]$.) Then,

$$\int_{\Omega_\pm} (u_0 - C_\pm)^2 dx = \int_\Omega \left(\frac{1 \pm z}{2}\right)^2 (u_0 - C_\pm)^2 dx.$$

In combination, we thus propose to approximate the reduced Mumford-Shah model (1) by the following Γ-convergence energy:

$$\begin{aligned} E_\epsilon[z, C_+, C_- \mid u_0] = \sigma \textstyle\int_\Omega \left(9\epsilon|\nabla z|^2 + \frac{(1-z^2)^2}{64\epsilon}\right) dx + \\ \lambda \textstyle\int_\Omega \left(\frac{1+z}{2}\right)^2 (u_0 - C_+)^2 dx + \lambda \int_\Omega \left(\frac{1-z}{2}\right)^2 (u_0 - C_-)^2 dx. \end{aligned} \tag{3}$$

One minimizes the energy by some optimal phase field z and means $C_\pm$'s.

Notice that all the four terms involve the field function z, but only the last two contain the mean fields $C_\pm$'s. Denote the sum of the last two terms by the "conditional" energy $E[C_+, C_- \mid u_0, z]$ given any z. Then the standard property of weighted least square approximation explicitly yields the conditional optima.

Theorem 1 (Optimal Means). *Given any square integrable phase field z on a finite domain Ω, as long as z is not constant, the optimal means $C_\pm$'s to a given image u_0 in terms of $E[C_+, C_- \mid u_0, z]$ are given by:*

$$C_\pm = C_\pm[z] = \frac{\int_\Omega (1 \pm z(x))^2 u_0(x) dx}{\int_\Omega (1 \pm z(x))^2 dx}. \tag{4}$$

On the other hand, by the direct method of Calculus of Variations using minimizing sequences [9], one can establish the existence of minimizers to (3).

Theorem 2 (Existence of Optimal Phase Fields). *Let u_0 be a square integrable image on a bounded domain Ω. Then there exists an optimal triple (z^*, C_+^*, C_-^*) which achieves the minimum energy of $E_\epsilon[z, C_+, C_- \mid u_0]$ among the admissible class of Sobolev phase fields [9].*

To compute an optimal minimizer, one could apply the conditional mean field formulae (4) to reduce the triple energy $E_\epsilon[z, C_+, C_- \mid u_0]$ to an energy solely depending upon z:

$$E_\epsilon[z \mid u_0] = E_\epsilon[z, C_+[z], C_-[z] \mid u_0].$$

But this energy is no longer quadratic in z and complexities multiply due to the denominators involving z.

Thus in practice, one employs the *alternating minimization* technique prevailing in multivariable optimization problems [8,20]. For given z^n at step n, one computes the optimal means $C_\pm^n = C_\pm[z^n]$ by the formulae (4), and then updates z^n to z^{n+1} by treating $C_\pm$'s as known and minimizing

$$\begin{aligned} E_\epsilon[z \mid u_0, C_+, C_-] = \sigma \int_\Omega \left(9\epsilon|\nabla z|^2 + \tfrac{(1-z^2)^2}{64\epsilon}\right) dx + \\ \lambda \int_\Omega \left(\tfrac{1+z}{2}\right)^2 (u_0 - C_+)^2 dx + \lambda \int_\Omega \left(\tfrac{1-z}{2}\right)^2 (u_0 - C_-)^2 dx. \end{aligned} \tag{5}$$

4 Fast and Robust Numerical Implementation; Examples

Computationally, the optimization problem (5) is solved via its Euler-Lagrange equation. Write $e_\pm = u_0 - C_\pm$ as the residuals on $\Omega_\pm$, which are independent of z since $C_\pm$ are given. Let $\mu = \lambda/(4\sigma)$. Then the Euler-Lagrange equation of $E_\epsilon[z \mid u_0, C_+, C_-]$ is given by

$$0 = -9\epsilon\Delta z - \frac{(1-z^2)z}{32\epsilon} + \mu e_+^2(1+z) - \mu e_-^2(1-z), \tag{6}$$

with the Neumann adiabatic boundary condition. One further rewrites it to:

$$-9\epsilon\Delta z + \left(\frac{z^2}{32\epsilon} + \mu(e_+^2 + e_-^2)\right) z = \mu e_-^2 - \mu e_+^2 + \frac{z}{32\epsilon},$$

or simply $-9\epsilon\Delta z + R(z)z = f(z)$ with R and f denoting the corresponding terms. The latter can be solved iteratively by having the z's in R and f frozen:

$$z_m \to z_{m+1}: \quad -9\epsilon\Delta z_{m+1} + R_m z_{m+1} = f_m, \tag{7}$$

where $R_m = R(z_m) \geq 0$ and $f_m = f(z_m)$. Thus at each step it suffices to solve this linear Poisson equation on $z_{m+1}(x_1, x_2)$, which can be implemented efficiently in Matlab due to many fast elliptic solvers. Our computational experiments show that even ordinary Guass-Jacobi type of iteration schemes [12] lead to fast and robust convergence, including starting from any random initial guess.

The following flow summarizes our entire algorithm:

$$\hookrightarrow z^n \overset{\text{by}(4)}{\longrightarrow} [C_+^n, C_-^n] \to \left[z_m^{n+1} \overset{\text{by}(7)}{\hookrightarrow} z_{m+1}^{n+1}\right] \to z^{n+1} \to$$

The examples in the next section have all been generated from this algorithm. Below we briefly discuss how to properly choose the parameters in the model.

(a) The Γ-convergence parameter ϵ should be in the order of $O(h)$, where h denotes the grid scale of a discrete image domain, for example $\epsilon = 4h$.
(b) Generally σ (or the tension parameter) is of order $O(1)$, while the fitting Lagrange multiplier λ should be inversely proportional to the variance of the Gaussian noise embedded in the observed image u_0 [2,3,4,5,19].

In Figure 1, we have demonstrated the performance of our new model and algorithm on three generic test images: peppers, the Milky Way, and the Pathfinder on the Mars by NASA (USA). For the images of peppers and the Pathfinder, we have shown the Γ-convergence output z's, while for the Milky Way in the middle, the zero level curve (i.e., the sharp transition curve) of the output z has been superimposed upon the original image u_0. (The associated mpeg movies are available from the author upon request.) The numerical performance (e.g., topological merging and splitting) is comparable with Chan and Vese's level-set approach [6], while the computational complexity is substantially lower without intermediate tasks like normal extension and level-set reinitialization [6,18].

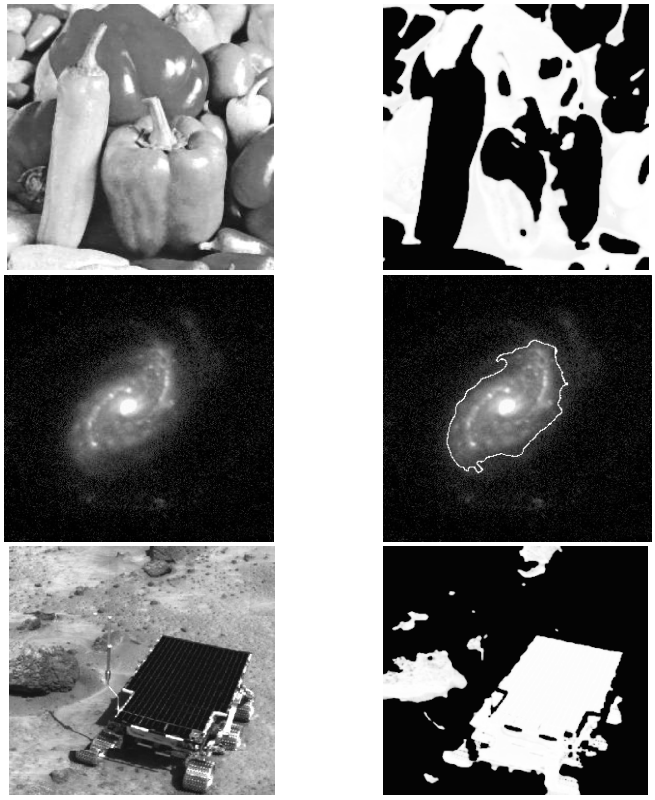

Fig. 1. Left: three generic images u_0's: peppers, the Milky Way, and the Pathfinder landed on the Mars (NASA, USA); Right: the output z's or their zero-level curves

References

1. L. Ambrosio and V. M. Tortorelli. Approximation of functionals depending on jumps by elliptic functionals via Γ-convergence. *Comm. Pure Appl. Math.*, 43:999–1036, 1990.
2. T. F. Chan, S. Osher, and J. Shen. The digital TV filter and nonlinear denoising. *IEEE Trans. Image Process.*, 10(2):231–241, 2001.
3. T. F. Chan and J. Shen. Variational restoration of non-flat image features: models and algorithms. *SIAM J. Appl. Math.*, 61(4):1338–1361, 2000.
4. T. F. Chan and J. Shen. Mathematical models for local nontexture inpaintings. *SIAM J. Appl. Math.*, 62(3):1019–1043, 2001.
5. T. F. Chan and J. Shen. *Image Processing and Analysis: variational, PDE, wavelet, and stochastic methods.* SIAM Publisher, Philadelphia, 2005.
6. T. F. Chan and L. A. Vese. Active contours without edges. *IEEE Trans. Image Process.*, 10(2):266–277, 2001.
7. T. M. Cover and J. A. Thomas. *Elements of Information Theory.* John Wiley & Sons, Inc., New York, 1991.
8. S. Esedoglu and J. Shen. Digital inpainting based on the Mumford-Shah-Euler image model. *European J. Appl. Math.*, 13:353–370, 2002.
9. L. C. Evans. *Partial Differential Equations.* Amer. Math. Soc., 1998.
10. S. Geman and D. Geman. Stochastic relaxation, Gibbs distributions, and the Bayesian restoration of images. *IEEE Trans. Pattern Anal. Machine Intell.*, 6:721–741, 1984.
11. V. L. Ginzburg and L. D. Landau. On the theory of superconductivity. *Soviet Phys. JETP*, 20:1064–1082, 1950.
12. G. H. Golub and C. F. Van Loan. *Matrix Computations.* The Johns Hopkins University Press, Baltimore, 1983.
13. J. Lie, M. Lysaker, and X.-C. Tai. A binary level set model and some applications to Mumford-Shah image segmentation. *UCLA CAM Tech. Report*, 04-31, 2004.
14. R. March. Visual reconstruction with discontinuities using variational methods. *Image Vision Comput.*, 10:30–38, 1992.
15. J.-M. Morel and S. Solimini. *Variational Methods in Image Segmentation*, volume 14 of *Progress in Nonlinear Differential Equations and Their Applications.* Birkhäuser, Boston, 1995.
16. D. Mumford. *Geometry Driven Diffusion in Computer Vision*, chapter "The Bayesian rationale for energy functionals", pages 141–153. Kluwer Academic, 1994.
17. D. Mumford and J. Shah. Optimal approximations by piecewise smooth functions and associated variational problems. *Comm. Pure Applied. Math.*, 42:577–685, 1989.
18. S. Osher and J. A. Sethian. Fronts propagating with curvature-dependent speed: Algorithms based on Hamilton-Jacobi formulations. *J. Comput. Phys.*, 79(12), 1988.
19. L. Rudin, S. Osher, and E. Fatemi. Nonlinear total variation based noise removal algorithms. *Physica D*, 60:259–268, 1992.
20. J. Shen. Bayesian video dejittering by BV image model. *SIAM J. Appl. Math.*, 64(5):1691–1708, 2004.
21. L. A. Vese and T. F. Chan. A multiphase level set framework for image segmentation using the Mumford-Shah model. *Int. J. Comput. Vision*, 50(3):271–293, 2002.