Problem Set 1, Math8601-Real Analysis

Assigned on Friday, Sept 17; Due on Friday, Sept 24. Autumn 2004

"Real Analysis for Real People in Real Time" – Jackie Shen ©

(1) (Set Theory Meets Bioinformatics; 20pts) Let $X = \{$all expressed human genes: past, present, and future$\}$. Take a reference initial year in history, say 100,000 years ago, and label it as $n = 1$. Then let n increase in years. Define for each n,

$$E_n = \{\text{all expressed human genes in the reference year } n\} \subseteq X.$$

As in the lecture, define the limit superior and inferior by

$$\limsup_{n\to\infty} E_n = \cap_{n=1}^{\infty} \cup_{k\geq n} E_k, \quad \text{and} \quad \liminf_{n\to\infty} E_n = \cup_{n=1}^{\infty} \cap_{k\geq n} E_k.$$

(1.a) Use plain English to explain *separately* the meaning that a particular gene (i.e., an active segment of DNAs) $g \in \limsup_n E_n$, and $g \in \liminf_n E_n$. For example, which scenario corresponds to the situation when the gene g emerges infinitely often (maybe on and off)? (1.b) Suppose Darwin tells you that a particular gene f is never expressed in all reference years n that are multiples of 11 (i.e., n=11, 22, ...), what can you say *for sure* regarding the relationship between f and the above two limit sets?

(2) (Sets v.s. Numbers; 10pts) Suppose a, b, c are three real numbers. Then the distributive law holds: $\boxed{a \times (b + c) = a \times b + a \times c}$. Applying the following number-set correspondence

$$a, b, c \leftrightarrow A, B, C, \quad + \leftrightarrow \cup, \quad \text{and} \quad \times \leftrightarrow \cap,$$

Lisa conjectures that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Can you help her prove it using definitions?

(3) (σ-Algebra; 20pts) (3.a) Given a set X, define a collection of two subsets: $\Sigma_0 = \{\phi, X\}$. Show that Σ_0 is a σ-algebra. (3.b) Let $\mathcal{A} = \{\phi\}$ denote the collection only containing the empty set. What is the σ-algebra $\Sigma_{\mathcal{A}}$ generated by $\mathcal{A}$? (3.c) Suppose $x \in X$ and X contains at least two elements. Let $\mathcal{B} = \{\{x\}\}$ denote the collection only containing the singleton set $\{x\}$. What is the σ-algebra $\Sigma_{\mathcal{B}}$ generated by $\mathcal{B}$?

(4) (σ-Algebra v.s. a Drunk's Random Walk; 10pt) Imagine a 1-D creature named Jackie who can only appear on the sites of the integer set $\mathbb{Z}$. One day after getting drunken at a bar located at $n = 0$, Jackie starts to walk home. Since he is drunken, his walking manner is random as follows: suppose currently (at time t) Jackie is at a general site $m \in \mathbb{Z}$, then at the next time step $t + 1$,

$$40\% \text{ chance moving to } m + 1, \; 40\% \text{ chance moving to } m - 1, \text{ and } 20\% \text{ remaining at } m.$$

The good news is that *Real Analysis* spares you from studying such unpredictable random manner. You are asked to study it in *Probability Theory* or *Stochastic Modelling*. However here are some σ-algebras that have been precisely motivated by Jackie's random walk. For each time step t (t =0, 1, 2, ...), define a collection Σ_t of subsets (or "events" as in probability)

$$\Sigma_t = \{A \subseteq \mathbb{Z} \mid A \text{ or } A^c \subseteq [-t, t]\}.$$

Show that (4.a) as time evolves forward, the world of events expands: $\Sigma_t \subseteq \Sigma_{t+1}$ for any t; and (4.b) each world Σ_t is a σ-algebra.

(5) (Many Ways to Generate Borel σ-algebra; 20pts) Consider all real numbers $X = \mathbb{R}$. Define the following two collections of subsets based on intervals:

$$\mathcal{A} = \{[a, b) \mid a, b \in \mathbb{R}, a < b\} \quad \text{and} \quad \mathcal{B} = \{(a, b] \mid a, b \in \mathbb{R}, a < b\}.$$

Let $\Sigma_{\mathcal{A}}$ and $\Sigma_{\mathcal{B}}$ denote separately the σ-algebras generated by $\mathcal{A}$ and $\mathcal{B}$. Show that $\Sigma_{\mathcal{A}} = \Sigma_{\mathcal{B}}$. (Hint: by direct construction show that $\mathcal{B} \subseteq \Sigma_{\mathcal{A}}$ and $\mathcal{A} \subseteq \Sigma_{\mathcal{B}}$.) This σ-algebra is called the Borel algebra on $\mathbb{R}$.

(6) (Dirac's Delta; 10pts) Given a nonempty set X and any of its element x_0, the Dirac (yes, the giant in *quantum mechanics*) delta measure δ_0 is defined by

$$\delta_0(A) = 1, \quad \text{if } x_0 \in A; \quad 0, \quad \text{otherwise},$$

for any subset A in the power σ-algebra $\Sigma = 2^X$. Verify that δ_0 is indeed a measure.

(7) (Countable Additivity; 10pts) Let $X = \mathbb{N}$ and $\Sigma = 2^{\mathbb{N}}$. Define $\mu(A) = 0$ if A is a finite subset, and $\mu(A) = \infty$ if A is an infinite set. Show that μ satisfies the *finite-additivity* condition but NOT *countable additivity*, and is thus not a measure.