

## Problem Set 2, Math8601-Real Analysis

Assigned on Friday, Sept 24; Due on Monday, Oct 4. Autumn 2004

“Real Analysis for Real People in the Real World” – Jackie Shen ©

- (1) (Challenging Measure Principles. 10pts) An important part of graduate training is to learn how to challenge given principles. (Einstein challenged our conventional notions of space and time!) So let us question the basic principles for a set function  $\mu$  to become a measure defined in the recent lectures.

Let  $X$  denote the entire collection of *microscopic* particles of this universe, including for example, protons (each with one positive charge), neutrons, electrons (each with one negative charge), etc. Let  $\Sigma$  denote the  $\sigma$ -algebra generated by all *macroscopic* objects in this universe (in some vague sense which is nonessential for the current problem). That is, for example, one can treat Jackie as a set of microscopic particles.

Define for each object  $A$  a value  $\mu(A)$  by

$$\mu(A) = \text{net charges in the body } A.$$

For example, most time Jackie is neutral, with zero net charges. But it could happen that after jogging around Lake Superior,  $\mu(\text{Jackie}) = -100$  since there are many negative ions dancing in the air near a lake. (a) Explain in one sentence that what  $\mu$  measures could be *negative*, which is our first attack to the nonnegativity principle. Moreover, (b) illustrate (using English) a scenario where  $A \subseteq B$ , but  $\mu(A) > \mu(B)$ , which challenges the monotonicity property of a (classical) measure proved in class.

- (2) (Measure or Non-Measure: Bioinformatics Revisited (from PS 1). 20pts) As in Problem Set 1, define

$$X = \{\text{all expressed genes of human beings: past, present, and future}\}.$$

By an expressed gene, we mean a specific DNA segment that actively contributes to the development of the human body. For convenience, let us assume  $X$  is a finite set (which however could be huge). Let  $\Sigma = 2^X$  denote the power  $\sigma$ -algebra of  $X$ . For any given subset  $G \subseteq X$ , define

$$\mu_*(G) = \{\text{number of people who carry at least one gene from } G\}.$$

(We could specify a fixed observation time, say, at noon 12:00pm London time on September 30, 2004, with total Earth population 6,000,000,000) (a) Suppose all human beings share at least two distinct (core) genes  $g_1$  and  $g_2$ . Show that  $\mu_*$  cannot be a measure. (b) Show that in any situation  $\mu_*$  is an outer measure. (This is the outer measure in the real world discovered by Jackie Shen, not in any textbook!)

- (3) (Measure or Non-Measure. 10pts) Let  $X$  denote all natural numbers no greater than 100 billion, and  $\Sigma = 2^X$  its power  $\sigma$ -algebra. For each subset  $A \subseteq X$ , define

$$\mu(A) = \text{percentage of even numbers in } A.$$

Thus for instance,

$$\mu(\{1, 2, 3\}) = 1/3, \quad \text{and} \quad \mu(\{24, 25, 26, 27, 28\}) = 3/5.$$

Is  $\mu$  a measure on the measurable space  $(X, \Sigma)$ ? Clearly explain why.

- (4) (Singular Measure Pairs. 20pts) Given a measurable space  $(X, \Sigma)$ , two measures  $\mu_+$  and  $\mu_-$  are said to be *singular* to each other if there exists a bi-partition of  $X$ :  $X = A_+ \cup A_-$ ,  $A_+ \cap A_- = \phi$  (empty), such that

$$\mu_+(A_-) = 0 \quad \text{and} \quad \mu_-(A_+) = 0.$$

That is,  $\mu_+$  only cares about the world of  $A_+$  but is blind to  $A_-$ . The same could be said about  $\mu_-$ . (a) Show that for any  $A \in \Sigma$ ,  $\mu_+(A) \leq \mu_+(A_+)$  and  $\mu_-(A) \leq \mu_-(A_-)$ . (b) Let  $X = \mathbb{N}$  be all natural numbers, and  $\Sigma = 2^{\mathbb{N}}$  the power  $\sigma$ -algebra. Define for any given subset  $A \in 2^{\mathbb{N}}$ ,

$$\mu_+(A) = \text{number of even integers in } A \quad \text{and} \quad \mu_-(A) = \text{number of odd integers in } A.$$

- (b.1) What are the largest sets  $B$  so that  $\mu_-(B) = 0$  and  $C$  so that  $\mu_+(C) = 0$ ? (b.2) Show that  $\mu_+$  and  $\mu_-$  are singular to each other. (Thus a singular pair must measure different features.)

- (5) ( $d$ -Dimensional Hausdorff Outer Measure at Scale  $\varepsilon$ . 20pts) Let  $X = \mathbb{R}^2$  denote all points on a 2-D plane. Fixing any  $d > 0$  and  $\varepsilon > 0$ , we define a collection  $\Omega_\varepsilon$  of subsets by

$$\Omega_\varepsilon = \{A \subseteq \mathbb{R}^2 \mid \text{diam}(A) \leq \varepsilon\} = \text{all subsets whose diameters are no greater than } \varepsilon.$$

The diameter is defined by  $\text{diam}(A) = \sup_{\mathbf{p}, \mathbf{q} \in A} d(\mathbf{p}, \mathbf{q})$ , i.e., the furthest paired distance in  $A$ . For example, the diameter of a disk of radius 2 is 4, and that of a rectangle of sides 3 and 4 is  $5 = \sqrt{3^2 + 4^2}$ . The diameter of an empty set  $\phi$  is treated as zero. (a) Show that  $\Omega_\varepsilon$   $\sigma$ -covers the entire plane. (b) For any subset  $A$ , define

$$\rho_d(A) = (\text{diam}(A))^d.$$

State the definition of the outer measure  $\mu_d^*$  (denoted by  $\mu_{\rho_d}^*$  if we are to follow the notations in the lectures) induced by  $(\Omega_\varepsilon, \rho_d)$ . (c) Let  $I = [a, b] \times \{0\}$  denote a closed interval on the horizontal axis from  $a$  to  $b$  (with  $a < b$ ). Show that (c.1) if  $d = 1$ , then  $\mu_1^*(I) \leq b - a$ , while (c.2) for any  $d > 1$ ,  $\mu_d^*(I) = 0$ . (Thus for an interval, it is uninteresting to treat it as a  $d$ -dimensional object with  $d > 1$ .)

- (6) (Lebesgue-Stieltjes Outer Measure. 20pts) Consider  $X = \mathbb{R}$ , and a collection of subsets defined by

$$\Omega = \{(a, b] \mid -\infty < a \leq b < \infty\},$$

i.e., all left-open-and-right-closed *finite* intervals. (a) Show that  $\Omega$   $\sigma$ -covers  $X = \mathbb{R}$ . (b) Let  $x_+ = f(x)$  denote the following (mountain-side) function:

$$x_+ = x, \quad \text{if } x \geq 0; \quad 0, \quad \text{if } x < 0.$$

Define

$$\rho((a, b]) = f(b) - f(a) = b_+ - a_+, \quad \text{for any } a \leq b.$$

State the definition of the outer measure  $\mu_\rho^*$  induced by  $\rho$ . (c) Show that for any subset  $A \subseteq (-\infty, 0]$ ,  $\mu_\rho^*(A) = 0$ . (d) Show that for any finite  $a \leq b$ ,  $\mu_\rho^*([a, b]) \leq b - a$ . (For any general *monotonically increasing* function  $f$ , this outer measure is called the Lebesgue-Stieltjes outer measure, which by Carathéodory's Characterization Theorem, leads to the Lebesgue-Stieltjes measure. We will learn more later.)

**Note.** No lecture on Friday, Oct. 1, which is why this PS2 will be due on Monday, Oct. 4. However, PS3 will be posted online (at [www.math.umn.edu/~jhshen](http://www.math.umn.edu/~jhshen)) on Friday, Oct. 1 on time, and will be due according to the normal schedule (i.e., Friday, Oct. 8).